Cardinal invariants above the continuum

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Outline

1. Some cardinal invariants at regular cardinals
2. Consistency results
3. A ZFC result
Some cardinal invariants at regular cardinals

Definition

Let $\kappa \geq \omega$ be a regular cardinal. Let $f, g \in \kappa^\kappa$. $f \leq^* g$ means that
$$|\{\alpha < \kappa : g(\alpha) < f(\alpha)\}| < \kappa$$

Definition

We say that $F \subseteq \kappa^\kappa$ is $^*$-unbounded if $\neg \exists g \in \kappa^\kappa \forall f \in F [f \leq^* g]$.

Definition

$$b(\kappa) = \min\{|F| : F \subseteq \kappa^\kappa \land F \text{ is } ^*-\text{unbounded}\}.$$
Definition

We say that $F \subseteq \kappa^\kappa$ is **-dominating if $\forall g \in \kappa^\kappa \exists f \in F \left[ g \leq^* f \right]$.

Definition

$d(\kappa) = \min \{|F| : F \subseteq \kappa^\kappa \text{ and } F \text{ is } * \text{-dominating}\}$. 
**Definition**

We say that $F \subseteq \kappa^\kappa$ is *-dominating if $\forall g \in \kappa^\kappa \exists f \in F \ [g \leq^* f]$.

**Definition**

$d(\kappa) = \min \{|F| : F \subseteq \kappa^\kappa \text{ and } F \text{ is } * \text{-dominating}\}$.

**Theorem**

For any regular $\kappa \geq \omega$, $\kappa^+ \leq \text{cf}(b(\kappa)) = b(\kappa) \leq \text{cf}(d(\kappa)) \leq d(\kappa) \leq 2^\kappa$.

These are the only relations between $b(\kappa)$ and $d(\kappa)$ that are provable in ZFC (Hechler for $\omega$; Cummings and Shelah for $\kappa > \omega$).
When $\kappa > \omega$, we can also use the club filter.

**Definition**

Let $\kappa > \omega$ be a regular cardinal. $f \leq_{cl} g$ means that $\{\alpha < \kappa : g(\alpha) < f(\alpha)\}$ is non-stationary. For $F \subseteq \kappa^\kappa$, we say that:

- $F$ is **cl-unbounded** if $\neg \exists g \in \kappa^\kappa \forall f \in F \left[ f \leq_{cl} g \right]$, and
- $F$ is **cl-dominating** if $\forall g \in \kappa^\kappa \exists f \in F \left[ g \leq_{cl} f \right]$.

**Definition**

We define

$$b_{cl}(\kappa) = \min\{|F| : F \subseteq \kappa^\kappa \land F \text{ is cl-unbounded}\},$$

$$d_{cl}(\kappa) = \min\{|F| : F \subseteq \kappa^\kappa \text{ and } F \text{ is cl-dominating}\}.$$
Theorem (Cummings and Shelah)

For every regular cardinal $\kappa > \omega$, $b(\kappa) = b_{cl}(\kappa)$. 
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For every regular cardinal $\kappa > \omega$, $b(\kappa) = b_{cl}(\kappa)$.

Theorem (Cummings and Shelah)

If $\kappa \geq \beth_\omega$ is regular, then $d(\kappa) = d_{cl}(\kappa)$.

Question

Does $d(\kappa) = d_{cl}(\kappa)$, for every regular uncountable $\kappa$? In particular, does $d(\omega_1) = d_{cl}(\omega_1)$?
Definition

Let $\kappa \geq \omega$ be regular.

- For $A, B \in \mathcal{P}(\kappa)$, $A$ splits $B$ if $|B \cap A| = |B \cap (\kappa \setminus A)| = \kappa$.
- $F \subseteq \mathcal{P}(\kappa)$ is called a splitting family if $\forall B \in [\kappa]^\kappa \exists A \in F \ [A \text{ splits } B]$.

$$s(\kappa) = \min\{|F| : F \subseteq \mathcal{P}(\kappa) \land F \text{ is a splitting family}\};$$

Theorem (Solomon)

$$\omega_1 \leq s(\omega) \leq d(\omega).$$
Theorem (Suzuki)

For a regular $\kappa > \omega$, $s(\kappa) \geq \kappa$ iff $\kappa$ is strongly inaccessible and $s(\kappa) \geq \kappa^+$ iff $\kappa$ is weakly compact.

