Big Ramsey degrees of the universal triangle-free graph

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Ramsey Theory, Small and Big

Property	**Examples**
$\mathcal{K}$ has Ramsey Property $\forall A \le B\ \exists C\ \forall k,\ C \to (B)^A_k$	finite: linear orders, Boolean algebras finite ordered: graphs, hypergraphs, graphs omitting k-cliques
$\mathcal{K}$ has Small Ramsey Degrees $\forall A\ \exists t_{\mathcal{K}}(A)\ \forall B\ \exists C\ \forall k,$ $C \to (B)^A_{k,t_{\mathcal{K}}(A)}$	finite: graphs, hypergraphs graphs omitting k-cliques hypergraphs omitting irreducibles
$\mathcal{K}$ has Big Ramsey Degrees $\forall A\ \exists T_{\mathcal{K}}(A)\ \forall k,$ $\mathbf{K} \to (\mathbf{K})^A_{k,T_{\mathcal{K}}(A)}$	the rationals, Rado graph, dense local order $\mathbf{S}(2)$, tournament $\mathbf{S}(3)$ $\mathbb{Q}_n,\ n \ge 2$.

$\mathcal{K}$ a Fraïssé class.

$\mathbf{K}$ = Fraïssé limit of $\mathcal{K}$, called a Fraïssé structure.

Missing Pieces: Forbidden Configurations

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No Fraïssé structures with forbidden configurations had a complete analysis of its Big Ramsey Degrees.

This was due to lack of tools for representing such Fraïssé structures and lack of a viable Ramsey theory for such (non-existent) representations.

We address this lack of representations and techniques, starting with the universal triangle-free graph in my submitted paper, *The Ramsey theory of the universal homogeneous triangle-free graph*, 48 pp.

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Reviews from *Annals of Mathematics* call this work 'a remarkable achievement', as it is 'the first major advance towards the Ramsey theory on Fraïssé limits that have forbidden configurations.'
(reviews linked on my website)

The beginning of a more general theory

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- Hypergraphs with forbidden configurations (in progress)
- Homogeneous bowtie-free graph (in progress, with Hubička)

Connections with Topological Dynamics

Ramsey Theory	corr.	Topological Dynamics
Ramsey Property $\mathcal{K}$ an order class with RP	KPT $\longleftrightarrow$	Extreme Amenability G is EA
Small Ramsey Degrees precpct expansion $\mathcal{K}^*$ has RP	NVT $\longleftrightarrow$	Computation of UMF of G X^* is UMF of G
Big Ramsey Degrees $\mathbf{K}$ admits a big R structure	Zucker $\longleftrightarrow$	Universal Completion Flow Big Ramsey Flow $=$ UCF of G

$\mathcal{K}$ a Fraïssé class
$\mathbf{K} = \mathrm{Flim}(\mathcal{K})$
$G = \mathrm{Aut}(\mathbf{K})$

A Brief (incomplete) Intro to Graph Colorings

Example: Ordered graph A embeds into ordered graph B.

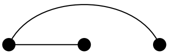


Figure: Ordered Graph A

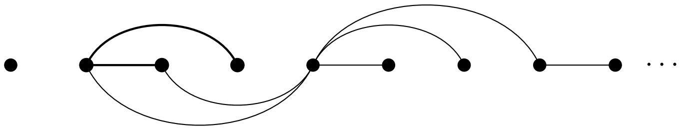
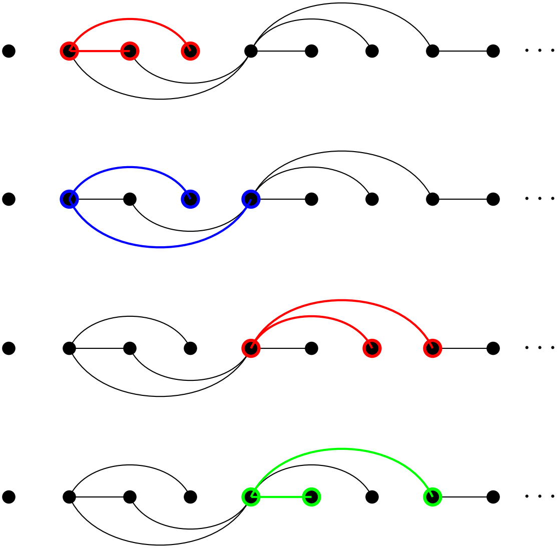


Figure: Ordered Graph B

Some copies of A in B



Thm. (Nešetřil/Rödl 1977 and Abramson/Harrington 1978)
The class of finite ordered graphs has the Ramsey property.

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It follows that the class of finite graphs has small Ramsey degrees.

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Actual degrees were structurally found in [LSV 2006] and computed in [J. Larson 2008].

