Divisibility in βN and $^*N$

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The motivation

N - discrete topological space on the set of natural numbers

βN - the set of ultrafilters on N

Principal ultrafilters $\{A \subseteq N : n \in A\}$ are identified with respective elements $n \in N$

Idea: extend the divisibility relation $|$ to βN to get results in number theory

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$\widetilde{|}$-divisibility

$\mathcal{U} = \{S \subseteq N : S \text{ is upward closed for } |\}$

Definition

For $\mathcal{F}, \mathcal{G} \in \beta N$

$$\mathcal{F} \,\widetilde{|}\, \mathcal{G} \text{ iff } \mathcal{F} \cap \mathcal{U} \subseteq \mathcal{G}$$

The restriction of $\widetilde{|}$ to N^2 is the usual $|$

$\widetilde{|}$ is reflexive and transitive, but not antisymmetric. Hence it is an order on $\beta N / \sim$, where

$$\mathcal{F} \sim \mathcal{G} \Leftrightarrow \mathcal{F} \,\widetilde{|}\, \mathcal{G} \wedge \mathcal{G} \,\widetilde{|}\, \mathcal{F}.$$

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Prime ultrafilters

$P \subseteq N$ - the set of primes

Prime ultrafilters: $\mathcal{P} \in \beta N \setminus \{1\}$ divisible only by 1 and themselves

Lemma

$\mathcal{P} \in \beta N$ is prime iff $P \in \mathcal{P}$.

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There are $2^{\mathfrak{c}}$ prime ultrafilters.

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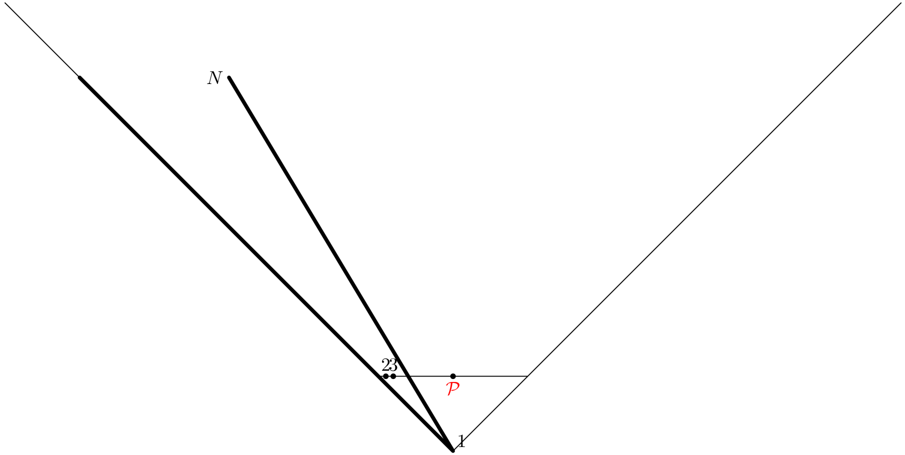

The second level

$A^2 = \{a^2 : a \in A\}$

The only ultrafilter above $\mathcal{P}$ containing P^2 is

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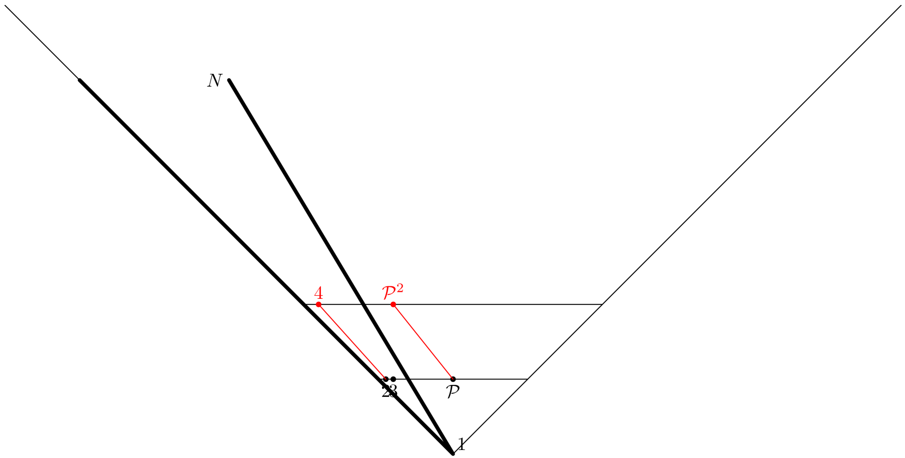

