Coimbra Daily

SPECIAL ISSUE!

Coimbra, Tuesday, September 8, 2015 €4.50

MAT TRIAD '15

TREASURE HUNT!

FIND THE KEY!

A wider convergence area for the MSTMAOR iteration methods for LCP

Ljiljana Cvetković, Vladimir Kostić, Ernest Šanca

SPECIAL ISSUE! Coimbra, Tuesday, September 8, 2015 €4.50

MAT TRIAD '15 *TREASURE HUNT!*

FIND THE KEY!

A wider convergence area for the MSTMAOR iteration methods for LCP

Ljiljana Cvetković, Vladimir Kostić, Ernest Šanca

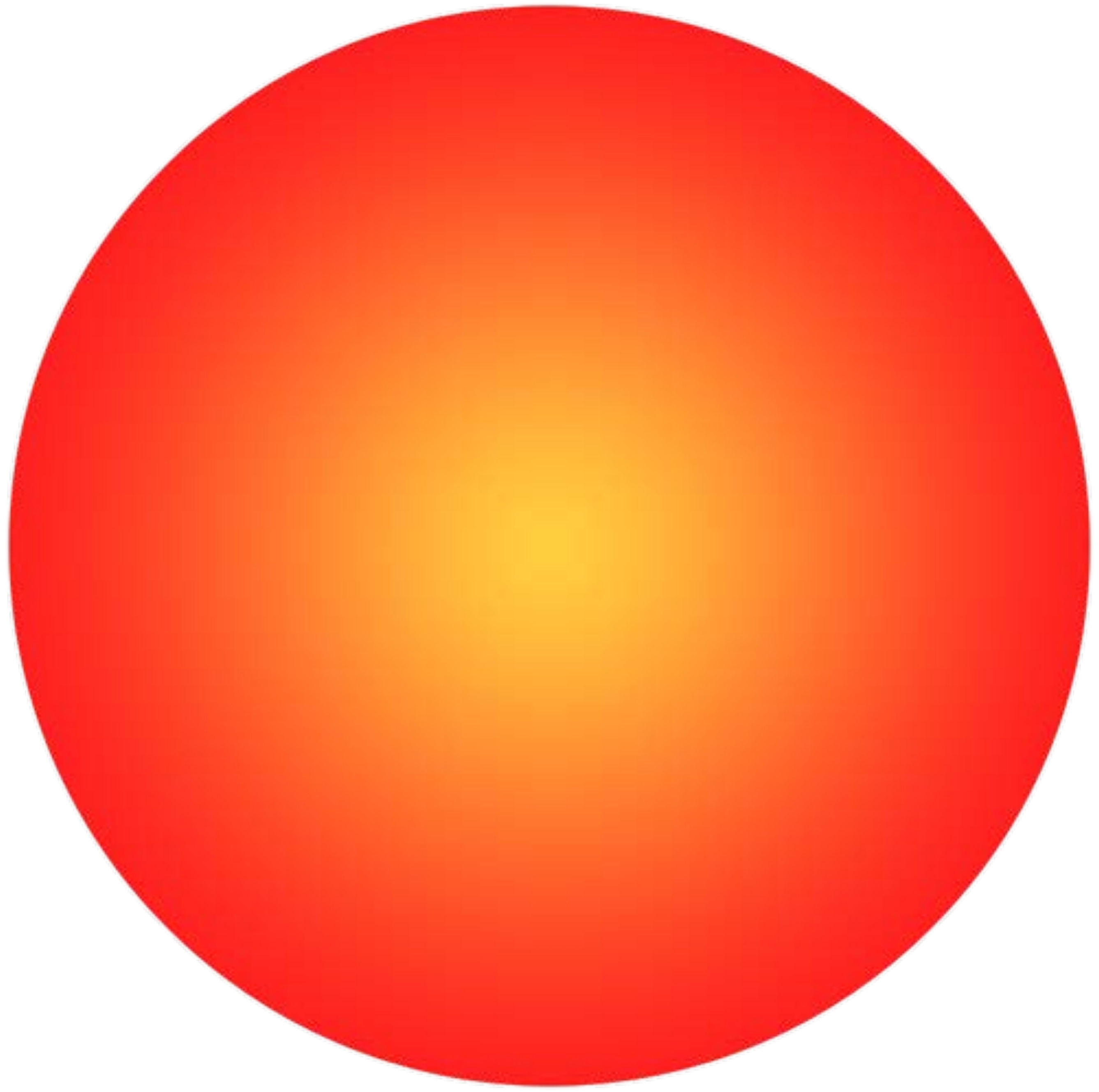
In order to solve large sparse linear complementarity problems on parallel multiprocessor systems, modulus-based synchronous two-stage multisplitting iteration methods based on two-stage multisplittings of the system matrices were constructed and investigated by Bai and Zhang (Numerical Algorithms 62, 59–77 2013).These iteration methods include the multisplitting relaxation methods such as Jacobi, Gauss-Seidel, SOR and AOR of the modulus type as special cases. In the same paper the convergence theory of these methods is developed, under the following assumptions: (i) the system matrix is an H+-matrix and (ii) one acceleration parameter is greater than the other. Here we show that the second assumption can be avoided, thus enabling us to obtain an improved convergence area. The result is obtained using the similar technique proposed by Cvetković and Kostić (Numerical