Other Structures known to have big Ramsey degrees

• the natural numbers (Ramsey 1929)

• the rationals (Galvin, Laver, Devlin 1979)

• the Rado graph and similar binary relational structures (Sauer 2006)

• the countable ultrametric Urysohn space (Nguyen Van Thé 2008)

• the dense local order, circular tournament, $\mathbb{Q}_n$ (Laflamme, NVT, Sauer 2010).

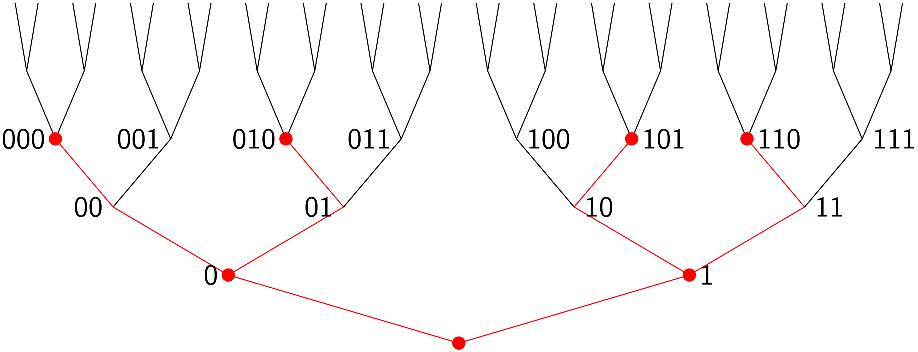
The crux of all but two of these proofs is a theorem of Milliken.

(The Urysohn space result uses Ramsey's Theorem.)

Strong Trees and Milliken's Theorem

A tree $T \subseteq 2^{<\omega}$ is a strong tree iff it is isomorphic either to $2^{<\omega}$ or to $2^{\leq k}$ for some finite k.

Strong Subtree $\cong 2^{\leq 2}$, Ex. 1



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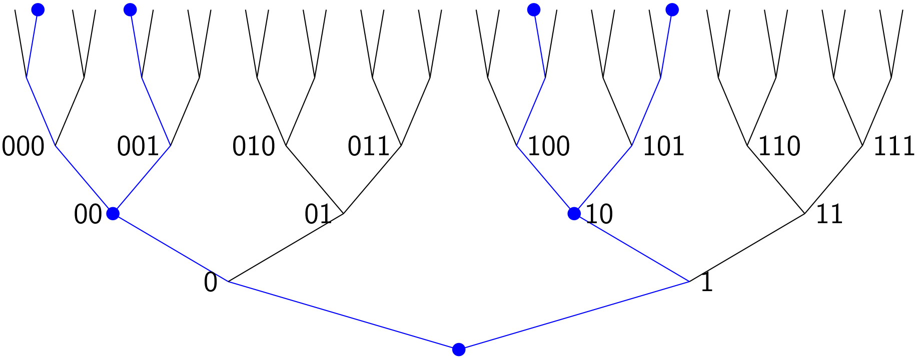
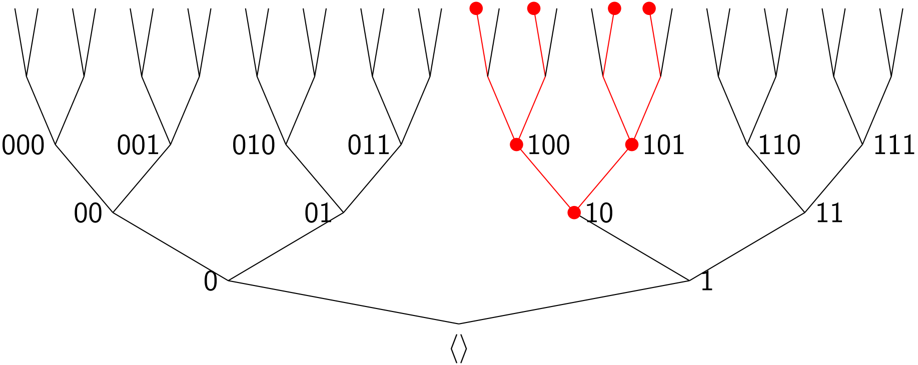


Strong Subtree $\cong 2^{\leq 2}$, Ex. 3



A Ramsey Theorem for Strong Trees

Thm. (Milliken 1979) Let $k \geq 0$, $l \geq 2$, and a coloring of all the subtrees of $2^{<\omega}$ which are isomorphic to $2^{\leq k}$ into l colors. Then there is an infinite strong subtree $S \subseteq 2^{<\omega}$ such that all copies of $2^{\leq k}$ in S have the same color.

