Lelek fan and Poulsen simplex as Fraïssé limits

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joint work with Wiesław Kubiś

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Definitions

- $\mathcal{C}$ a category whose objects are non-empty compact second countable metric spaces
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- so the arrows are **retractions** onto \( K \)
Assume that each \( K \in \text{Ob}(\mathcal{C}) \) is equipped with a metric \( d_K \).

Given two \( \mathcal{C} \)-arrows \( f, g : K \to L \), \( f = \langle e, p \rangle \), \( g = \langle i, q \rangle \), we define

\[
d(f, g) = \begin{cases} 
\max_{y \in L} d_K(p(y), q(y)) & \text{if } e = i, \\
+\infty & \text{otherwise}.
\end{cases}
\]
Definitions - metric

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  +\infty & \text{otherwise}.
  \end{cases}$$

- $\mathcal{C}$ equipped with the metric $d$ on each $\text{Hom}(K, L)$ is a metric category if $d(f_0 \circ g, f_1 \circ g) \leq d(f_0, f_1)$ and $d(h \circ f_0, h \circ f_1) \leq d(f_0, f_1)$, whenever the composition makes sense.
Definitions - amalgamation

- $\mathcal{C}$ is **directed** if for every $A, B \in \mathcal{C}$ there is $C \in \mathcal{C}$ such that there exist arrows from $A$ to $C$ and from $B$ to $C$. 
**Definitions - amalgamation**

- $C$ is **directed** if for every $A, B \in C$ there is $C \in C$ such that there exist arrows from $A$ to $C$ and from $B$ to $C$.

- $C$ has the **almost amalgamation property** if for every $C$-arrows $f : A \to B$, $g : A \to C$, for every $\varepsilon > 0$, there exist $C$-arrows $f' : B \to D$, $g' : C \to D$ such that $d(f' \circ f, g' \circ g) < \varepsilon$.

- $C$ has the **strict amalgamation property** if we can have $f'$ and $g'$ as above satisfying $f' \circ f = g' \circ g$. 

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• $\mathcal{C}$ has the **strict amalgamation property** if we can have $f'$ and $g'$ as above satisfying $f' \circ f = g' \circ g$. 
$\mathcal{C}$ is separable if there is a countable subcategory $\mathcal{F}$ such that

1. for every $X \in \text{Ob}(\mathcal{C})$ there are $A \in \text{Ob}(\mathcal{F})$ and a $\mathcal{C}$-arrow $f : X \rightarrow A$;

2. for every $\mathcal{C}$-arrow $f : A \rightarrow Y$ with $A \in \text{Ob}(\mathcal{F})$, for every $\varepsilon > 0$ there exists an $\mathcal{C}$-arrow $g : Y \rightarrow B$ and an $\mathcal{F}$-arrow $u : A \rightarrow B$ such that $d(g \circ f, u) < \varepsilon$. 
The general setting
The Lelek fan
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More applications to the Lelek fan

Definitions - Fraïssé sequence

$\mathcal{C}$-sequence $\bar{U} = \langle U_m; u^n_m \rangle$ is a Fraïssé sequence if the following holds:

$(F)$ Given $\varepsilon > 0$, $m \in \omega$, and an arrow $f : U_m \to F$, where $F \in \text{Ob}(\mathcal{C})$, there exist $m < n$ and an arrow $g : F \to U_n$ such that $d(g \circ f, u^n_m) < \varepsilon$. 
Theorem (Kubiś)

Let $\mathcal{C}$ be a directed metric category with objects and arrows as before that has the almost amalgamation property. The following conditions are equivalent:

(a) $\mathcal{C}$ is separable.
(b) $\mathcal{C}$ has a Fraïssé sequence.
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Consequences

Theorem (Kubiś)

Under assumptions of the previous theorem and separability we have:

1. **Uniqueness** There exists exactly one Fraïssé sequence \( \vec{U} \) (up to an isomorphism).
2. **Universality** For every sequence \( \vec{X} \) in \( C \) there is an arrow \( f : \vec{X} \to \vec{U} \).
3. **Almost homogeneity** For every \( A, B \in \text{Ob}(C) \) and for all arrows \( i : A \to \vec{U}, j : B \to \vec{U} \), for every \( C \)-arrow \( f : A \to B \), for every \( \varepsilon > 0 \), there exists an isomorphism \( H : \vec{U} \to \vec{U} \) such that \( d(j \circ f, H \circ i) < \varepsilon \).