A wider convergence area for the MSTMAOR iteration methods for LCP

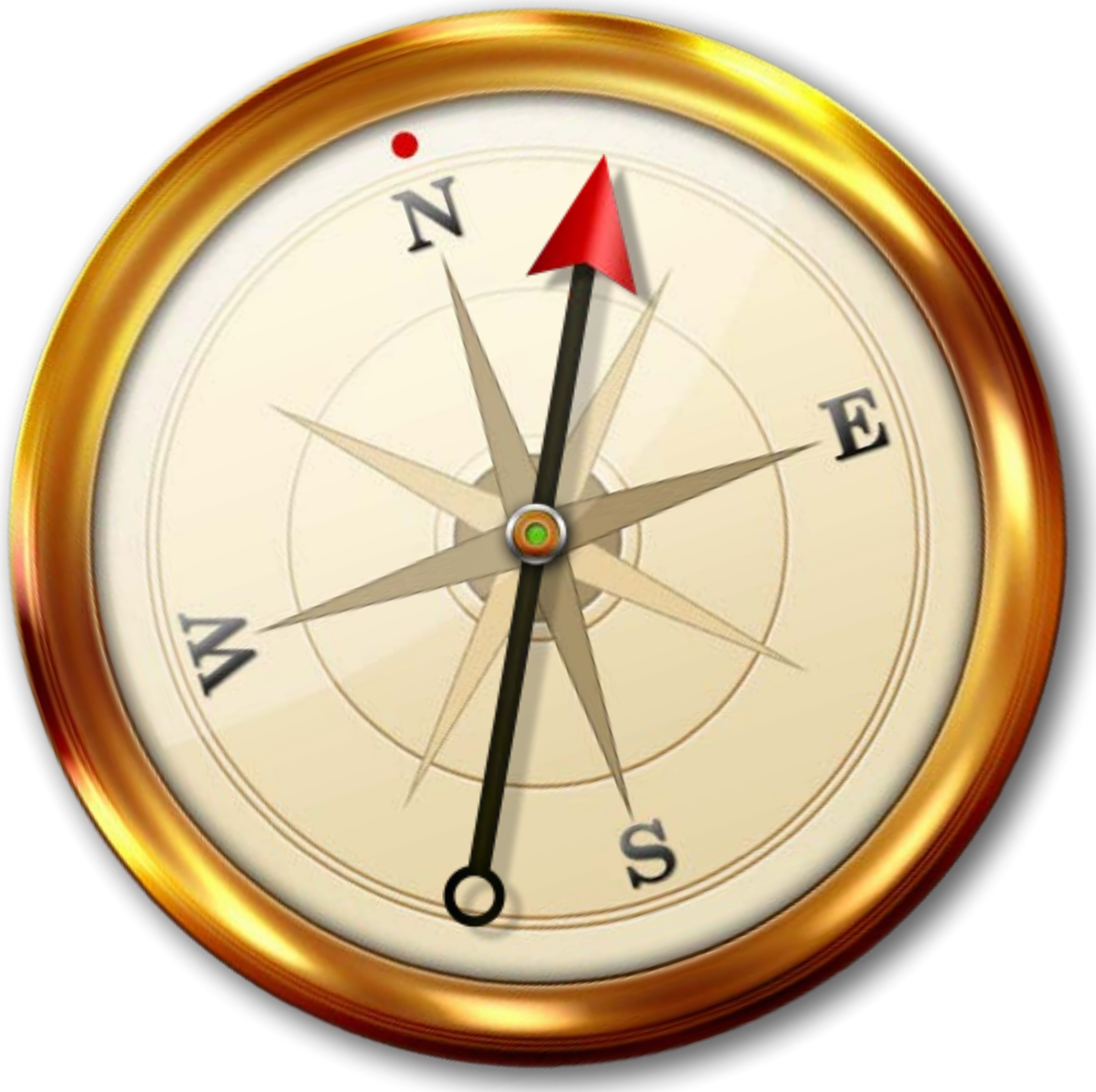
Ljiljana Cvetković, Vladimir Kostić, Ernest Šanca