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Thm. (Halpern-Läuchli 1966) Let $d \geq 1$, $l \geq 2$, and $T_i = 2^{<\omega}$ for $i < d$. Given a coloring of the product of level sets of the T_i into l colors,

$$f : \bigcup_{n<\omega} \prod_{i<d} T_i(n) \to l,$$

there are infinite strong trees $S_i \leq T_i$ and an infinite sets of levels $M \subseteq \omega$ where the splitting in S_i occurs, such that f is constant on $\bigcup_{m \in M} \prod_{i<d} S_i(m)$.

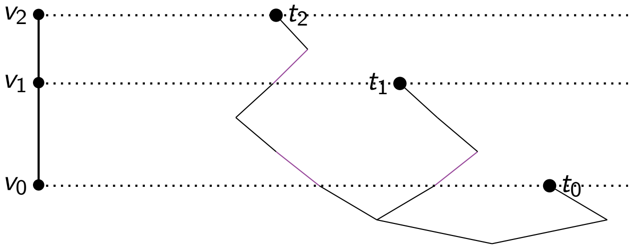
Nodes in Trees can Code Graphs

Let A be a graph. Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

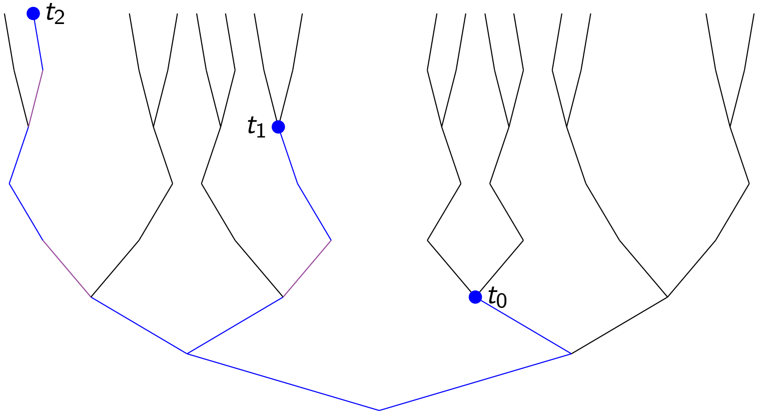
A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1.$$

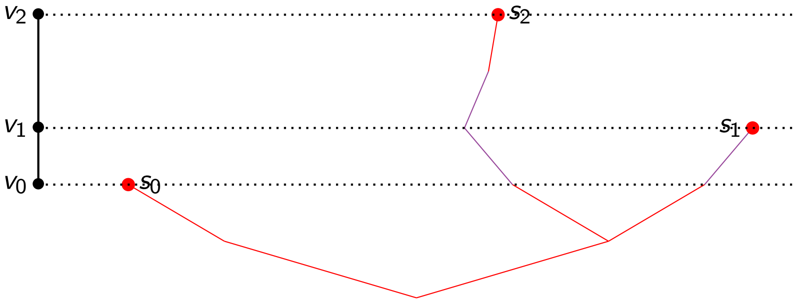
The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



A Strong Tree Envelope



A Different Antichain Coding a Path



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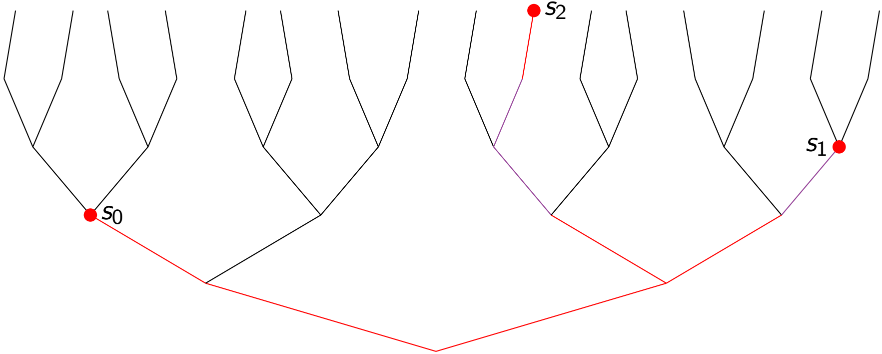

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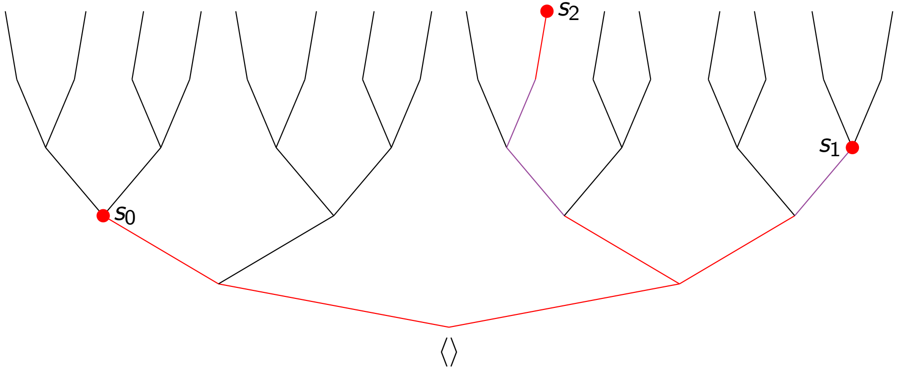

Outline of Sauer's Proof: $\mathcal{R}$ has finite big Ramsey degrees

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6. Show that each type persists in each subgraph which is random to obtain exact numbers.