In our examples we will have almost homogeneity for sequences in \( C \) as well.
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Lelek fan

- $C$ – the Cantor set
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- **Cantor fan** $V$ is the cone over the Cantor set:
  $C \times [0, 1]/C \times \{1\}$
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Lelek fan

- \( C \) – the Cantor set
- **Cantor fan** \( V \) is the cone over the Cantor set: \( C \times [0, 1]/C \times \{1\} \)
- **Lelek fan** \( \mathbb{L} \) is a non-trivial closed connected subset of \( V \) containing the top point, which has a dense set of endpoints in \( \mathbb{L} \)
Lelek fan

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About the Lelek fan

- Lelek fan was constructed by Lelek in 1960
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- Lelek fan is unique: any two are homeomorphic (Bula-Oversteegen 1990 and Charatonik 1989)
Geometric fans

Definition

A geometric fan is a closed connected subset of the Cantor fan containing the top point.
The category $\mathcal{F}$

- **Objects** are finite geometric fans, metric inherited from $\mathbb{R}^2$. 
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- $f : F \to G$ is **affine** if $f(\lambda \cdot x) = \lambda \cdot f(x)$ for every $x \in F$, $\lambda \in [0, 1)$.
- $f : F \to G$ is a **stable embedding** if it is a one-to-one affine map such that endpoints are mapped to endpoints.
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- $f : F \rightarrow G$ is a **stable embedding** if it is a one-to-one affine map such that endpoints are mapped to endpoints.
- An arrow from $F$ to $G$ is a pair $\langle e, p \rangle$ such that $e : F \rightarrow G$ is a stable embedding, $p : G \rightarrow F$ is a 1-Lipschitz affine surjection and $p \circ e = \text{id}_F$. 
Properties

- Geometric fans = inverse limits of sequences in $\mathcal{F}$
- The category $\mathcal{F}$ is directed and has the strict amalgamation property
- $\mathcal{F}$ is a separable metric category
Fraïssé sequences

Theorem (Kubiś - K)

Let $\tilde{U}$ be a sequence in $\mathfrak{F}$ and let $U_\infty$ be its inverse limit. The following properties are equivalent:

(a) The set of endpoints $E(U_\infty)$ is dense in $U_\infty$.

(b) $\tilde{U}$ is a Fraïssé sequence.
### Consequences

- **uniqueness** of a Fraïssé sequence
  The Lelek fan is a unique smooth fan whose set of end-points is dense.

- **universality** with respect to all geometric fans
  For every geometric fan $F$ there are a stable embedding $e$ into the Lelek fan $\mathbb{L}$ and a 1-Lipschitz affine retraction $p$ from $\mathbb{L}$ onto $F$ such that $p \circ e = \text{id}_F$. 
almost homogeneity with respect to all geometric fans
Let $F$ be a geometric fan and let $f, g : \mathbb{L} \to F$ be continuous affine surjections. Then for every $\varepsilon > 0$ there is a homeomorphism $h : \mathbb{L} \to \mathbb{L}$ such that for every $x \in \mathbb{L}$, $d_F (f \circ h(x), g(x)) < \varepsilon$. 
Consequences

- **almost homogeneity** with respect to all geometric fans

Let $F$ be a geometric fan and let $f, g : \mathbb{L} \to F$ be continuous affine surjections. Then for every $\varepsilon > 0$ there is a homeomorphism $h : \mathbb{L} \to \mathbb{L}$ such that for every $x \in \mathbb{L}$, $d_F(f \circ h(x), g(x)) < \varepsilon$.

**Remark**

In 2015, Bartošová and Kwiatkowska obtained uniqueness, universality, and almost homogeneity of the Lelek fan in the context of the projective Fraïssé theory.
Extreme points

**Definition**

A point $x$ in a compact convex set $K$ of a topological vector space is an **extreme point** if whenever $x = \lambda y + (1 - \lambda)z$ for some $\lambda \in [0, 1]$, $y, z \in K$, then $\lambda = 0$ or $\lambda = 1$.

The set of extreme points of $K$ is denoted by $\text{ext } K$. 

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Simplices

Definition

A **simplex** is a non-empty compact convex and metrizable set $K$ in a locally convex linear topological space such that every $x \in K$ has a unique probability measure $\mu$ supported on $\text{ext } K$ and such that

$$f(x) = \int_K f \, d\mu$$

for every continuous affine function $f : K \to \mathbb{R}$. 
Finite-dimensional simplices

Example

Finite-dimensional simplex $\Delta_n$

$$\{ x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x(i) = 1 \text{ and } x(i) \geq 0 \text{ for every } i = 1, \ldots, n+1 \}$$

In particular, $\Delta_0$ is a singleton, $\Delta_1$ is a closed interval, and $\Delta_2$ is a triangle.
The Poulsen simplex

Definition

The **Poulsen simplex** is a simplex that has a dense set of extreme points.
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Remark

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*Uniqueness was proved by Lindenstrauss, Olsen, and Sternfeld in '78.*
The category $\mathcal{S}$

- **Objects** are finite-dimensional simplices.
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- $p: L \to K$ is **affine** if for any $x, y \in L$ and $\lambda \in [0, 1]$ we have $p(\lambda x + (1 - \lambda)y) = \lambda p(x) + (1 - \lambda)p(y)$.
- **Stable embedding** is a one-to-one affine map such that extreme points are mapped to extreme points.
The category \( \mathcal{S} \)