- So if $\kappa$ is not weakly compact, then $s(\kappa) < \kappa^+ \leq b(\kappa)$. 
Theorem (Suzuki)

For a regular \( \kappa > \omega \), \( s(\kappa) \geq \kappa \) iff \( \kappa \) is strongly inaccessible and \( s(\kappa) \geq \kappa^+ \) iff \( \kappa \) is weakly compact.

So if \( \kappa \) is not weakly compact, then \( s(\kappa) < \kappa^+ \leq b(\kappa) \).

Theorem (Zapletal)

If it is consistent to have a regular uncountable cardinal \( \kappa \) such that \( s(\kappa) \geq \kappa^{++} \), then it is also consistent that there is a \( \kappa \) with \( o(\kappa) \geq \kappa^{++} \).

Theorem (Ben-Neria and Gitik)

If \( o(\kappa) = \kappa^{++} \), then there is a forcing extension in which \( s(\kappa) = \kappa^{++} \).

\( \kappa \) does not remain measurable in their model.
Question

What is the consistency strength of the statement that $\kappa$ is a measurable cardinal and $s(\kappa) = \kappa^{++}$?
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What is the consistency strength of the statement that $\kappa$ is a measurable cardinal and $s(\kappa) = \kappa^{++}$?

- $s(\omega)$ and $b(\omega)$ are independent.

Theorem (Baumgartner and Dordal)

*It is consistent to have $s(\omega) < b(\omega)$.*

Theorem (Shelah)

*It is consistent to have $\omega_1 = b(\omega) < s(\omega) = \omega_2$.*

- Historically, Shelah’s result was the first published use of creature forcing.
It turns out the $\omega$ is the **only regular cardinal** for which the statement $b(\kappa) < s(\kappa)$ is consistent.

**Theorem (R. and Shelah[2])**

*For any regular uncountable cardinal $\kappa$, $s(\kappa) \leq b(\kappa)$.***
- It turns out the $\omega$ is the **only regular cardinal** for which the statement $b(\kappa) < s(\kappa)$ is consistent.

**Theorem (R. and Shelah[2])**

For any regular uncountable cardinal $\kappa$, $s(\kappa) \leq b(\kappa)$.

- $b(\omega)$ and $d(\omega)$ are dual to each other
- The dual of $s(\omega)$ is $t(\omega)$.

**Definition**

*For a family $F \subseteq [\kappa]^\kappa$ and a set $B \in \mathcal{P}(\kappa)$, $B$ is said to **reap** $F$ if for every $A \in F$, $|A \cap B| = |A \cap (\kappa \setminus B)| = \kappa$. We say that $F \subseteq [\kappa]^\kappa$ is **unreaped** if there is no $B \in \mathcal{P}(\kappa)$ that reaps $F$.***
- $F \subseteq [\kappa]^\kappa$ is unreaped iff each $B \in \mathcal{P}(\kappa)$ is decided by some member of $F$.

**Definition**

$$r(\kappa) = \min \{|F| : F \subseteq [\kappa]^\kappa \text{ and } F \text{ is unreaped}\}.$$
$F \subseteq [\kappa]^\kappa$ is unreaped iff each $B \in \mathcal{P}(\kappa)$ is decided by some member of $F$.

**Definition**

$$r(\kappa) = \min \{|F| : F \subseteq [\kappa]^\kappa \text{ and } F \text{ is unreaped}\}.$$  

The proof of $s(\omega) \leq d(\omega)$ dualizes to the proof of $b(\omega) \leq r(\omega)$.

Also $r(\omega)$ and $d(\omega)$ are independent.

Not clear if there is a good theory of duality at uncountable regular cardinals too.

For example, Suzuki’s theorem says that $s(\kappa)$ is small unless $\kappa$ is weakly compact.