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There it lay for 19 years.

Main Obstacles to Big Ramsey Degrees of $\mathcal{H}_3$

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"So far, the lack of tools to represent ultrahomogeneous structures is the major obstacle towards a better understanding of their infinite partition properties." (Nguyen Van Thé, Habilitation 2013)

Main Theorem: $\mathcal{H}_3$ has Finite Big Ramsey Degrees

Theorem. (D.) For each finite triangle-free graph A, there is a positive integer $T_{\mathcal{K}_3}(\mathrm{A})$ such that for any coloring of all copies of A in $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H} \le \mathcal{H}_3$, again universal triangle-free, such that all copies of A in $\mathcal{H}$ take no more than $T_{\mathcal{K}_3}(\mathrm{A})$ colors.

Structure of Proof: Three Main Parts

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II Prove a Ramsey Theorem for strictly similar finite antichains.

III Apply Ramsey Theorem for strictly similar antichains finitely many times. Then take an antichain of coding nodes coding $\mathcal{H}_3$.

Part I: Strong Coding Trees

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Problem: How to make sure triangles are never encoded but branching is as thick as possible?

First Approach: Strong Triangle-Free Trees

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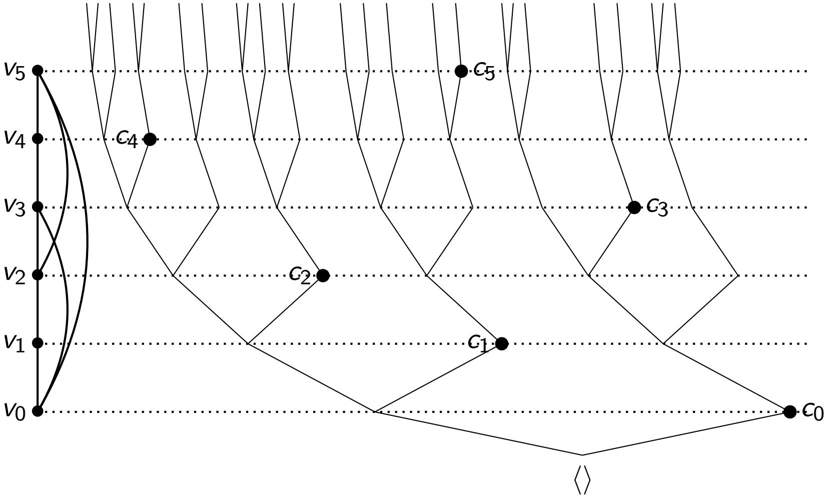
• Make a Branching Criterion so that nodes that split are exactly those nodes which do not extend to code a triangle with coding nodes at or below the splitting level.

Strong triangle-free tree $\mathbb{S}$



Almost sufficient

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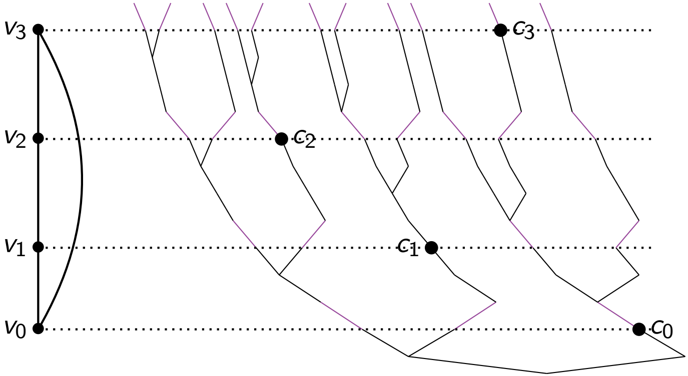
except for vertex colorings: there is a bad coloring of coding nodes.

Refined Approach: Strong coding tree $\mathbb{T}$



Skew the levels of interest.