The second level

$A^{(2)} = \{ab : a, b \in A, GCD(a,b) = 1\}$

$F_{(\mathcal{P},2)} = \{A^{(2)} : A \in \mathcal{P}, A \subseteq P\}$

Ultrafilters containing $F_{(\mathcal{P},2)}$ are also divisible only by 1, $\mathcal{P}$ and themselves

Example. $\mathcal{P} \cdot \mathcal{P} \supseteq F_{(\mathcal{P},2)}$

where

$$\mathcal{F} \cdot \mathcal{G} = \{A \in P(N) : \{n \in N : \{m \in N : mn \in A\} \in \mathcal{G}\} \in \mathcal{F}\}.$$

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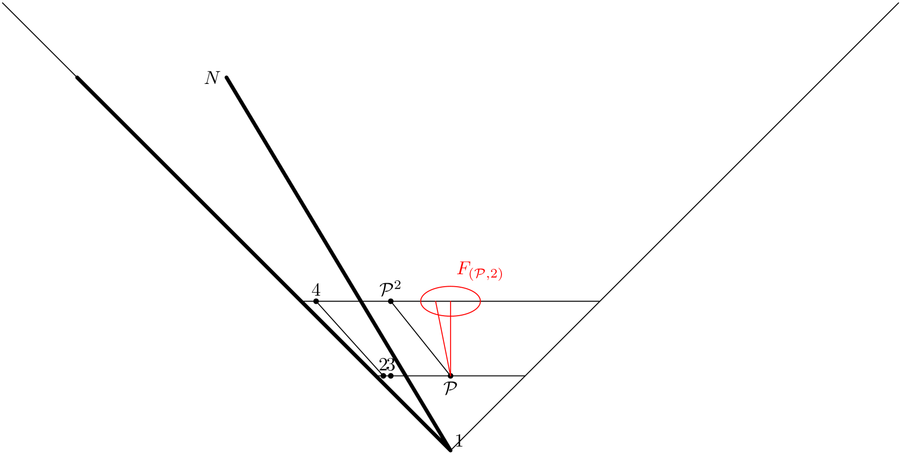

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Theorem

Let $\mathcal{P}$ be prime. There is unique ultrafilter $\mathcal{F} \supseteq F_{(\mathcal{P},2)}$ if and only if $\mathcal{P}$ is Ramsey.

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$AB = \{ab : a \in A, b \in B, GCD(a,b) = 1\}$
$F_{(\mathcal{P},1),(\mathcal{Q},1)} = \{AB : A \in \mathcal{P}, B \in \mathcal{Q}, A, B \subseteq P$ are disjoint$\}$

Ultrafilters containing $F_{(\mathcal{P},1),(\mathcal{Q},1)}$ are divisible only by 1, $\mathcal{P}$, $\mathcal{Q}$ and themselves

They are exactly ultrafilters containing AB for some disjoint $A, B \subseteq P$

Example. $\mathcal{P} \cdot \mathcal{Q}, \mathcal{Q} \cdot \mathcal{P} \supseteq F_{(\mathcal{P},1),(\mathcal{Q},1)}$

Bears similarities to another kind of product of filters

$$\mathcal{F} \times \mathcal{G} = \{X \in P(N^2) : (\exists A \in \mathcal{F})(\exists B \in \mathcal{G}) A \times B \subseteq X\}.$$

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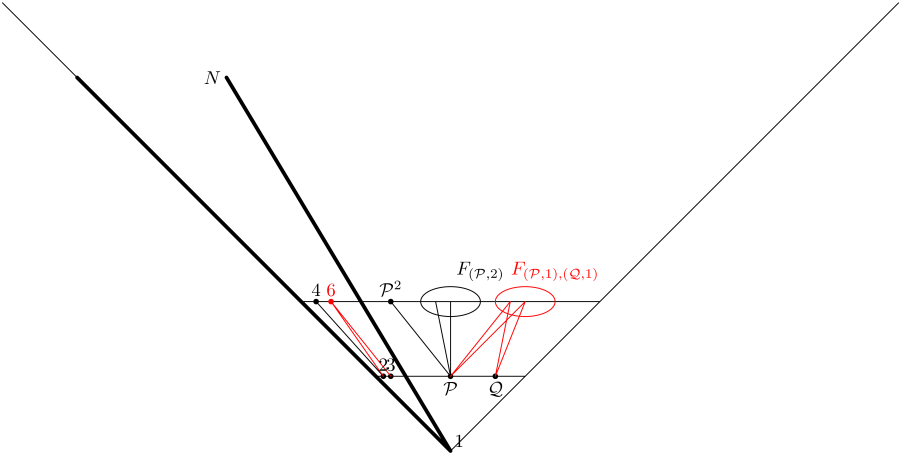

