Topological isomorphism of oligomorphic groups

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Outline

- The setting
- Profinite groups
- Oligomorphic groups
- Open questions

The setting

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All isomorphisms of groups will be topological isomorphisms.

Two opposite classes

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- Oligomorphic:
  for each k, the action on $\mathbb{N}^k$ has only finitely many orbits
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  for each k, the action on $\mathbb{N}^k$ has only finitely many orbits
  These are the automorphism groups of ω-categorical structures with domain $\mathbb{N}$.
- Profinite: each orbit of the action on $\mathbb{N}$ is finite.
  These are the compact subgroups of S_∞ and up to isomorphism, the inverse limits of finite groups.

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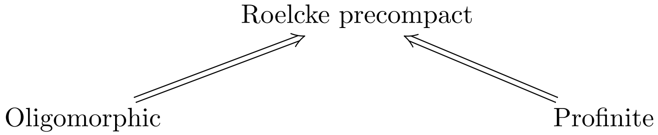

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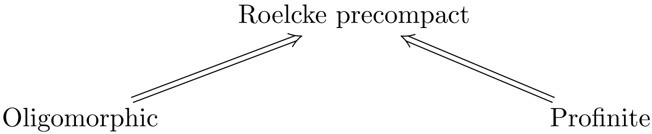
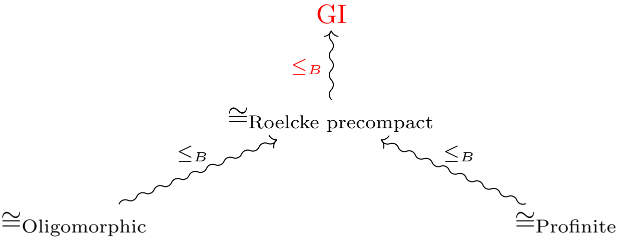


Background on Roelcke precompact groups:

- Tsankov,
  Unitary representations of oligomorphic groups
  Geom. Funct. Anal. 22 (2012)

Previous results on profinite groups



Theorem (Kechris, Nies, Tent)

Isomorphism of Roelcke precompact groups is Borel below graph isomorphism.

Graph isomorphism (GI) is universal for S_∞ orbit equivalence relations. Result independently by Rosendal and Zielinski, JSL 2018

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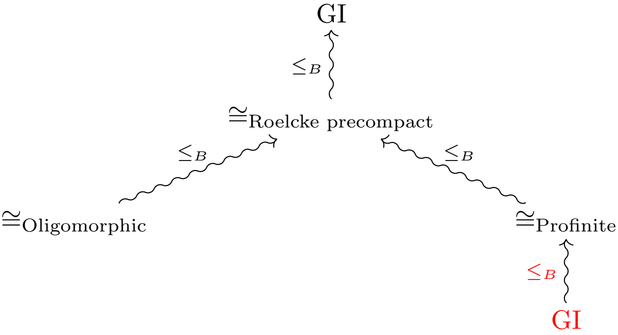
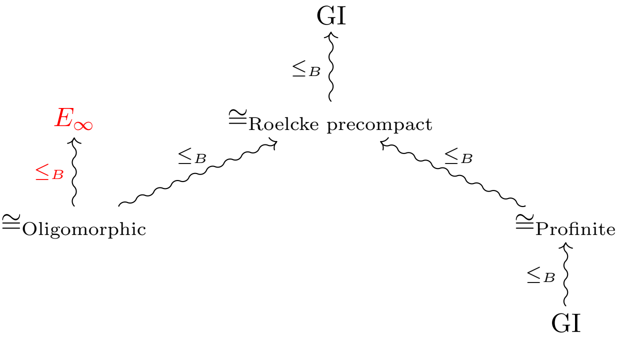


Theorem (Kechris, Nies, Tent)

Graph isomorphism is Borel below isomorphism of profinite groups.

Isomorphism of oligomorphic groups is between $=_{\mathbb{R}}$ and GI.

Main result (Nies, Tent, S.)



Isomorphism of oligomorphic groups is Borel below E_∞.

E_∞ denotes a universal countable Borel equivalence relation.
To be countable means: each class is countable.

