Nonamalgamation in the generic multiverse

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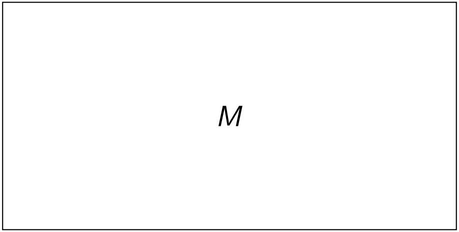
SETTOP 2018

The contents of this talk are joint with Joel David Hamkins, Lukas Daniel Klausner, Jonathan Verner, and Kameryn Williams.

The generic multiverse

Let M be a countable (transitive) model of set theory. The *generic multiverse* of M is the smallest collection of models containing M and closed under taking (set) forcing extensions and ground models.



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$$
\begin{tikzcd}[row sep=small]
 & & M[G] \\
N[H] = M \arrow[urr, dash] \arrow[dr, dash] & & \\
 & N &
\end{tikzcd}
$$

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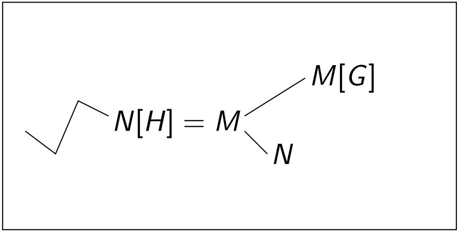

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The multiverse as an order

Order the multiverse by $\subseteq$ to get a poset of size 2^ω.

- Depending on the theory of M, the multiverse may or may not have minimal elements. (Reitz, 2007)
- Any countable chain in the multiverse, arising from a sequence of forcing notions of uniformly bounded size, has an upper bound. (Fuchs–Hamkins–Reitz, 2015)
- The multiverse is downward directed. (Usuba, 2017)

We will focus on studying the complexity of the multiverse via the posets that embed into it.

Ordinary embeddings

Theorem

Any finite poset embeds into the generic multiverse.

Proof.

For each $p \in P$ fix a Cohen real c_p, all mutually generic over M. Then just map p to $M[\bigoplus_{q \leq p} c_q]$. $\square$

Theorem

Any locally finite poset of size 2^ω embeds into the generic multiverse.

Amalgamability

The multiverse has interesting properties that are not captured just by looking at mutually generic extensions.

Definition

A family of forcing extensions $\mathcal{E}$ is *amalgamable* (over M) if there is another forcing extension $M[G]$ extending each model in $\mathcal{E}$, i.e. if $\mathcal{E}$ has an upper bound in the multiverse of M.

Theorem (Mostowski, 1976)

There are two Cohen reals c, d over M such that $M[c]$ and $M[d]$ do not amalgamate.

Proof.

Fix a catastrophic real z for M...

Two nonamalgamable reals

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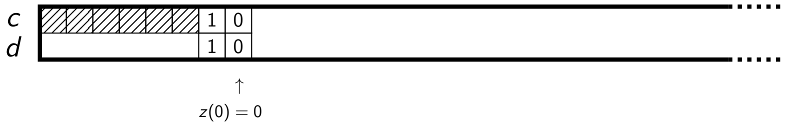
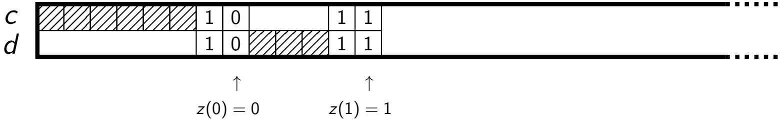


Two nonamalgamable reals



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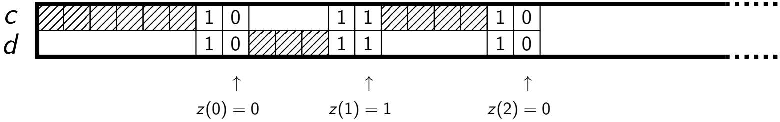

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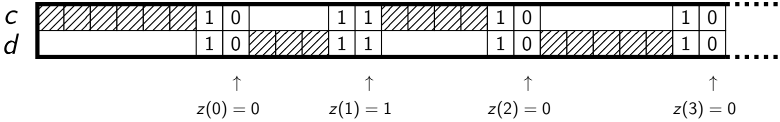

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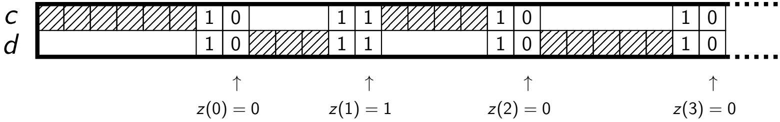

Mostowski's blockchain

This is the prototypical example of a *blockchain construction*. Mostowski used it to essentially prove the following.