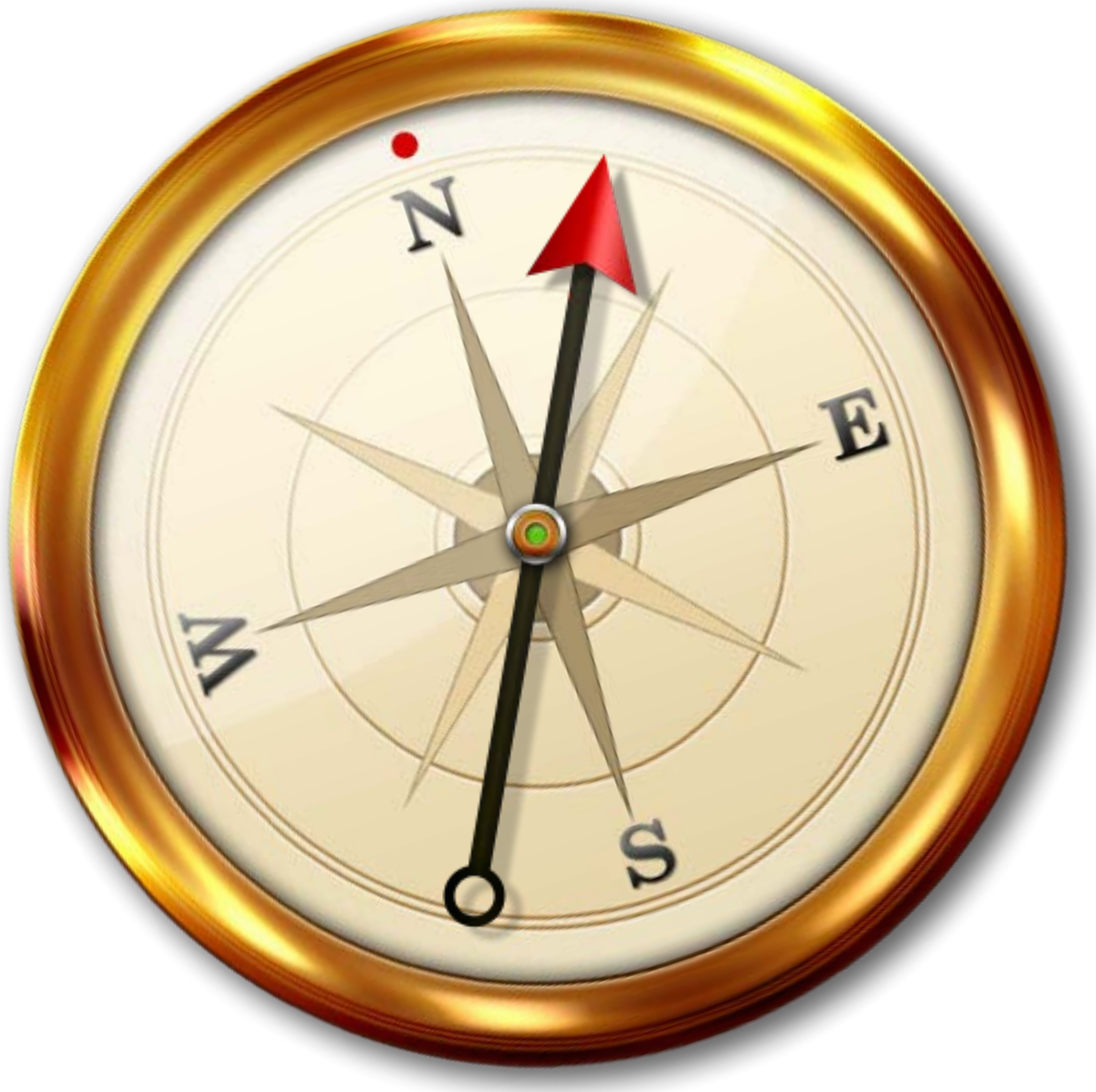
A wider convergence area for the MSTMAOR iteration methods for LCP