The Space of Strong Coding Trees: $\mathcal{T}_3$

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The criteria guaranteeing this are

1. Parallel 1's Criterion: All new sets of parallel 1's in A are witnessed by a coding node in A 'nearby'.
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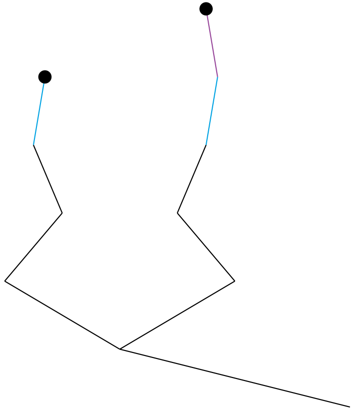
The space $\mathcal{T}_3$ of strong coding trees is very near a topological Ramsey space.

A subtree of $\mathbb{T}$ in which P1C fails

It has parallel 1's not witnessed by a coding node (P1C fails).



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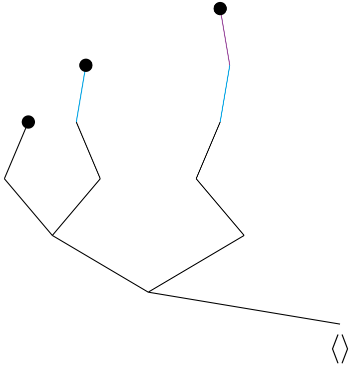

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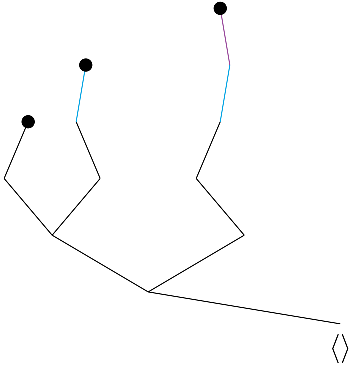

This gives the basic idea of P1C, though there are more subtleties involved.

Part II: A Ramsey Theorem for Strictly Similar Finite Antichains.

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It persists upon taking subtrees in $\mathcal{T}_3$.

Ramsey Theorem for Strong Coding Trees

Theorem. (D.) Let A be a finite subtree of a strong coding tree T, and let c be a coloring of all strictly similar copies of A in T.

Then there is a strong coding tree $S \leq T$ in which all strictly similar copies of A in S have the same color.

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Strict similarity is an equivalence relation.

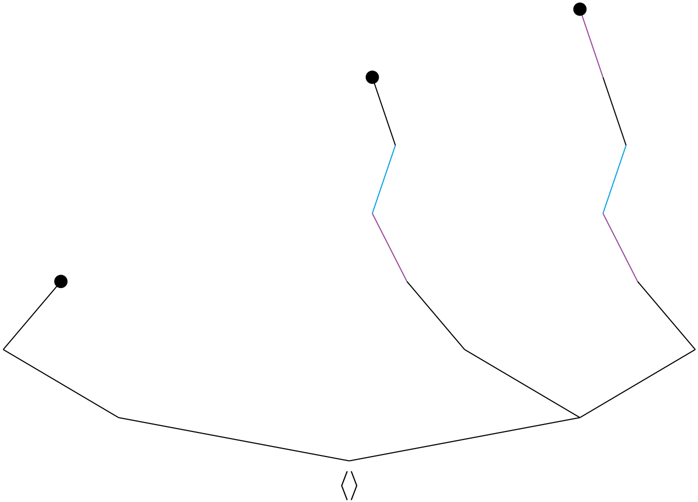
Some Examples of Strict Similarity Types

Let G be the graph with three vertices and no edges.

We show some distinct strict similarity types of trees coding G.

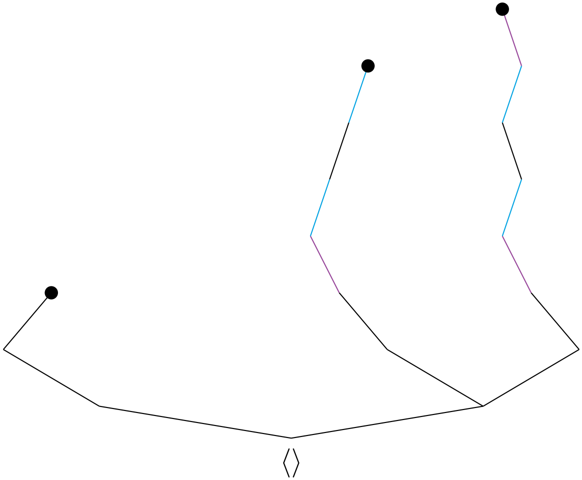
G a graph with three vertices and no edges