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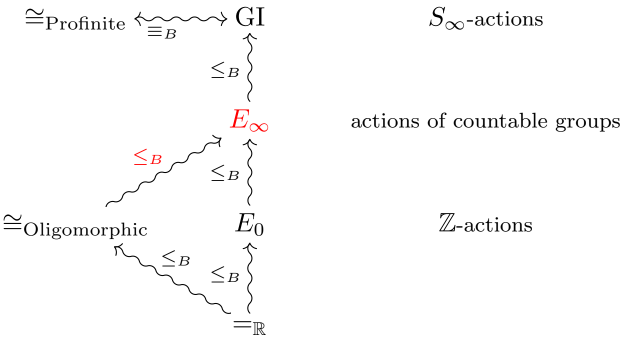

Isomorphism of oligomorphic groups is Borel below E_∞.

E_0 denotes equality with finite error on $2^\mathbb{N}$.

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To define the Borel sets, we start with sets of the form

$$\{G \leq_c S_\infty \colon G \cap N_\sigma \neq \emptyset\},$$

where $G \leq_c S_\infty$ means that G is a closed subgroup of S_∞.

The Borel sets are generated from these basic sets by complementation and countable union.

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Example: for every $f \in S_\infty$, the set $\bigcap_k \{H \colon H \cap N_{f \upharpoonright k} \neq \emptyset\}$ is Borel. It expresses that a closed subgroup of S_∞ contains α.

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Assume that E, F are binary relations on standard Borel spaces X, Y.

Definition. E is *Borel reducible to* F, or $E \leq_B F$, if there is a Borel measurable $r\colon X \to Y$ with

$$(x, y) \in E \iff (r(x), r(y)) \in F.$$

Complexity of the isomorphism relation

for Roelcke precompact and profinite groups

Roelcke precompactness

A closed subgroup G of S_∞ is called Roelcke precompact if
for each open subgroup U of G
there is a finite set $F \subseteq G$ such that $UFU = G$.

This condition is Borel because it suffices to check it for the basic open subgroups $U_n = \{\rho \in G \colon \forall i < n\,[\rho(i) = i]\}$; further, we can pick F from a countable dense set predetermined from G in a Borel way.

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Proof. Each open subgroup U contains a basic open subgroup U_n.
U_n has finitely many double cosets, and U is the union of some of them.
In fact, from G we can in a Borel way determine a listing $A_0, A_1, \ldots$. without repetition of all open cosets.

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Proof.

- Let $\mathcal{M}(G)$ be the structure with domain the open cosets. Via the listing $A_0, A_1, \ldots$ above, we can identify its domain with ω.
- The ternary predicate $R(A, B, C)$ holds in $\mathcal{M}(G)$ if $AB \subseteq C$.
- The main work is to show that for Roelcke precompact $G, H \leq_c S_\infty$,

$$G \cong H \iff \mathcal{M}(G) \cong \mathcal{M}(H).$$

□

Definition

A topological group G is called profinite if one of the following equivalent conditions holds.

(a) G is compact, and the clopen sets form a basis for the topology.
(b) G is the inverse limit of a system of finite groups.
(c) G is isomorphic to a closed subgroup of S_∞ with all orbits finite.

Graph isomorphism $\leq_B$ isomorphism of profinite groups

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Let $\mathcal{N}_2^p$ denote the class of nilpotent-2 groups of exponent p.

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Sketch of proof: A result of Alan Mekler (1981) implies the theorem for countable groups.
A symmetric and irreflexive countable graph is called *nice* if it has no triangles, no squares, and for each pair of distinct vertices x, y, there is a vertex z joined to x and not to y.

Easy fact: Graph isomorphism $\leq_B$ nice graph isomorphism.

Mekler's construction

Isomorphism of nice graphs $\leq_B$ isomorphism of countable groups in $\mathcal{N}_2^p$.

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- Let F be the free $\mathcal{N}_2^p$ group on free generators $x_0, x_1, \ldots$.
- For $r \neq s$ we write $x_{r,s} = [x_r, x_s]$.
- Given a graph with domain $\mathbb{N}$ and edge relation A, let

$$G(A) = F / \langle x_{r,s} \colon rAs \rangle_{\text{normal closure}}.$$

Show that A can be reconstructed from $G(A)$. Then for nice graphs A, B:

$A \cong B$ iff $G(A) \cong G(B)$.