So we might expect that $r(\kappa)$ is large below the first weakly compact cardinal.
**Question**

*Is it consistent (relative to large cardinals) that there is some uncountable regular cardinal $\kappa$ below the first weakly compact cardinal such that $r(\kappa) < 2^\kappa$?*

- My conjecture is yes (so Suzuki’s theorem has no dual).
Question

Is it consistent (relative to large cardinals) that there is some uncountable regular cardinal \( \kappa \) below the first weakly compact cardinal such that \( r(\kappa) < 2^\kappa \)?

- My conjecture is yes (so Suzuki’s theorem has no dual).
- The proof that for all \( \kappa > \omega, \ s(\kappa) \leq b(\kappa) \) does not dualize.
- But the theorem does have a partial dual:
Question

Is it consistent (relative to large cardinals) that there is some uncountable regular cardinal $\kappa$ below the first weakly compact cardinal such that $r(\kappa) < 2^\kappa$?

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- The proof that for all $\kappa > \omega$, $s(\kappa) \leq b(\kappa)$ does not dualize.
- But the theorem does have a partial dual:

**Theorem (R. + Shelah [3])**

For all regular cardinals $\kappa \geq \bethomega$, $d(\kappa) \leq r(\kappa)$.

- So for sufficiently large $\kappa$, $s(\kappa) \leq b(\kappa) \leq d(\kappa) \leq r(\kappa)$ provably in ZFC.
Question

Is $d(\aleph_1) \leq r(\aleph_1)$ provable? Is $d(\kappa) \leq r(\kappa)$ provable for all regular $\kappa < \beth_\omega$?

Question

Is it consistent (relative to large cardinals) that $r(\omega_1) < 2^{\aleph_1}$?
Question

Is $d(\aleph_1) \leq r(\aleph_1)$ provable? Is $d(\kappa) \leq r(\kappa)$ provable for all regular $\kappa < \beth_\omega$?

Question

Is it consistent (relative to large cardinals) that $r(\omega_1) < 2^{\aleph_1}$?

- This is related to an old question of Kunen about bases for uniform ultrafilters.

Definition

Let $\kappa \geq \omega$ be regular. Let $\mathcal{U}$ be an ultrafilter on $\kappa$. We say that:

- $\mathcal{U}$ is uniform if every element of $\mathcal{U}$ has cardinality $\kappa$;
- $F \subseteq \mathcal{P}(\kappa)$ is a base for $\mathcal{U}$ if $\mathcal{U} = \{B \subseteq \kappa : \exists A \in F [A \subseteq B]\}$. 
Definition

\[ u(\kappa) = \min\{|F| : F \text{ is a base for a uniform ultrafilter on } \kappa \} . \]

- Clearly \( r(\kappa) \leq u(\kappa) . \)
- \( u(\omega) \) and \( s(\omega) \) are independent.
- However for \( \kappa > \omega , \ s(\kappa) \leq b(\kappa) \leq r(\kappa) . \)
Definition

\[ u(\kappa) = \min\{|F| : F \text{ is a base for a uniform ultrafilter on } \kappa\} \].

- Clearly \( r(\kappa) \leq u(\kappa) \).
- \( u(\omega) \) and \( s(\omega) \) are independent.
- However for \( \kappa > \omega \), \( s(\kappa) \leq b(\kappa) \leq r(\kappa) \).

Question (Kunen)

Is it consistent that \( u(\omega_1) < 2^{\aleph_1} \)?

Theorem (Garti and Shelah)

If \( \kappa \) is supercompact, then \( u(\kappa) < 2^{\kappa} \) is consistent.
Definition

Let $\kappa \geq \omega$ be a regular cardinal.

- $A, B \in [\kappa]^\kappa$ are said to be almost disjoint or a.d. if $|A \cap B| < \kappa$.
- A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is said to be almost disjoint or a.d. if the members of $\mathcal{A}$ are pairwise a.d.
- Finally $\mathcal{A} \subseteq [\kappa]^\kappa$ is called maximal almost disjoint or m.a.d. if $\mathcal{A}$ is an a.d. family, $|\mathcal{A}| \geq \kappa$, and $\mathcal{A}$ cannot be extended to a larger a.d. family in $[\kappa]^\kappa$. 
Definition

Let $\kappa \geq \omega$ be a regular cardinal.