A tree A coding G - not P1C but still a valid strict similarity type



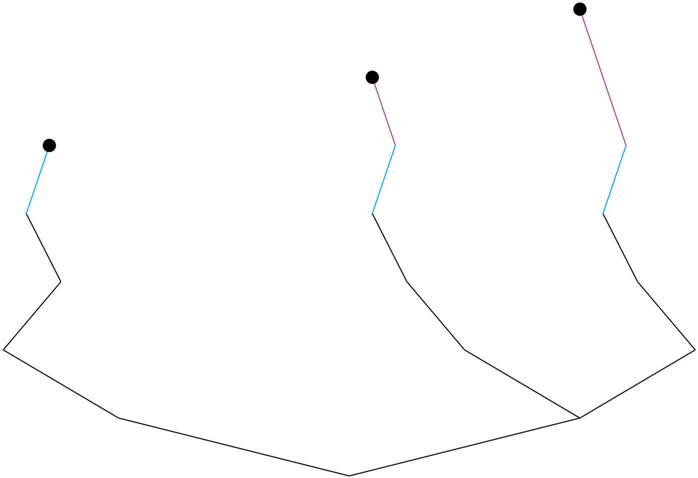
G a graph with three vertices and no edges

B codes G and is strictly similar to A.



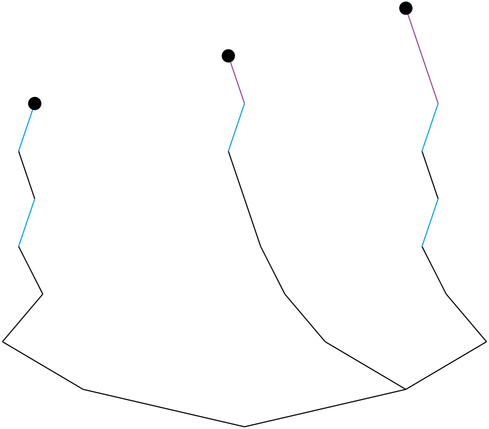
The tree C codes G

C is not strictly similar to A.

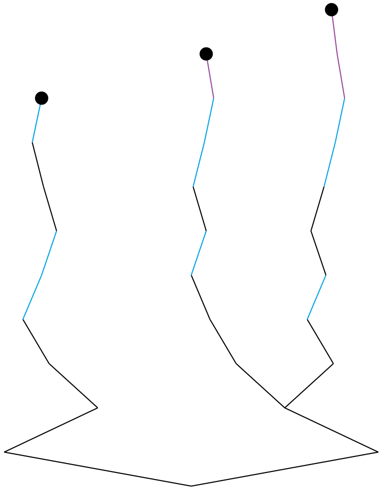


The tree D codes G

D is not strictly similar to either A or C.

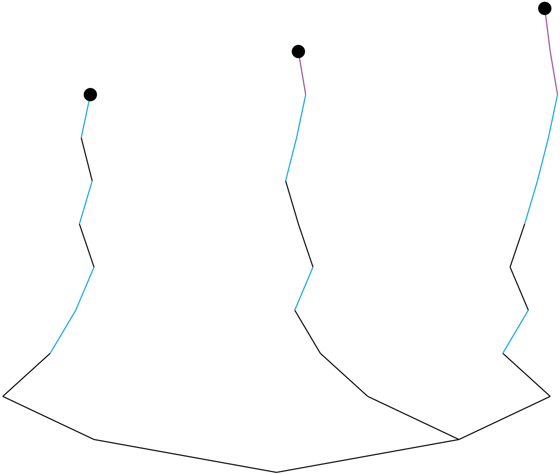


The tree E codes G and is not strictly similar to A - D



E is incremental. More on that later.

The tree F codes G and is strictly similar to E



F is also incremental.

Part III: Apply the Ramsey Theorem to Strictly Similarity Types of Antichains to obtain the Main Theorem.

Bounds for $T_{\mathcal{K}_3}(G)$

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6. Then f has no more colors on the copies of G in $\mathcal{H}'$ than the number of (incremental) strict similarity types of antichains coding G.

Reducing the Upper Bounds

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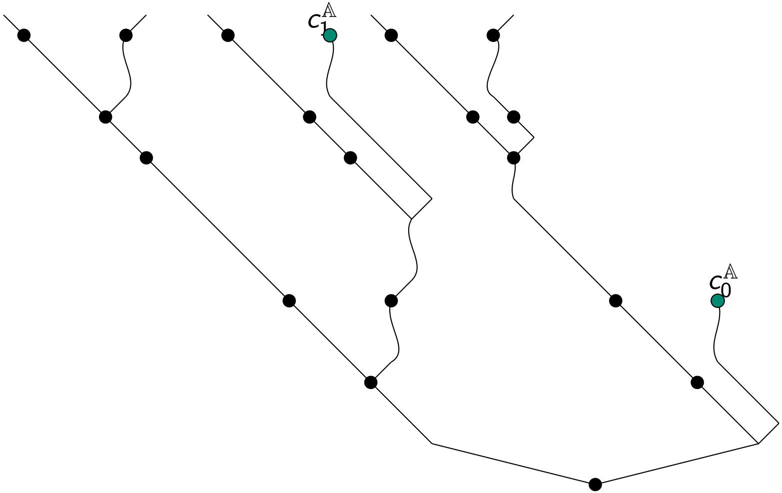
We can take S in the previous slide to be an incremental strong coding tree.

