# Reversible sequences of natural numbers and reversibility of some disconnected binary structures

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3rd July 2018

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- Reversible structures have *the property Cantor-Schröder-Bernstein* (shorter CSB) for condensations (bijective homomorphisms)
- each class of reversible posets yields the corresponding class of reversible topological spaces
- reversibility is related to the size of the classes $[\rho]_{\cong}$, and to the shape and structure of certain suborders of the condensational order

# Disconnected binary structures

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In this presentation we investigate reversibility in the class of binary structures, that is models of the relational language $L_b = \langle R \rangle$, where $\mathrm{ar}(R) = 2$, and, moreover, we restrict our attention to the class of *disconnected $L_b$-structures*.

# Disconnected binary structures

In this presentation we investigate reversibility in the class of binary structures, that is models of the relational language $L_b = \langle R \rangle$, where $\mathrm{ar}(R) = 2$, and, moreover, we restrict our attention to the class of *disconnected $L_b$-structures*.

If $\mathbb{X} = \langle X, \rho \rangle$ is an $L_b$-structure and $\sim_\rho$ the minimal equivalence relation on $X$ containing $\rho$, then the corresponding equivalence classes are called the *connectivity components* of $\mathbb{X}$ and $\mathbb{X}$ is said to be *disconnected* if it has more than one component, that is, if $\sim_\rho \neq X^2$). The prototypical disconnected structures are, of course, equivalence relations themselves; other prominent representatives of that class are some countable ultrahomogeneous graphs and posets, non-rooted trees, etc.

# Reversible sequences of cardinals

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If $\mathbb{X}$ is a binary structure, and $\mathbb{X}_i$, $i \in I$, are its connectivity components, then, clearly, the sequence of cardinal numbers $\langle |X_i| : i \in I \rangle$ is an isomorphism-invariant of the structure, and in some classes of structures (for example, in the class of equivalence relations) that cardinal invariant characterizes the structure up to isomorphism.

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So we isolate the following property of sequences of cardinals (called reversibility as well) which characterizes reversibility in the class of equivalence relations:

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So we isolate the following property of sequences of cardinals (called reversibility as well) which characterizes reversibility in the class of equivalence relations: a sequence of non-zero cardinals $\langle \kappa_i : i \in I \rangle$ is defined to be *reversible* iff

$$\neg \exists f \in \mathrm{Sur}(I) \setminus \mathrm{Sym}(I) \;\; \forall j \in I \;\; \sum_{i \in f^{-1}[\{j\}]} \kappa_i = \kappa_j,$$

where $\mathrm{Sym}(I)$ (resp. $\mathrm{Sur}(I)$) denotes the set of all bijections (resp. surjections) $f : I \to I$.

# Reversible sequences of natural numbers

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Next, we characterize reversible sequences of cardinals. First, we reduce the problem to characterizing the reversible sequences of natural numbers.

**Proposition**

A sequence of nonzero cardinals $\langle \kappa_i : i \in I \rangle$ is reversible iff it is a finite-one-sequence or a reversible sequence of natural numbers.

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In order to give the characterization of reversible sequences of natural numbers, we first recall some definitions. If $\langle n_i : i \in I \rangle \in {}^{I}\mathbb{N}$, then $I = \bigcup_{m \in \mathbb{N}} I_m$, where

$$I_m := \{i \in I : n_i = m\}.$$

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A set $K \subseteq \mathbb{N}$ is called *independent* iff $n \notin \langle K \setminus \{n\} \rangle$, for all $n \in K$, where $\langle K \setminus \{n\} \rangle$ is the subsemigroup of the semigroup $\langle \mathbb{N}, + \rangle$ generated by $K \setminus \{n\}$; by $\gcd(K)$ we denote the greatest common divisor of the numbers from $K$.

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**Theorem**

A sequence $\langle n_i : i \in I \rangle \in {}^I\mathbb{N}$ is reversible iff either it is a finite-to-one sequence, or $K = \{m \in \mathbb{N} : |I_m| \geq \omega\}$ is a nonempty independent set and $\gcd(K)$ divides at most finitely many elements of the set $\{n_i : i \in I\}$.

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For example, if $I$ is a nonempty set of arbitrary size, and $\langle n_i : i \in I\rangle \in {}^I\mathbb{N}$, then we have:

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- if $K = \emptyset$ (which is possible if $|I| \leq \omega$), the sequence $\langle n_i \rangle$ is reversible;
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- if $K = \{2, 5\}$, the sequence $\langle n_i\rangle$ is reversible iff $\{n_i : i \in I\}$ is finite;
- if $K = \{4, 10\}$, then the sequence $\langle n_i\rangle$ is reversible iff the set $\{n_i : i \in I\}$ contains at most finitely many even numbers;

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**Proposition**

(a) $({}^{\mathbb{N}}\mathbb{N})_{\text{rev}}$ is a dense $F_{\sigma\delta\sigma}$-subset of the Baire space ${}^{\mathbb{N}}\mathbb{N}$;

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(a) $({}^\mathbb{N}\mathbb{N})_{\text{rev}}$ is a dense $F_{\sigma\delta\sigma}$-subset of the Baire space ${}^\mathbb{N}\mathbb{N}$;
(b) $({}^\mathbb{N}\mathbb{N})_{\text{rev}}$ is not a subsemigroup of the semigroup $\langle {}^\mathbb{N}\mathbb{N}, \circ \rangle$.