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Now a profinite group $\overline{G}(A)$ in $\mathcal{N}_2^p$ is constructed from $G(A)$ in such a way that A can be recovered from $\overline{G}(A)$.

$A \cong B$ iff $\overline{G}(A) \cong \overline{G}(B)$.

$A \to \overline{G}(A)$ is Borel. So GI $\leq_B$ isomorphism of profinite $\mathcal{N}_2^p$ groups.

Complexity of the isomorphism relation for oligomorphic subgroups of S_∞

Oligomorphic groups

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- This is the opposite of profinite, where each orbit is finite.
- Intuitively, oligomorphic groups are big, profinite groups are small.
- Unlike for profinite groups, G being oligomorphic depends on the way G is embedded into S_∞.

Oligomorphic groups as automorphism groups

Fact

$G \leq_c S_\infty$ is oligomorphic $\iff$ G is the automorphism group of an ω-categorical structure S with domain $\mathbb{N}$.

Proof.

$\Leftarrow$: this follows from the Ryll-Nardzewski Theorem.
$\Rightarrow$: $G = \mathrm{Aut}(S)$ where S is the structure with a k-ary relation symbol for each orbit of G on $\mathbb{N}^k$. □

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- There are 2^ω many Henson digraphs. Their automorphism groups are pairwise non-isomorphic.

Conjugacy of oligomorphic groups

The conjugacy relation for oligomorphic groups is smooth.

To see this,

- Given a closed subgroup G of S_∞, let V_G be the corresponding orbit equivalence structure: for each $k > 0$ introduce a $2k$-ary relation that holds for two k-tuples of distinct elements if they are in the same orbit of $\mathbb{N}^k$.
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- V_G is ω-categorical.
- One checks that for oligomorphic groups G, H

$$G \text{ and } H \text{ are conjugate in } S_\infty \iff V_G \cong V_H.$$

- Isomorphism of ω-categorical structures M, N for the same language is smooth because $M \cong N \iff \mathrm{Th}(M) = \mathrm{Th}(N)$.

Interpretability

An *interpretation* of a structure A in a structure B is a representation of A as a definable set of k-tuples in B quotiented by a definable equivalence relation. The relations and functions of A are also represented in a definable way.

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- If $\Vdash_{\mathbb{P}} \varphi$, then $\mathsf{ZFC} + \varphi$ is interpretable in ZFC.

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Examples:

- Let M be a countably infinite structure in the empty language. Let N be a countably infinite structure with an equivalence relation with all classes of size 2.

  Then M can be interpreted in N. Conversely, N can be interpreted in M: quotient M^3 by the equivalence relation: $(a, b, c) \sim (a', b', c') \Leftrightarrow ((a = b) \wedge (a' = b') \wedge (c = c')) \vee ((a \neq b) \wedge (a' \neq b') \wedge (c = c'))$.

Bi-interpretability

Structures A, B are *bi-interpretable* if there are

- interpretations Γ, of A in B, and Δ, of B in A (all formulas without parameters)
- isomorphisms $\gamma\colon A \cong \Gamma(B)$, $\delta\colon B \cong \Delta(A)$ such that $\delta \circ \gamma$ is definable in A (without parameters) and similarly for $\gamma \circ \delta$.

(Note that $\hat{A} = \Gamma(\Delta(A))$ consists of equivalence classes of tuples from A.)

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Coquand[1] showed that for ω-categorical A, B we have

$\mathrm{Aut}(A) \cong \mathrm{Aut}(B) \iff A$ and B are bi-interpretable.

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  Then M can be interpreted in N and conversely.

Since $\mathrm{Aut}(M) \ncong \mathrm{Aut}(N)$, M, N are not bi-interpretable.

The space of theories

- Theories in a countable language can be identified with elements of $2^{\mathbb{N}}$ via an enumeration of formulas.
- The complete theories form a closed set.
- To be ω-categorical is a Π^0_3 property of theories, because by Ryll-Nardzewski this property is equivalent to saying that for each n, the Boolean algebra of formulas with at most n free variables modulo T-equivalence is finite.

Bi-interpretability of structures via their theories

We can express bi-interpretability of ω-categorical structures A, B in terms of their theories:

- $A \cong \Gamma(B)$ means that $\mathrm{Th}(B)$ says "the structure interpreted in B via Γ satisfies $\mathrm{Th}(A)$"
- similar for $B \cong \Delta(A)$
- also express that some $\gamma : A \cong \Gamma(\Delta(A))$ is defined by a particular first order formula.