An antichain $\mathbb{A}$ of coding nodes of S coding $\mathcal{H}_3$



The tree minus the antichain of $c_n^{\mathbb{A}}$'s is isomorphic to $\mathbb{T}$.

Part II Expanded: Ideas behind the proof of
the Ramsey Theorem for Strictly Similar Finite Trees

(a) Prove new Halpern-Läuchli styleTheorems for strong coding trees.
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 – An analogue of Milliken's Theorem.

(c) New notion of envelope.
 – Turns an antichain into a tree satisfying Strict P1C.

(a) Halpern-Läuchli-style Theorem

Thm. (D.) Given a strong coding tree T and

1. B a finite, valid strong coding subtree of T;
2. A a finite subtree of B with $\max(A) \subseteq \max(B)$; and
3. X a level set extending A into T with $A \cup X$ satisfying the P1C and valid in T.

Color all end-extensions Y of A in T for which $A \cup Y$ is strictly similar to $A \cup X$ into finitely many colors.

Then there is a strong coding tree $S \leq T$ end-extending B such that all level sets Y in S with $A \cup Y$ strictly similar to $A \cup X$ have the same color.

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Remark. The proof uses three different forcings and Harrington-style ideas. The forcings are best thought of as conducting unbounded searches for finite objects in ZFC.

(b) Ramsey Theorem for Finite Trees satisfying the SP1C

Thm. (D.) Let T be a strong coding tree, and let A be a finite valid subtree of T satisfying the Strict P1C. Suppose all the strictly similar copies of A in T are colored in finitely many colors.

Then there is a strong coding subtree $S \leq T$ such that all strictly similar copies of A in S have the same color.

A tree A satisfies the strict P1C if each new set of parallel 1's is witnessed by a coding node before anything else happens (other occurrences of new parallel 1's, splits, or coding nodes).

(c) Envelopes and Witnessing Coding Nodes

Envelopes add some neutral coding nodes to a finite tree to make it satisfy the strict Parallel 1's Criterion.

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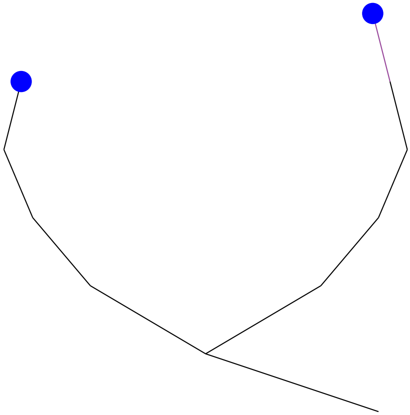
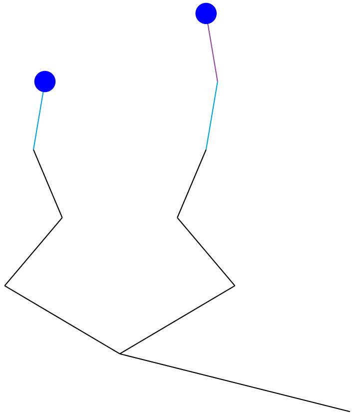
We now give some examples of envelopes.

A codes a non-edge



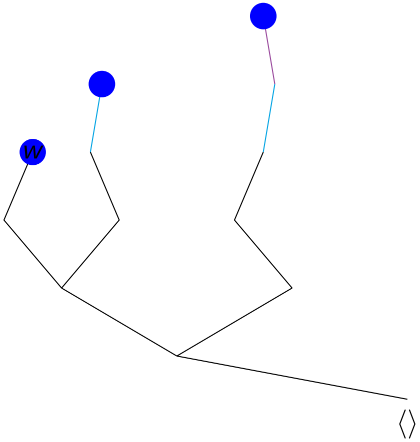
This satisfies the Strict Parallel 1's Criterion, so A is its own envelope.

B codes a non-edge



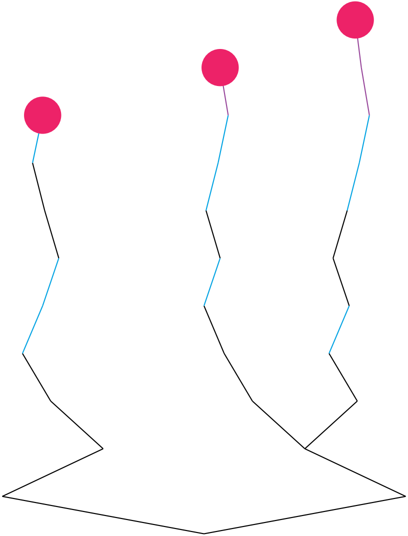
B does not satisfy the Parallel 1's Criterion.