- $A, B \in [\kappa]^\kappa$ are said to be **almost disjoint** or a.d. if $|A \cap B| < \kappa$.
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Definition

$a(\kappa) = \min \{|\mathcal{A}| : \mathcal{A} \subseteq [\kappa]^\kappa \text{ and } \mathcal{A} \text{ is m.a.d.}\}$. 
Theorem (Rothberger)

For any regular $\kappa \geq \omega$, $b(\kappa) \leq a(\kappa)$.

Theorem (Shelah)

It is consistent to have $\aleph_1 = b(\omega) < a(\omega) = \aleph_2 = s(\omega)$. It is also consistent to have $\aleph_1 = b(\omega) = a(\omega) < s(\omega)$. 
Theorem (Rothberger)
For any regular $\kappa \geq \omega$, $b(\kappa) \leq a(\kappa)$.

Theorem (Shelah)
It is consistent to have $\aleph_1 = b(\omega) < a(\omega) = \aleph_2 = s(\omega)$. It is also consistent to have $\aleph_1 = b(\omega) = a(\omega) < s(\omega)$.

- It turns out that $\omega$ is the only regular $\kappa$ where $b(\kappa) = \kappa^+ < \kappa^{++} = a(\kappa)$ is consistent.

Theorem (R. + Shelah)
If $\kappa > \omega$ is regular, then $b(\kappa) = \kappa^+$ implies $a(\kappa) = \kappa^+$. 
Theorem (Blass, Hyttinen, and Zhang)

Let $\kappa > \omega$ be regular. If $d(\kappa) = \kappa^+$, then $a(\kappa) = \kappa^+$. 
Theorem (Blass, Hyttinen, and Zhang)

Let $\kappa > \omega$ be regular. If $d(\kappa) = \kappa^+$, then $a(\kappa) = \kappa^+$.

Question (Roitman)

Does $d(\omega) = \aleph_1$ imply that $a(\omega) = \aleph_1$?

Theorem (Shelah)

It is consistent to have $\aleph_2 = d(\omega) < a(\omega) = \aleph_3$.

- He actually gave two different proofs of $\text{Con}(d(\omega) < a(\omega))$.
- The first proof used ultrapowers and needed a measurable cardinal $\theta$ to produce a model with $\theta < d(\omega) < a(\omega)$.
- The other proof used templates and produced a model with $d(\omega) = \aleph_2$. 
The first proof also works for $\mathfrak{u}(\omega)$.

**Theorem (Shelah)**

Suppose there is a measurable cardinal $\theta$. Then there is a c.c.c. forcing extension in which $\theta < \mathfrak{u}(\omega) < \mathfrak{a}(\omega)$. 

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The first proof also works for $u(\omega)$.

\textbf{Theorem (Shelah)}

Suppose there is a measurable cardinal $\theta$. Then there is a c.c.c. forcing extension in which $\theta < u(\omega) < a(\omega)$.

\textbf{Question}

What is the consistency strength of $u(\omega) < a(\omega)$?
Some cardinal invariants at regular cardinals

Consistency results

A ZFC result

Bibliography

Consistency results

- R. + Shelah used the method of Boolean ultrapowers to get several consistency results involving $\alpha(\kappa)$.

Theorem (R. + Shelah [1])

For any regular $\kappa > \omega$, $\mathfrak{d}(\kappa) < \alpha(\kappa)$ is consistent relative to a supercompact cardinal.

- This is analogous to Shelah’s first result that $\mathfrak{d}(\omega) < \alpha(\omega)$ is consistent relative to a measurable.
- The consistency of $\mathfrak{b}(\kappa) < \alpha(\kappa)$ for uncountable $\kappa$ was also unknown before this result.
Theorem

More specifically, suppose \( \aleph_0 < \kappa = \kappa^{<\kappa} < \theta \) and that \( \theta \) is supercompact. Then there is a forcing extension in which \( \theta < b(\kappa) = d(\kappa) < a(\kappa) \).