- **Objects** are finite-dimensional simplices.
- \( p: L \to K \) is **affine** if for any \( x, y \in L \) and \( \lambda \in [0, 1] \) we have
  \[
  p(\lambda x + (1 - \lambda)y) = \lambda p(x) + (1 - \lambda)p(y).
  \]
- **Stable embedding** is a one-to-one affine map such that extreme points are mapped to extreme points.
- **An arrow** from \( K \) to \( L \) is a pair \( \langle e, p \rangle \) such that \( e: K \to L \) is a stable embedding, \( p: L \to K \) is an affine projection and \( p \circ e = \text{id}_K \).
Theorem (Lazar-Lindenstrauss ’71)

Metrizable simplices are, up to affine homeomorphisms, precisely the limits of inverse sequences in $\mathcal{S}$.

- The category $\mathcal{S}$ is directed and has the strict amalgamation property
- $\mathcal{S}$ is a separable metric category
Theorem (Kubiś - K)

Let $\tilde{U}$ be a sequence in $\mathfrak{S}$ and let $K$ be its inverse limit. The following properties are equivalent:

(a) The set $\text{ext } K$ is dense in $K$.
(b) $\tilde{U}$ is a Fraïssé sequence.
Consequences

- **uniqueness** of a Fraïssé sequence
  The Poulsen simplex $\mathbb{P}$ is unique, up to affine homeomorphisms.

- **universality** with respect to all simplices
  Every metrizable simplex is affinely homeomorphic to a face of $\mathbb{P}$.
Consequences

- **almost homogeneity** with respect to all simplices
  Let $F$ be a simplex and let $f, g : \mathbb{P} \rightarrow F$ be affine and continuous. Then for every $\varepsilon > 0$ there is an affine homeomorphism $H : \mathbb{P} \rightarrow \mathbb{P}$ such that for every $x \in \mathbb{P}$, $d_F(f \circ H(x), g(x)) < \varepsilon$, where $d_F$ is a fixed compatible metric on $F$.

**Remark**

*Uniqueness, universality, and homogeneity of $\mathbb{P}$ were proved by Lindenstrauss, Olsen, and Sternfeld in '78.*
Homogeneity results

Remark

Let $f : S \to T$ be a bijection, such that $S, T \subseteq E(\mathbb{L})$ are finite sets. Then there exists an affine homeomorphism $h : \mathbb{L} \to \mathbb{L}$ such that $h \upharpoonright S = f$. 
Remark

Let \( f : S \to T \) be a bijection, such that \( S, T \subseteq E(\mathbb{L}) \) are finite sets. Then there exists an affine homeomorphism \( h : \mathbb{L} \to \mathbb{L} \) such that \( h \upharpoonright S = f \).

Theorem (Kubiś - K)

Let \( A, B \subseteq E(\mathbb{L}) \) be countable dense sets. Then there exists an affine homeomorphism \( h : \mathbb{L} \to \mathbb{L} \) such that \( h[A] = B \).
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There exists a homeomorphism $h : E(\mathbb{L}) \rightarrow E(\mathbb{L})$ such that for no homeomorphism $f : \mathbb{L} \rightarrow \mathbb{L}$, we have $f \upharpoonright E(\mathbb{L}) = h$. 
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Generalization of the category \( \mathcal{F} \)

- \( F \) be a geometric fan
- \( E(F) \) - the set of endpoints of \( F \)
- A **skeleton** in \( F \) is a convex set \( D \subseteq F \) such that \( E(D) \) is countable, contained in \( E(F) \) and dense in \( E(F) \).
Let $\mathcal{F}^d$ be the category whose **objects** are pairs of finite geometric fans $(F^1, F^2)$ with $F^1 = F^2$. 
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An arrow from $(F^1, F^2)$ to $(G^1, G^2)$ is a pair $\langle e, p \rangle$ such that $e: F^1 \to G^1$ is a stable embedding, $p: G^2 \to F^2$ is a 1-Lipschitz affine retraction and $p \circ e = \text{id}_F$. 
The category $\mathcal{F}^d$ is directed and has the strict amalgamation property.

$\mathcal{F}^d$ is a separable metric category, therefore it has a unique up to isomorphism Fraïssé sequence.

Its limit is $(D, \mathbb{L})$ for some skeleton $D$ in $\mathbb{L}$. 
Generalization of the category $\mathcal{F}$

To show the main theorem we need the following lemma:

**Lemma**

Let $L$ be a geometric fan and let $D$ be a skeleton in $L$. Then there exist a geometric fan $L'$, a skeleton $D'$ of $L'$, and an affine (not necessarily 1-Lipschitz) homeomorphism $h: L \to L'$ with $h(D) = D'$ such that there is a sequence $\vec{F}$ in $\mathcal{F}^d$ satisfying $L' = \lim \leftarrow \vec{F}$ and $D' = \lim \rightarrow \vec{F}$. 