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Theorem

Let $\mathcal{P}, \mathcal{Q}$ be primes. If there is unique $\mathcal{F} \supseteq F_{(\mathcal{P},1),(\mathcal{Q},1)}$ then both $\mathcal{P}$ and $\mathcal{Q}$ are P-points.

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For every prime $\mathcal{P}$ there is a prime $\mathcal{Q}$ such that there are $2^{\mathfrak{c}}$ ultrafilters $\mathcal{F} \supseteq F_{(\mathcal{P},1),(\mathcal{Q},1)}$.

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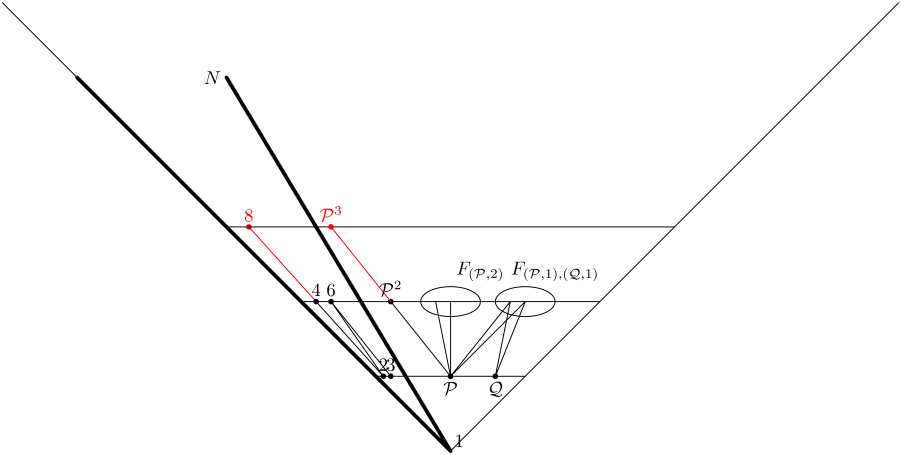

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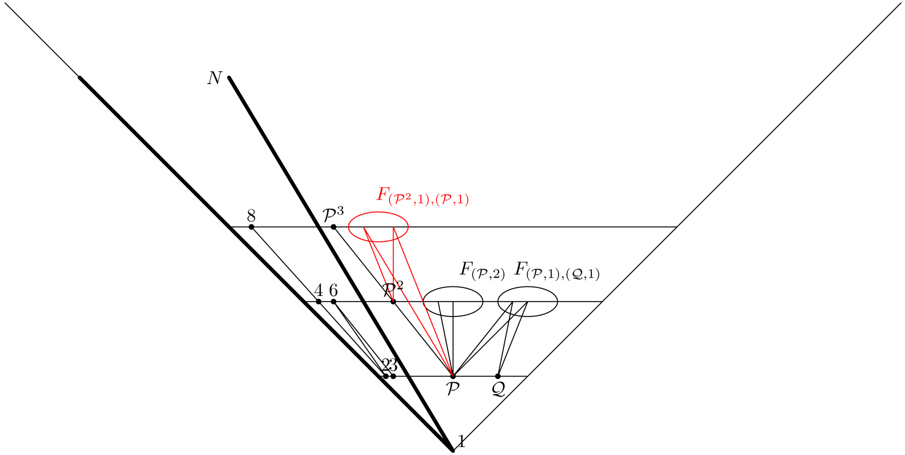