Theorem (Mostowski, 1976)

Any finite poset embeds into the generic multiverse in a way that preserves nonamalgamability.

$*$-embeddings

Definition

Let P, Q be posets with least elements $0_P, 0_Q$. A map $f\colon P \to Q$ is a $*$-*embedding* if

1. $x \le y \iff f(x) \le f(y)$,
2. Any finite $X \subseteq P$ has an upper/nonzero-lower bound if and only if $f[X]$ does.

Which posets $*$-embed into the generic multiverse?

$*$-embeddings into the multiverse

Definition

A family of sets $\mathcal{A}$ has the *finite obstruction property* if for any $B \notin \mathcal{A}$ there is a finite $B' \subseteq B$ with $B' \notin \mathcal{A}$.

Theorem

Let $\mathcal{A} \in M$ be a family of subsets of a set I, containing all singletons, closed under subsets and with the finite obstruction property in M. Then there are Cohen reals $\{c_i; i \in I\}$ over M such that

1. *If $A \in \mathcal{A}$ then $\langle c_i; i \in A \rangle$ is generic over M;*
2. *If $B \notin \mathcal{A}$ then $\{M[c_i]; i \in B\}$ does not amalgamate;*
3. *If $A, A' \in \mathcal{A}$ then $M[c_A] \cap M[c_{A'}] = M[c_{A \cap A'}]$.*

Corollary

Let $\mathcal{A}$ be as above. Then $(\mathcal{A}, \subseteq)$ $*$-embeds into the generic multiverse. In particular, every finite poset $*$-embeds into the generic multiverse.

Proof sketch — a better blockchain

Build the generics as a blockchain, making sure that columns are only simultaneously active at coding points or if they should be amalgamable. The construction has three types of steps:

1. Genericity steps: give the columns above some $A \in \mathcal{A}$ a chance to be generic;
2. Coding steps: code a bit of z in the columns above some minimal $B \notin \mathcal{A}$;
3. Intersection steps: given an $\mathrm{Add}(\omega, A)$-name σ and an $\mathrm{Add}(\omega, A')$-name τ, try to force them to differ.

Beyond Cohen forcing

Can we realize $*$-embeddings using something other than just Cohen forcing?
Not in general: some combinations of forcing notions always amalgamate their generic extensions.

- The posets might have wildly different sizes (e.g. $\mathrm{Add}(\omega, 1)$ and $\mathrm{Add}(\omega_1, 1)$).
- A poset might be very tight and rigid (e.g. a Suslin tree which is Suslin-off-the-generic-branch).

Definition

A poset $\mathbb{P}$ is *wide* if it is not $|\mathbb{P}|$-cc below any condition.

$*$-embedding into the wide multiverse

Theorem

Let $\{\mathbb{P}_i; i \in I\} \in M$ be wide posets, all of the same size $\kappa \geq |I|$ in M. Let $\mathcal{A} \in M$ be a family of subsets of I as before. Then there are generic filters $G_i \subseteq \mathbb{P}_i$ over M such that:

1. *If $A \in \mathcal{A}$ then $\prod_{i \in A} G_i$ is generic over M;*
2. *If $B \notin \mathcal{A}$ then $\{M[G_i]; i \in B\}$ does not amalgamate;*
3. *If $A, A' \in \mathcal{A}$ then $M[\prod_{i \in A} G_i] \cap M[\prod_{i \in A'} G_i] = M[\prod_{i \in A \cap A'} G_i]$.*

Corollary

Let $\{\mathbb{P}_i; i \in I\}$ and $\mathcal{A}$ be as above. Then $(\mathcal{A}, \subseteq)$ $*$-embeds into the generic multiverse given by products of the $\mathbb{P}_i$.

Extension of embeddings

Question

Given posets $P \leq Q$, with P sitting "nicely" in Q, and a $*$-embedding f of P into the generic multiverse, does f extend to a $*$-embedding of Q?

Some very partial results for the Cohen multiverse:

1. Given countably many Cohen reals, there is another Cohen real which amalgamates with all of them.
2. Given a Cohen real, there is another Cohen real which does not amalgamate with it.

Thank you.