We can also arrange \( b(\kappa) \) and \( d(\kappa) \) to be different.

Theorem (R. + Shelah [1])

Suppose \( \aleph_0 < \kappa = \kappa^{<\kappa} < \theta \) and that \( \theta \) is supercompact. Then there is a forcing extension in which \( \theta < b(\kappa) < d(\kappa) < a(\kappa) \).
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More specifically, suppose \( \aleph_0 < \kappa = \kappa^{<\kappa} < \theta \) and that \( \theta \) is supercompact. Then there is a forcing extension in which \( \theta < b(\kappa) = d(\kappa) < a(\kappa) \).

- We can also arrange \( b(\kappa) \) and \( d(\kappa) \) to be different.

Theorem (R. + Shelah [1])

Suppose \( \aleph_0 < \kappa = \kappa^{<\kappa} < \theta \) and that \( \theta \) is supercompact. Then there is a forcing extension in which \( \theta < b(\kappa) < d(\kappa) < a(\kappa) \).

Question

What is the consistency strength of the statement that there is an uncountable regular cardinal \( \kappa \) for which \( d(\kappa) < a(\kappa) \), or even \( b(\kappa) < a(\kappa) \)?
Question

For uncountable regular $\kappa$, does $b(\kappa) = \kappa^{++}$ imply that $\alpha(\kappa) = \kappa^{++}$?
Question

For uncountable regular \( \kappa \), does \( b(\kappa) = \kappa^{++} \) imply that \( a(\kappa) = \kappa^{++} \) ?

Theorem (R. + Shelah [1])

If \( \kappa \) is a Laver indestructible supercompact cardinal, then \( u(\kappa) < a(\kappa) \) is consistent relative to a supercompact cardinal above \( \kappa \). More specifically, suppose that \( \kappa < \theta \), that \( \theta \) is supercompact, and that \( \kappa \) is Laver indestructible supercompact. Then there is a forcing extension in which \( \theta < u(\kappa) < a(\kappa) \).

This is analogous to Shelah’s that \( \theta < u(\omega) < a(\omega) \) is consistent if \( \theta \) is measurable.
Definition

Suppose $\theta$ supercompact, $\theta \leq \mu = \mu^{<\theta} < \mu^+ < \chi$. $\mathcal{B}_{\chi,\mu,\theta}$ is the completion of $\text{Fn}(\chi, \mu, \theta) = \{f : \text{dom}(f) \in [\chi]^{<\theta} \text{ and } \text{ran}(f) \subseteq \mu\}$ ordered by reverse inclusion.
Definition

Suppose \( \theta \) supercompact, \( \theta \leq \mu = \mu^{<\theta} < \mu^+ < \chi \). \( \mathcal{B}_{\chi, \mu, \theta} \) is the completion of \( \text{Fn}(\chi, \mu, \theta) = \{ f : \text{dom}(f) \in [\chi]^{<\theta} \text{ and } \text{ran}(f) \subseteq \mu \} \) ordered by reverse inclusion.

- Build a \( \theta \)-complete optimal ultrafilter \( D \) on \( \mathcal{B}_{\chi, \mu, \theta} \) (using the fact that \( \theta \) is supercompact).
- For getting a model with \( b(\kappa) = d(\kappa) < a(\kappa) \), fix the usual iteration \( P \) for forcing \( b(\kappa) = d(\kappa) = \mu^+ \).
Definition

Suppose \( \theta \) supercompact, \( \theta \leq \mu = \mu^{<\theta} < \mu^+ < \chi \). \( \mathcal{B}_{\chi,\mu,\theta} \) is the completion of \( \text{Fn}(\chi, \mu, \theta) = \{ f : \text{dom}(f) \in [\chi]^{<\theta} \text{ and } \text{ran}(f) \subseteq \mu \} \) ordered by reverse inclusion.