For ω-categorical theories, it suffices that one of the compositions (of the interpretations) is definable.

Bi-interpretability of ω-categorical theories

Theorem (Nies, Tent, S.)

There is a Σ_2^0 relation which coincides with bi-interpretability on the Π_3^0 set of ω-categorical theories.

Given ω-categorical theories S, T. We have an initial block of existential quantifiers fixing the dimensions of the interpretations and asserting the existence of the definable isomorphism γ.

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- The rest is easy if the signature if finite
- In general, we have to express that a certain tree computed from S, T is infinite. The tree is matching types of S and types of T in a way consistent with γ being an isomorphism.
- The branching of the tree is bounded depending on S, T, because the dimensions are fixed, and for each arity there are only so many types. So it is Π^0_1 in S, T to say that the tree is infinite.

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Corollary

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Proof of Corollary.

- There is a finer Polish topology with the same Borel sets in which the set of ω-categorical theories is closed.
- Then the Σ^0_2 relation above yields a $\boldsymbol{\Sigma^0_2}$ description of bi-interpretability on this closed set.

□

Oligomorphic groups “are” countable models

Theorem

Isomorphism of oligomorphic groups is Borel bi-reducible with the orbit equivalence relation of a Borel action $S_\infty \curvearrowright \mathcal{B}$; where

- *$\mathcal{B}$ is an invariant Borel set of models with domain $\mathbb{N}$ for the language with one ternary relation symbol,*
- *the action of S_∞ is the natural one.*

A Borel equivalence relation on a Polish space is called *countable* if every equivalence class is countable.

Corollary

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Proof.

- Above we proved that isomorphism of oligomorphic groups is Borel reducible to a $\mathbf{\Sigma}^0_2$ equivalence relation on a Polish space.
- So the isomorphism relation on $\mathcal{B}$ in the foregoing Theorem is Borel reducible to a $\mathbf{\Sigma}^0_2$ equivalence relation.
- By Hjorth and Kechris (1995; Theorem 3.8): If an S_∞ orbit equivalence relation is Borel reducible to a $\mathbf{\Sigma}^0_2$ equivalence relation, then it is reducible to a countable equivalence relation.

□

Theorem

Isomorphism of oligomorphic groups is Borel bi-reducible with the orbit equivalence relation of the natural action of S_∞ on an isomorphism invariant Borel set $\mathcal{B}$ of models.

For Roelcke precompact G, we defined a structure $\mathcal{M}(G)$ with domain consisting of the cosets of open subgroups. We can in a Borel way find a bijection of these cosets with $\mathbb{N}$. We showed

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We will define an "inverse" operation $\mathcal{G}$ of the operation $\mathcal{M}$ on a Borel set $\mathcal{B}$ of models. For oligomorphic G and $M \in \mathcal{B}$ we will have

$$\mathcal{G}(\mathcal{M}(G)) \cong G \text{ and } \mathcal{M}(\mathcal{G}(M)) \cong M$$

This suffices because it implies the converse reduction

$$\mathcal{G}(M) \cong \mathcal{G}(N) \iff M \cong N.$$

Axiomatizing the range of the map $\mathcal{M}$

- We actually define the map $\mathcal{G}$ on an invariant co-analytic set $\mathcal{D}$ of L-structures that contains range($\mathcal{M}$).
- Then range($\mathcal{M}$) $\subseteq \mathcal{B} \subseteq \mathcal{D}$ for an invariant Borel set $\mathcal{B}$.

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- Then range($\mathcal{M}$) $\subseteq \mathcal{B} \subseteq \mathcal{D}$ for an invariant Borel set $\mathcal{B}$.
- Since $\mathcal{M}(\mathcal{G}(M)) \cong M$ for each $M \in \mathcal{B}$, actually $\mathcal{B}$ equals the closure of range($\mathcal{M}$) under isomorphism.
- We will observe a number of properties, called axioms, of all the structures of the form $\mathcal{M}(G)$. They can be expressed in Π^1_1 form.
- $\mathcal{D}$ is the set of structures satisfying all the axioms.