Ljiljana Cvetković, Vladimir Kostić, Ernest Šanca



Treasure hunt


N
E
S
W

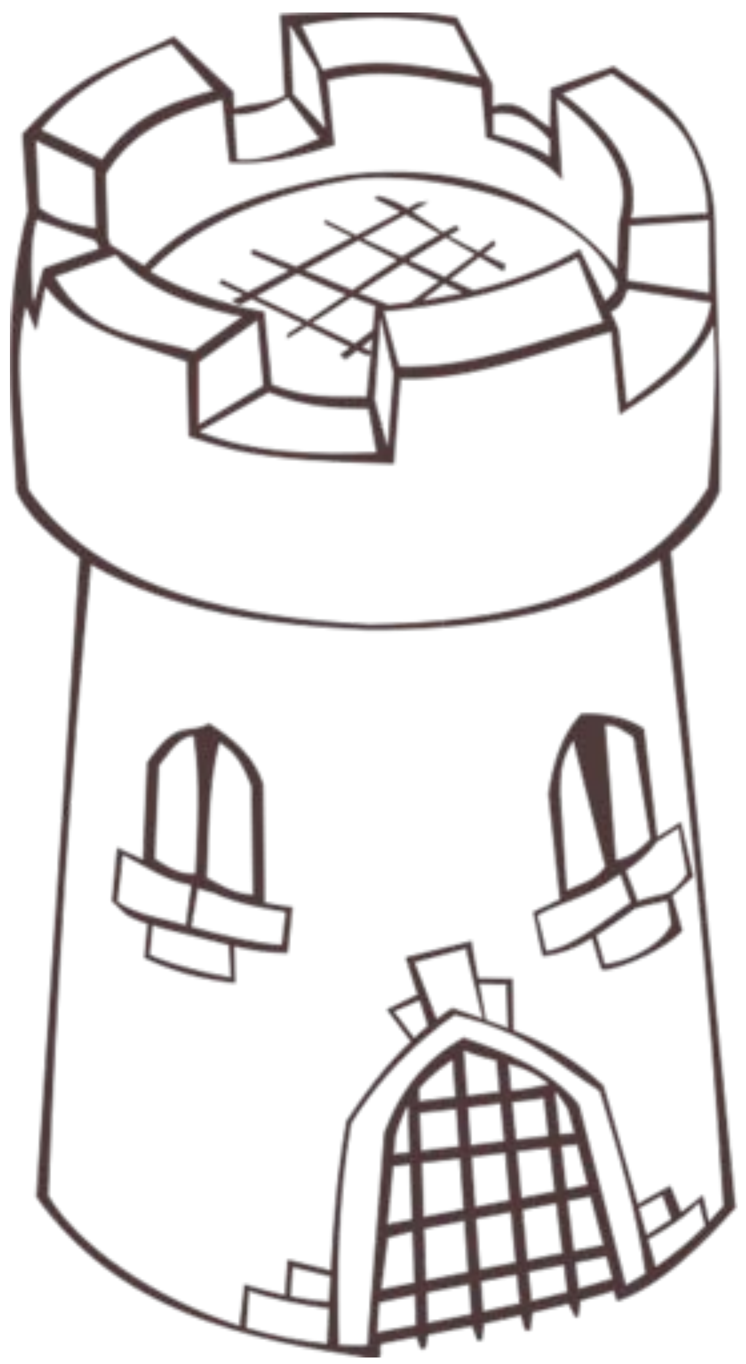

N
E
S
W


INTRO

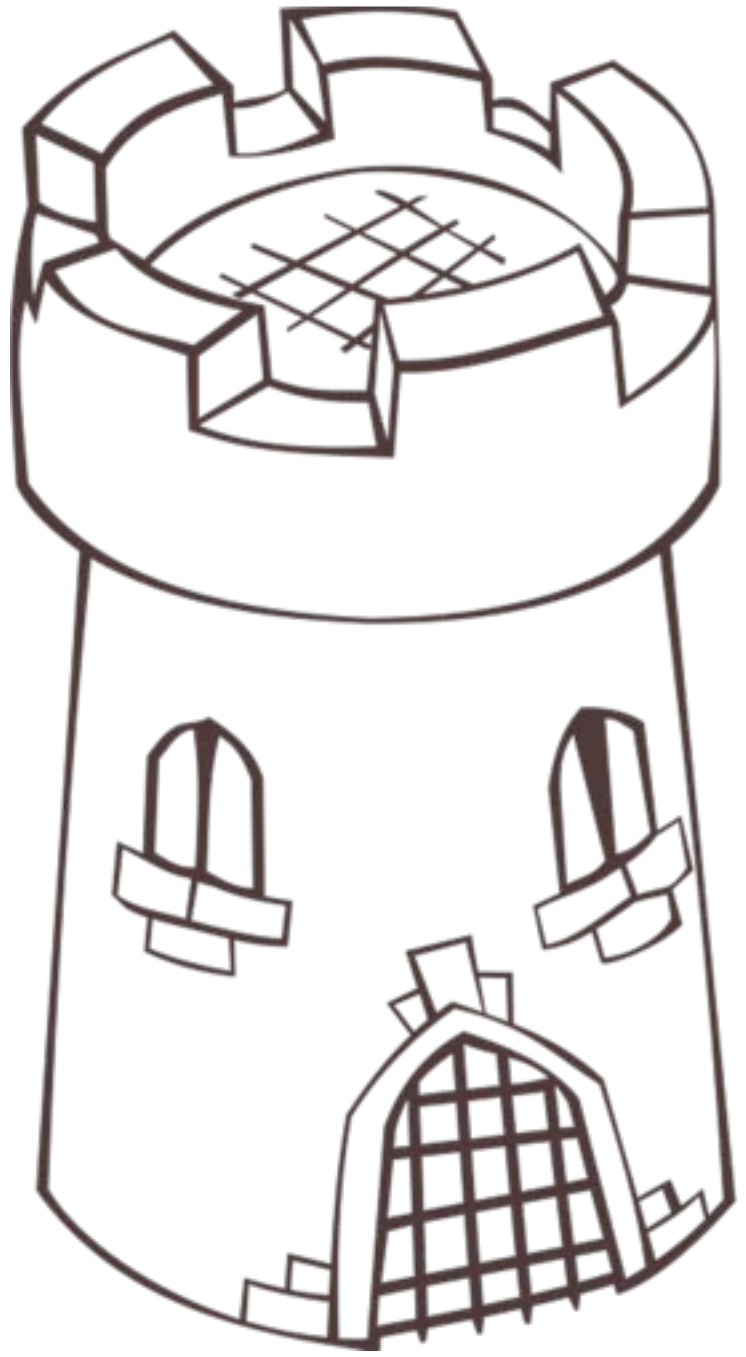

RETROSPECT
INTRO

Retrospect
Intro