# Reversible RFM structures

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We shalll say that a sequence of $L$-structures $\langle \mathbb{X}_i : i \in I \rangle$ is *rich for monomorphisms* iff

$$\forall i, j \in I \;\; \forall A \in [X_j]^{|X_i|} \;\; \exists g \in \mathrm{Mono}(\mathbb{X}_i, \mathbb{X}_j) \;\; g[X_i] = A.$$

Since the reversibility of the components is a necessary condition for the reversibility of a disconnected binary structure, by RFM we denote the class of sequences $\langle \mathbb{X}_i : i \in I \rangle$ (where $I$ is any non-empty set) of pairwise disjoint, connected and reversible $L_b$-structures, which are rich for monomorphisms.

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**Theorem**

If $\langle \mathbb{X}_i : i \in I \rangle \in \mathrm{RFM}$ then we have that $\bigcup_{i \in I} \mathbb{X}_i$ is a reversible structure if and only if $\langle |X_i| : i \in I \rangle$ is a reversible sequence of cardinals.

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**Theorem**

Let $\sim$ be an equivalence relation on a set $X$, $\mathbb{X} = \langle X, \sim \rangle$, and $\{X_i : i \in I\}$ the corresponding partition. Then the structure $\mathbb{X}$ is reversible iff $\langle |X_i| : i \in I \rangle$ is a reversible sequence of cardinals.
The same holds for the graphs (resp. posets) of the form $\mathbb{X} = \bigcup_{i \in I} \mathbb{X}_i$, where $\mathbb{X}_i$, $i \in I$, are pairwise disjoint complete graphs (resp. cardinals $\leq \omega$).

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**Remark**

There are $\mathfrak{c}$-many non-isomorphic countable reversible, as well as $\mathfrak{c}$-many non-isomorphic countable nonreversible equivalence relations. The same holds for the classes of graphs and posets from the previous theorem.

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By the well known characterization of Lachlan and Woodrow, each countable ultrahomogeneous graph is isomorphic to one of the following:

- $\mathbb{G}_{\mu\nu}$ - the union of $\mu$ disjoint copies of $\mathbb{K}_\nu$, where $\mu\nu = \omega$. $\mathbb{G}_{\mu\nu}$ is reversible iff $\mu < \omega$ or $\nu < \omega$;

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- Graph complements of these graphs. A graph is reversible iff its graph complement is.

# More reversible digraphs, posets and topological spaces

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If $\mathbb{X}_i$, $i \in I$, are disjoint tournaments and the sequence of cardinals $\langle |X_i| : i \in I \rangle$ is reversible, then the digraph $\bigcup_{i \in I} \mathbb{X}_i$ is reversible.
This statement holds if, in particular, $\mathbb{X}_i$, $i \in I$, are disjoint linear orders. Then $\bigcup_{i \in I} \mathbb{X}_i$ is a reversible disconnected partial order.

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Let us recall that if $\mathbb{P} = \langle P, \leq \rangle$ is a partial order and $\mathcal{O}$ the topology on the set $P$ generated by the base consisting of the sets of the form $B_p := \{q \in p : q \leq p\}$, then endomorphisms of $\mathbb{P}$ are exactly the continuous self mappings of the space $\langle P, \mathcal{O} \rangle$.

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# Reversible disjoint unions of ordinals

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Lastly, using the characterization of reversible sequences of natural numbers obtained above, we characterize reversible posets that are a disjoint union of ordinals $\alpha_i = \gamma_i + n_i$, $i \in I$, where $\gamma_i \in \mathrm{Lim} \cup \{0\}$, and $n_i \in \omega$.

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Let us, before that, define

$$\begin{aligned} I_\alpha &:= \{i \in I : \alpha_i = \alpha\}, \text{ for } \alpha \in \mathrm{Ord}, \\ J_\gamma &:= \{j \in I : \gamma_j = \gamma\}, \text{ for } \gamma \in \mathrm{Lim} \cup \{0\}. \end{aligned}$$

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**Theorem**

$\bigcup_{i \in I} \alpha_i$ is a reversible poset iff exactly one of the following is true

(I) $\langle \alpha_i : i \in I \rangle$ is a finite-to-one sequence,

(II) There is $\gamma = \max\{\gamma_i : i \in I\}$, for $\alpha \leq \gamma$ we have $|I_\alpha| < \omega$, and $\langle n_i : i \in J_\gamma \setminus I_\gamma \rangle$ is a reversible sequence of natural numbers, which is not finite-to-one.

The same holds for the poset $\bigcup_{i \in I} \alpha_i^*$.

# Reversible sequences of several things

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### Proposition

For each sequence of nonzero cardinals $\bar{\kappa} = \langle \kappa_i : i \in I \rangle$ the following conditions are equivalent:
(a) The poset $\bigcup_{i \in I} \kappa_i$ is a reversible structure;
(b) $\bar{\kappa}$ is a reversible sequence of ordinals;
(c) $\bar{\kappa}$ is a reversible sequence of cardinals;
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We remark that the equivalence (a) $\Leftrightarrow$ (c) of the above proposition shows that the characterization of reversible RFM structures from a previous slide holds in a class of (sequences of) structures which is larger than the class RFM (for example, $\langle \omega, \omega_1, \omega_2, \omega_3, \ldots \rangle \notin$ RFM).

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