- Build a \( \theta \)-complete optimal ultrafilter \( D \) on \( \mathcal{B}_{\chi,\mu,\theta} \) (using the fact that \( \theta \) is supercompact).
- For getting a model with \( b(\kappa) = d(\kappa) < a(\kappa) \), fix the usual iteration \( \mathbb{P} \) for forcing \( b(\kappa) = d(\kappa) = \mu^+ \).
- Let \( \mathbb{Q} = \mathbb{P}[\mathcal{B}_{\chi,\mu,\theta}] / D \).
- Forcing with \( \mathbb{Q} \) preserves \( b(\kappa) = d(\kappa) = \mu^+ \) and makes \( a(\kappa) = \text{cf}(\chi) \).
An application of PCF theory

**Theorem**

For any regular $\kappa \geq \beth_\omega$, $d(\kappa) \leq r(\kappa)$.

**Definition**

Let $\kappa > \omega$ be a regular cardinal. If $A \in [\kappa]^\kappa$, then we define a function $s_A : \kappa \to A$ by setting $s_A(\alpha) = \min(A \setminus (\alpha + 1))$, for each $\alpha \in \kappa$.

**Definition**

Let $E_2 \subseteq E_1$ both be clubs in $\kappa$. For each $\xi \in \kappa$, define $\text{set}(E_1, \xi) = \{\zeta < s_{E_1}(\xi) : \xi \leq \zeta\}$. Define $\text{set}(E_2, E_1) = \bigcup \{\text{set}(E_1, \xi) : \xi \in E_2\}$. 
Assume $\kappa \geq \beth_\omega$. Let $F \subseteq [\kappa]^\kappa$ be such that $F$ is unreaped and $|F| = r(\kappa)$.

We will need the revised GCH

**Definition**

Let $\kappa$ and $\lambda$ be cardinals. Define $\lambda^{[\kappa]}$ to be

$$\min \left\{ |P| : P \subseteq [\lambda]^\leq \kappa \text{ and } \forall u \in [\lambda]^\kappa \exists P_0 \subseteq P \left[ |P_0| < \kappa \text{ and } u = \bigcup P_0 \right] \right\}.$$  

The operation $\lambda^{[\kappa]}$ is sometimes referred to as the *weak power*. 
Assume \( \kappa \geq \beth_\omega \). Let \( F \subseteq [\kappa]^\kappa \) be such that \( F \) is unreaped and \( |F| = r(\kappa) \).

We will need the revised GCH.

**Definition**

Let \( \kappa \) and \( \lambda \) be cardinals. Define \( \lambda^{[\kappa]} \) to be

\[
\min \left\{ |\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\leq \kappa} \text{ and } \forall u \in [\lambda]^\kappa \exists \mathcal{P}_0 \subseteq \mathcal{P} \left[ |\mathcal{P}_0| < \kappa \text{ and } u = \bigcup \mathcal{P}_0 \right] \right\}.
\]

The operation \( \lambda^{[\kappa]} \) is sometimes referred to as the weak power.

- Easy exercise: GCH is equivalent to the statement that for all regular cardinals \( \kappa < \lambda \), \( \lambda^{[\kappa]} = \lambda \).
- The revised GCH, which is a theorem of ZFC says that for “lots of pairs” of regular cardinals we have \( \lambda^{[\kappa]} = \lambda \).
Theorem (Shelah; The Revised GCH)

If $\theta$ is a strong limit uncountable cardinal, then for every $\lambda \geq \theta$, there exists $\sigma < \theta$ such that for every $\sigma \leq \kappa < \theta$, $\lambda^{[\kappa]} = \lambda$.

Corollary

Let $\mu \geq \beth_\omega$ be any cardinal. There exists an uncountable regular cardinal $\theta < \beth_\omega$ and a family $P \subseteq [\mu]^{<\theta}$ such that $|P| \leq \mu$ and for each $u \in [\mu]^\theta$, there exists $v \in P$ with the property that $v \subseteq u$ and $|v| \geq \aleph_0$. 
Applying this with $\mu = r(\kappa)$, fix an uncountable regular cardinal $\theta < \aleph_\omega$ and a family $\mathcal{P} \subseteq [\theta \times F]^{\leq \theta}$ such that $|\mathcal{P}| \leq \mu$ and $\mathcal{P}$ has the property that for each $u \in [\theta \times F]^{\theta}$, there exists $v \in \mathcal{P}$ satisfying $v \subseteq u$ and $|v| \geq \aleph_0$. 
Applying this with $\mu = r(\kappa)$, fix an uncountable regular cardinal $\theta < \beth_\omega$
and a family $P \subseteq [\theta \times F]^{\leq \theta}$ such that $|P| \leq \mu$ and $P$ has the property that for each $u \in [\theta \times F]^\theta$, there exists $v \in P$ satisfying $v \subseteq u$ and $|v| \geq \aleph_0$.