Search

RETROSPECT
INTRO
SEARCH



Intro



INTRO

LCP

Problem statement

LCP

$$A \in \mathbb{R}^{n,n}$$

$$q \in \mathbb{R}^{n}$$

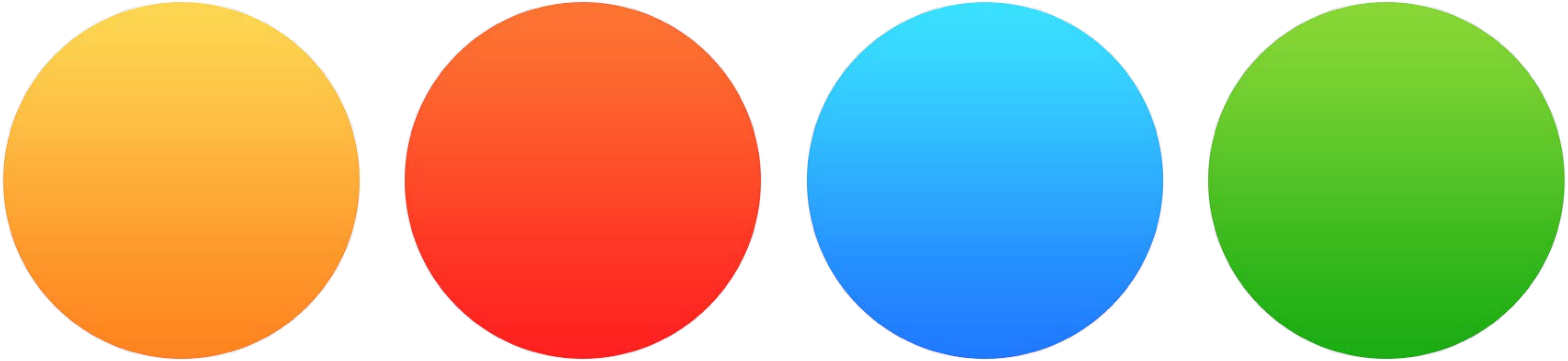
$$z, r \in \mathbb{R}^{n} \rightsquigarrow ?$$