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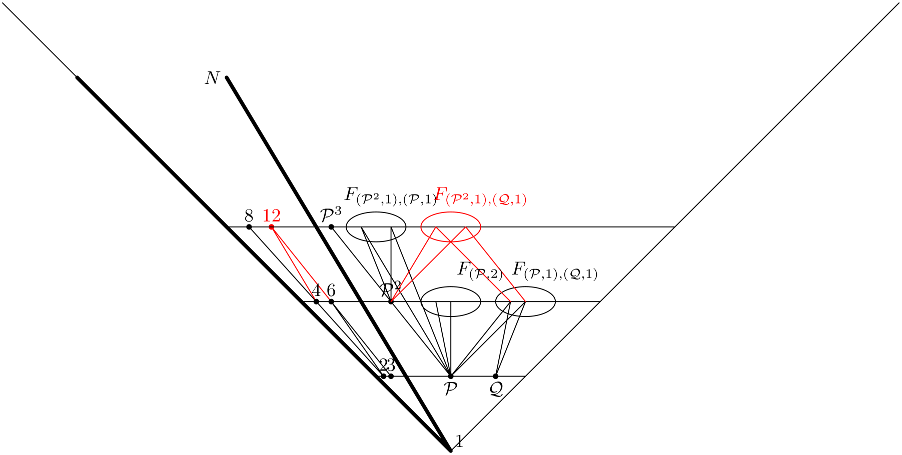

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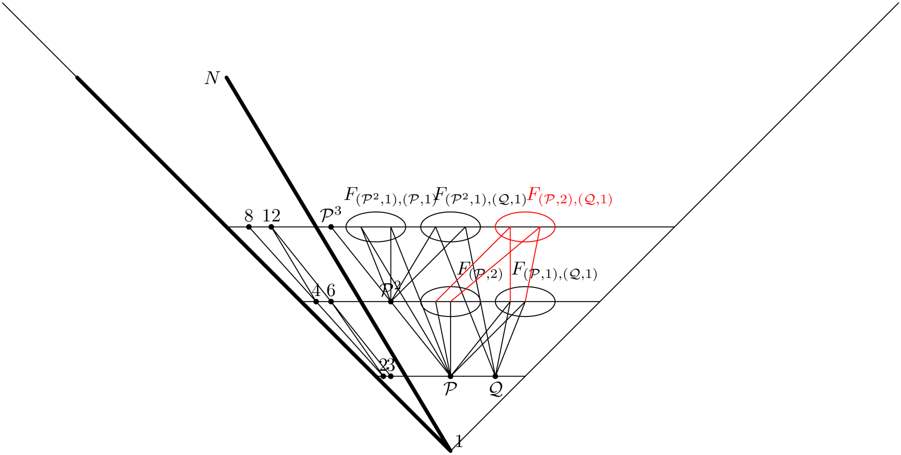

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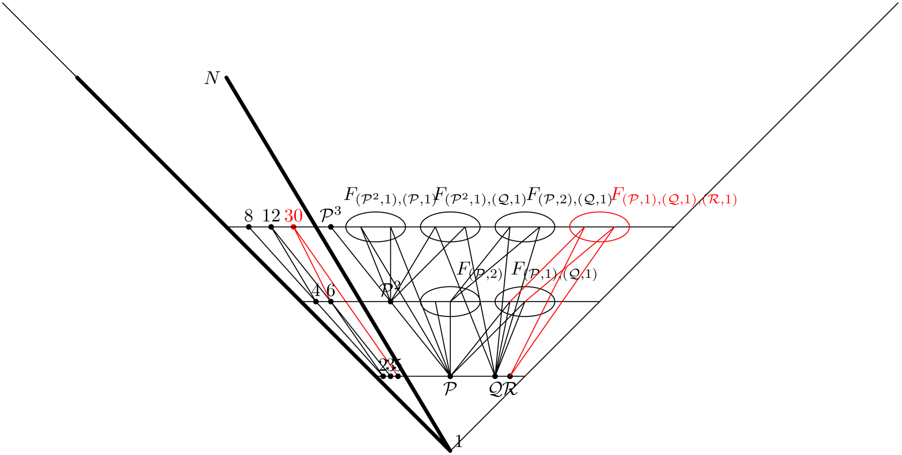

Nonstandard arithmetic

A superstructure over X:
$V_0(X) = X$,
$V_{n+1}(X) = V_n(X) \cup P(V_n(X))$,
$V(X) = \bigcup_{n\in\omega} V_n(X)$.

$V(Y)$ is a nonstandard extension of $V(X)$ if $X \subset Y$ and there is a rank-preserving function $* : V(X) \to V(Y)$ such that ${}^*X = Y$ and satisfying:

The Transfer Principle. For every bounded formula φ and every $a_1, a_2, \ldots, a_n \in V(X)$, $\varphi(a_1, a_2, \ldots, a_n)$ holds in $V(X)$ if and only if $\varphi({}^*a_1, {}^*a_2, \ldots, {}^*a_n)$ holds in $V(Y)$.