An Envelope E(B)

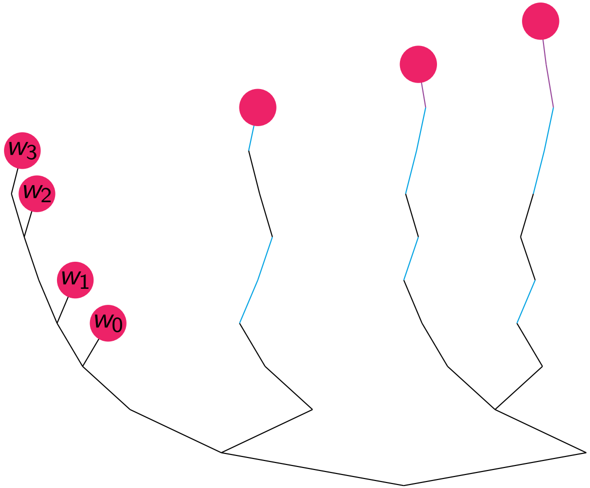


The envelope $E(B)$ satisfies the Strict Parallel 1's Criterion.

An incremental tree C coding three vertices with no edges



An envelope $E(C)$



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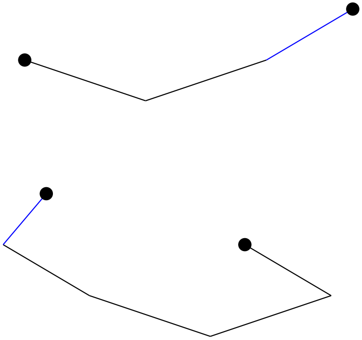
5. Then each copy of A in S has an envelop in T, by adding in some nodes from W.
6. Thus, each copy of A in S has the same color.

Proving the lower bounds in general for big Ramsey degrees of $\mathcal{H}_3$ is a work in progress.

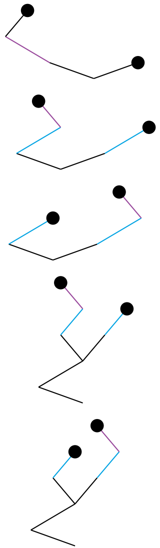
Big Ramsey degrees for edges and non-edges have been computed.

Edges have big Ramsey degree 2 in $\mathcal{H}_3$



Obtained in [Sauer 1998] by different methods.

Non-edges have 5 Strict Similarity Types (D.)



Remarks

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To adapt these methods to other structures $\mathcal{S}$ with forbidden configurations, one needs to find the correct Branching Criteria, Extension Criteria guaranteeing a finite subtree can be extended inside a tree coding $\mathcal{S}$, and Ramsey theorems for relevant structures.

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II(a) HL - Case (i): level set X contains a splitting node

List the immediate successors of $\max(A)$ as $s_0, \dots, s_d$, where s_d denotes the node which the splitting node in X extends.

Let $T_i = \{t \in T : t \supseteq s_i\}$, for each $i \leq d$.

Fix κ large enough so that $\kappa \rightarrow (\aleph_1)^{2d}_{\aleph_0}$ holds.

Such a κ is guaranteed in ZFC by a theorem of Erdős and Rado.

The forcing for Case (i)

$\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \upharpoonright l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

(ii) For each $i < d$, $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \upharpoonright l_p$.

(iii) $\mathrm{ran}(p)$ has no pre-determined new parallel 1's in T.

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(iii) $\text{ran}(p)$ has no pre-determined new parallel 1's in T.

$q \le p$ if and only if $\vec{\delta}_q \supseteq \vec{\delta}_p$, $l_q \ge l_p$, and

(i) $q(d) \supset p(d)$, and $q(i,\delta) \supset p(i,\delta)$ for each $\delta \in \vec{\delta}_p$ and $i < d$; and

(ii) The set $\{q(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{q(d)\}$ has no new sets of parallel 1's above $\{p(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{p(d)\}$.

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(3) The assumption that $A \cup X$ satisfies the Parallel 1's Criterion is necessary.

The rest of II

II(a) Case (ii): level set X contains a coding node.

This case is more complex and requires preliminary forcings to obtain cone-homogeneity, an induction proof to construct a cone-homogeneous strong coding tree, and another forcing to obtain the Halpern-Läuchli style theorem.

The rest of II

II(a) Case (ii): level set X contains a coding node.

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II(b): Ramsey Theorem for Finite Trees with Strict P1C.

This is obtained by induction using II(a).

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II(b): Ramsey Theorem for Finite Trees with Strict P1C.

This is obtained by induction using II(a).

II(c) was elaborated on already.