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Big Ramsey degrees of the universal triangle-free graph

It is a central question in the theory of homogeneous relational structures as to which structures have finite big Ramsey degrees. This question, of interest for several decades, has gained recent momentum as it was brought into focus by Kechris, Pestov, and Todorcevic in their 2005 paper, in which they proved a deep correspondence between Ramsey theory of Fraïssé limits and topological dynamics. An infinite structure S is **homogeneous** if any isomorphism between two finitely generated substructures of S can be extended to an automorphism of S. A homogeneous structure $\mathbf{S}$ is said to have **finite big Ramsey degrees** if for each finite substructure A of $\mathbf{S}$, there is a number $n(\mathrm{A})$ such that any coloring of the copies of A in $\mathbf{S}$ into finitely many colors can be reduced down to no more than n colors on some substructure $\mathbf{S}'$ isomorphic to $\mathbf{S}$. This is interesting not only as a Ramsey property for infinite structures, but also because of its implications for topological dynamics, following recent work of Zucker.

Prior to work of the speaker, finite big Ramsey degrees had been proved for a handful of homogeneous structures including the rationals (Devlin 1979), the Rado graph (Sauer 2006), ultrametric spaces (Nguyen Van Thé 2008), and enriched versions of the rationals and related circular directed graphs (Laflamme, Nguyen Van Thé, and Sauer 2010). According to Nguyen Van Thé, Sauer, and Todorcevic, the lack of tools to represent ultrahomogeneous structures with forbidden configurations, particularly the lack of any analogue of Milliken's Ramsey theorem for strong trees applicable to such structures, was a major obstacle towards a better understanding of their infinite partition properties.

The universal triangle-free graph, constructed by Henson in 1971 and denoted $\mathbf{H}_3$, is the simplest homogeneous structure with a forbidden configuration. Prior to my work, Komjáth and Rödl had proved in

1986 that vertex colorings have big Ramsey degree 1, and Sauer had proved in 1998 that edge colorings have big Ramsey degree 2. It was a major open problem in the Ramsey theory of homogeneous structures whether or not each finite triangle has a finite big Ramsey degree. Recently, I solved this problem in [1]. This work involved developing a new notion of trees coding $\mathbf{H}_3$; using forcing techniques to prove, in ZFC, Ramsey theorems for these trees, proving new Halpern-Läuchli and Milliken-style theorems; and deducing bounds from the tree structures. Work in progress is the theorem that all Henson graphs, the universal k-clique-free graphs, have finite big Ramsey degrees. The methods developed seem robust enough that correct modifications should likely apply to a large class of homogeneous structures with forbidden configurations.

[1] Dobrinen, N., The Ramsey theory of the universal homogeneous triangle-free graph. 48 pp. Submitted.