Fix $M < H(\chi)$ containing everything relevant with $|M| = \mu$ and $F \subseteq M$.

$M \cap \kappa^\kappa$ is a dominating family (this shows $d(\kappa) \leq \mu$).
Applying this with $\mu = r(\kappa)$, fix an uncountable regular cardinal $\theta < \beth_\omega$ and a family $\mathcal{P} \subseteq [\theta \times F]^{\leq \theta}$ such that $|\mathcal{P}| \leq \mu$ and $\mathcal{P}$ has the property that for each $u \in [\theta \times F]^\theta$, there exists $v \in \mathcal{P}$ satisfying $v \subseteq u$ and $|v| \geq \aleph_0$.

Fix $M < H(\chi)$ containing everything relevant with $|M| = \mu$ and $F \subseteq M$.

$M \cap \kappa^\kappa$ is a dominating family (this shows $\mho(\kappa) \leq \mu$).

It may be assumed that for any club $E_1 \subseteq \kappa$, there exists a club $E_2 \subseteq E_1$ such that for all $B \in F$, $B \not\subseteq^* \text{ set } (E_2, E_1)$ (otherwise there is an easy argument).
Applying this with $\mu = r(\kappa)$, fix an uncountable regular cardinal $\theta < \beth_\omega$ and a family $\mathcal{P} \subseteq [\theta \times F]^\leq\theta$ such that $|\mathcal{P}| \leq \mu$ and $\mathcal{P}$ has the property that for each $u \in [\theta \times F]^\theta$, there exists $v \in \mathcal{P}$ satisfying $v \subseteq u$ and $|v| \geq \aleph_0$.

Fix $M < H(\chi)$ containing everything relevant with $|M| = \mu$ and $F \subseteq M$.

$M \cap \kappa^\kappa$ is a dominating family (this shows $\mathfrak{d}(\kappa) \leq \mu$).

It may be assumed that for any club $E_1 \subseteq \kappa$, there exists a club $E_2 \subseteq E_1$ such that for all $B \in F$, $B \not\subseteq^* \) set (E_2, E_1$) (otherwise there is an easy argument).

Since $F$ is an unreaped family, it follows that for each club $E_1 \subseteq \kappa$, there exist a club $E_2 \subseteq E_1$ and a $B \in F$ such that $B \subseteq^* \kappa \setminus \) set (E_2, E_1)$. 
Let $f \in \kappa^\kappa$ be a fixed function.

Construct a sequence $\langle \langle E_i, E_i^1, B_i \rangle : i < \theta \rangle$ so that the following conditions are satisfied at each $i < \theta$:

1. $E_i$ and $E_i^1$ are both clubs in $\kappa$, $E_i^1 \subseteq E_i$, and $\forall j < i \left[ E_i \subseteq E_j^1 \right]$;
2. $B_i \in F$ and $B_i \subseteq^* \kappa \setminus \mathrm{set}\left( E_i^1, E_i \right)$;
3. if $i = 0$, then $E_i = \{ \alpha < \kappa : \alpha \text{ is closed under } f \}$. 
Let $f \in \kappa^\kappa$ be a fixed function.

Construct a sequence $\langle \langle E_i, E_i^1, B_i \rangle : i < \theta \rangle$ so that the following conditions are satisfied at each $i < \theta$:

1. $E_i$ and $E_i^1$ are both clubs in $\kappa$, $E_i^1 \subseteq E_i$, and $\forall j < i \left[ E_i \subseteq E_j^1 \right]$;
2. $B_i \in F$ and $B_i \subseteq^* \kappa \setminus \text{set}(E_i^1, E_i)$;
3. if $i = 0$, then $E_i = \{ \alpha < \kappa : \alpha \text{ is closed under } f \}$.