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Nonstandard arithmetic

By Transfer, for $x, y \in {}^*N$:

$$x \,{}^*| y \text{ iff } (\exists k \in {}^*N) y = kx.$$

Each element $n \in N$ is identified with ${}^*n$.
In every nonstandard extension $V({}^*N)$ of $V(N)$ holds a generalization of the Fundamental Theorem of Arithmetic. (Here p is the unique increasing function from N to P.)

Theorem

(a) For every $z \in {}^*N$ and every internal sequence $\langle h(n) : n \leq z \rangle$ there is unique $x \in {}^*N$ such that ${}^*p(n)^{h(n)} \,{}^*\| \, x$ for $n \leq z$ and ${}^*p(n) \,{}^*\nmid x$ for $n > z$; we denote such element by $\prod_{n \leq z} {}^*p(n)^{h(n)}$.
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Nonstandard arithmetic

By Transfer, for $x, y \in {}^*N$:

$$x \,{}^*| y \text{ iff } (\exists k \in {}^*N) y = kx.$$

Each element $n \in N$ is identified with ${}^*n$.

In every nonstandard extension $V({}^*N)$ of $V(N)$ holds a generalization of the Fundamental Theorem of Arithmetic. (Here p is the unique increasing function from N to P.)

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The connection

For every $x \in {}^*N$ the family $\{S \subseteq N : x \in {}^*S\}$ is an ultrafilter; it is denoted by $v(x)$.

Thus a function $v : {}^*N \to \beta N$ is obtained. v is onto if $V({}^*N)$ is an enlargement.

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Similarities between $V(^*N)$ and βN:
-for $n \in N$, $v(n) = n$ (the corresponding principal ultrafilter);
-$x \in {}^*N$ is prime iff $v(x)$ is a prime ultrafilter;
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The following conditions are equivalent for every two ultrafilters $\mathcal{F}, \mathcal{G} \in \beta N$:
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Lemma

Let $V({}^*N)$ be any nonstandard extension.
(a) $x \in {}^*N$ is of the form p^2 for some $p \in {}^*P$ if and only if $v(x) = \mathcal{P}^2$ for some prime ultrafilter $\mathcal{P}$.
(b) $x \in {}^*N$ is of the form $p \cdot q$ for two distinct primes p, q such that $v(p) = v(q) = \mathcal{P}$ if and only if $v(x) \supseteq F_{(\mathcal{P},2)}$.
(c) $x \in {}^*N$ is of the form $p \cdot q$ for two primes p, q such that $v(p) = \mathcal{P}$, $v(q) = \mathcal{Q}$ and $\mathcal{P} \neq \mathcal{Q}$ if and only if $v(x) \supseteq F_{(\mathcal{P},1),(\mathcal{Q},1)}$.

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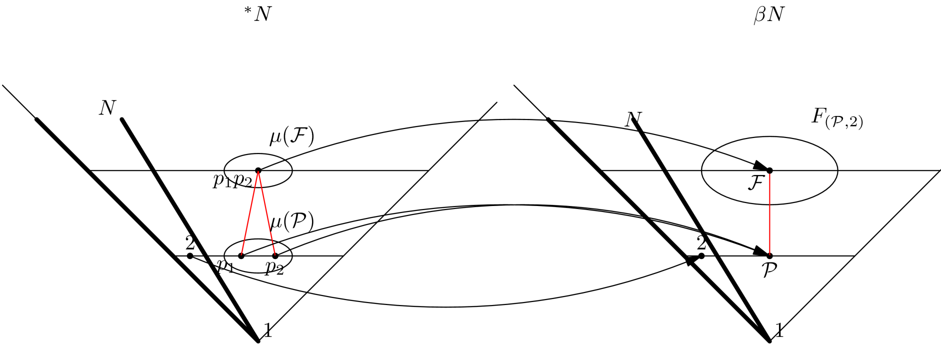

Above finite levels of the $\widetilde{\mid}$-hierarchy

Theorem

(a) There is the $\widetilde{\mid}$-greatest class MAX of ultrafilters.

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(c) Every $\mathcal{F} \in \beta N$ such that there are $p \in P$ and $n \in N \setminus \{0\}$ so that $p^{n*} \| \mathcal{F}$ has an immediate predecessor in $(\beta N, \widetilde{\mid})$.

(d) Every $\widetilde{\mid}$-ascending sequence of ultrafilters of length ω has the least upper bound.

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Thank you for your attention!