Definable relations in $\mathcal{M}(G)$

Recall that our language L only has one ternary relation $R(A, B, C)$ (which is interpreted by $AB \subseteq C$ for cosets A, B, C).

- The property of A to be a *subgroup* is definable in $\mathcal{M}(G)$ by the formula $AA \subseteq A$. That a subgroup A is contained in a subgroup B is definable by the formula $AB \subseteq B$.

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The first few axioms posit for a general L-structure M that the formulas behave reasonably. E.g., $\subseteq$ is transitive. We use terms like "*subgroup*", "*left coset* of" to refer to elements satisfying them.

The filter group $\mathcal{F}(M)$

Given a structure M, denote by $\mathcal{F}(M)$ the set of filters (for $\subseteq$) that contain a left and a right coset of each subgroup. (These cosets are unique because axioms require that distinct left cosets are disjoint etc.) We use letters x, y, z for filters.

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The filter of *subgroups* is the identity 1.

The filter group $\mathcal{F}(M)$

We can express by Π^1_1 axioms that these operations behave as a group: associativity, and $\forall x\, [x \cdot x^{-1} = 1]$.

The sets $\{x \colon U \in x\}$, where $U \in M$ is a *subgroup*, are declared a basis of neighbourhoods for the identity. Using the right axioms, we ensure that $\mathcal{F}(M)$ is a Polish group.

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Further, for each subgroup $V \in M$, there is an action $\mathcal{F}(M) \curvearrowright LC(V)$ given by

$$x \cdot A = B \text{ iff } \exists S \in x\,[SA \subseteq B],$$

where $LC(V)$ denotes the set of *left cosets* of V.

A faithful subgroup

- For oligomorphic G, there is an open subgroup V such that the action $G \curvearrowright LC(V)$ is oligomorphic:
  e.g. let $V = G_{\{n_1,\ldots,n_k\}}$ (the pointwise stabilizer) where the n_i represent the k many 1-orbits. Call such a V a *faithful* subgroup.

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  e.g. let $V = G_{\{n_1,\ldots,n_k\}}$ (the pointwise stabilizer) where the n_i represent the k many 1-orbits. Call such a V a *faithful* subgroup.
- As a further axiom for an abstract L-structure M, we require the existence of such V, and that the embedding of $\mathcal{F}(M)$ into S_∞ is topological (these axioms are Π^1_1 but not first-order).
- Then $\mathcal{F}(M)$ is oligomorphic and hence Roelcke precompact.

Showing that the coset structure of $\mathcal{F}(M)$ is isomorphic to M

Mainly, we have to show that each open subgroup $\mathcal{U}$ of $\mathcal{F}(M)$ has the form $\mathcal{U} = \{x \colon U \in x\}$ for some *subgroup* U in M.

- By definition of the topology, $\mathcal{U}$ contains a basic open subgroup $\hat{W} = \{x \colon W \in x\}$, for some *subgroup* $W \in M$.

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- Since $\mathcal{F}(M)$ is Roelcke precompact, $\mathcal{U}$ is a finite union of double cosets of $\hat{W}$.
- We require as an axiom for M that each such finite union that is closed under the group operations corresponds to an actual *subgroup* in M.

Turning $\mathcal{F}(M)$ into closed subgroup $\mathcal{G}(M)$ of S_∞

- By Π^1_1 uniformization (Addison/Kondo), from $M \in \mathcal{B}$ we can in a Borel way determine a faithful subgroup V.

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- Let $A_0, A_1, \ldots$ list $LC(V)$ in the natural order.
- Then the action $\mathcal{F}(M) \curvearrowright LC(V)$ yields a topological embedding of $\mathcal{F}(M)$ into S_∞.
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By the arguments above we have for each oligomorphic G and each $M \in \mathcal{B}$

$$\mathcal{G}(\mathcal{M}(G)) \cong G \text{ and } \mathcal{M}(\mathcal{G}(M)) \cong M.$$

Theorem

Isomorphism of oligomorphic groups is Borel bi-reducible with the orbit equivalence relation of the natural action of S_∞ on a Borel set $\mathcal{B}$ of models.

Outlook: model theoretic characterization of complexity

Let $\mathcal{C}$ be an invariant Borel set of countable structures.

Theorem (Hjorth, Kechris)

TFAE:

- $\cong_{\mathcal{C}}$ *is smooth.*
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References

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