Define $u : \theta \to F$ by setting $u(i) = B_i$ for all $i \in \theta$.

By the choice of $\mathcal{P}$ and $M$, we can find a sub-function $w \subseteq u$ in $M$ so that $\text{otp}(\text{dom}(w)) = \omega$.

Let $\langle i_n : n \in \omega \rangle$ be the strictly increasing enumeration of $\text{dom}(w)$.
By regularity of $\kappa$, there exists a function $g \in M \cap \kappa^\kappa$ with the property that for each $\alpha \in \kappa$, $\forall i \in \text{dom}(w) \left[ B_i \cap [\alpha, g(\alpha)) \neq 0 \right]$. 
By regularity of $\kappa$, there exists a function $g \in M \cap \kappa^\kappa$ with the property that for each $\alpha \in \kappa$, $\forall i \in \text{dom}(w) \ [B_i \cap [\alpha, g(\alpha)) \neq 0]$.

Find $\delta < \kappa$ so that for each $n \in \omega$:

1. $B_{i_n} \setminus \delta \subseteq \kappa \setminus \text{set}(E_{i_n}^1, E_{i_n})$;
2. $\min(E_{i_n}) < \delta$
By regularity of $\kappa$, there exists a function $g \in M \cap \kappa^\kappa$ with the property that for each $\alpha \in \kappa$, $\forall i \in \text{dom}(w) [B_i \cap [\alpha, g(\alpha)) \neq 0]$.

Find $\delta < \kappa$ so that for each $n \in \omega$:
1. $B_i \setminus \delta \subseteq \kappa \setminus \text{set}(E_{i_n}^1, E_{i_n})$;
2. $\min(E_{i_n}) < \delta$

We will show that for any $\alpha > \delta$, $f(\alpha) < g(\alpha)$.  

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Fix $\alpha > \delta$ and define $\xi_n = \sup(E_i \cap (\alpha + 1))$.

Then $\xi_n \in E_i$ and they are non-increasing.

There exist $\xi$ and $N \in \omega$ such that $\forall n \geq N [\xi_n = \xi]$.
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There exist $\xi$ and $N \in \omega$ such that $\forall n \geq N [\xi_n = \xi]$.

Fix $\beta \in B_{iN} \cap [\alpha, g(\alpha))$.

Then $\beta \notin \text{set}(E^1_{iN}, E_{iN})$.

Note $\xi = \xi_{N+1} \in E_{iN+1} \subseteq E^1_{iN}$.

Hence $\beta \notin \left[\xi, s_{E_{iN}}(\xi)\right)$.
Fix $\alpha > \delta$ and define $\xi_n = \sup(E_{i_n} \cap (\alpha + 1))$.

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Fix $\beta \in B_{i_N} \cap [\alpha, g(\alpha))$.

Then $\beta \notin \text{set}\left(E_{i_{i_N}}^1, E_{i_N}\right)$.

Note $\xi = \xi_{N+1} \in E_{i_{N+1}} \subseteq E_{i_N}^1$.

Hence $\beta \notin \left[\xi, s_{E_{i_N}}(\xi)\right]$.

On the other hand, $\xi \leq \alpha \leq \beta$. Hence $\xi \leq \alpha < s_{E_{i_N}}(\xi) \leq \beta < g(\alpha)$.
Fix $\alpha > \delta$ and define $\xi_n = \sup(E_{i_n} \cap (\alpha + 1))$.

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On the other hand, $\xi \leq \alpha \leq \beta$. Hence $\xi \leq \alpha < s_{E_{i_N}}(\xi) \leq \beta < g(\alpha)$.

Finally, since $s_{E_{i_N}}(\xi) \in E_{i_N} \subseteq E_0$, $s_{E_{i_N}}(\xi)$ is closed under $f$.

Therefore, $f(\alpha) < s_{E_{i_N}}(\xi) \leq \beta < g(\alpha)$. 
Bibliography