LCP

$$A \in \mathbb{R}^{n,n}$$
$$q \in \mathbb{R}^{n}$$
$$z, r \in \mathbb{R}^{n} \rightsquigarrow ?$$

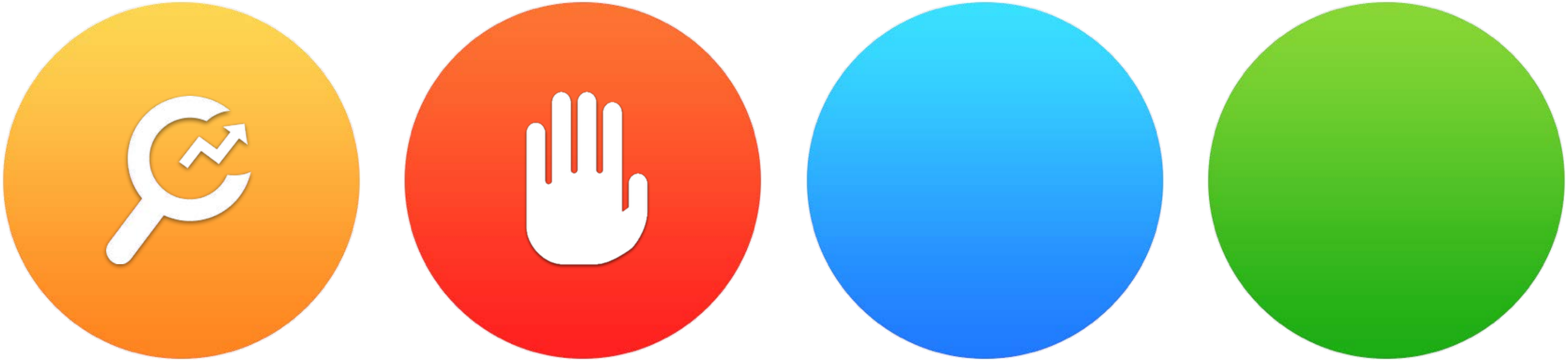
$$z \geq 0$$
$$r := Az + q \geq 0$$
$$z^T r = 0$$

Motivation



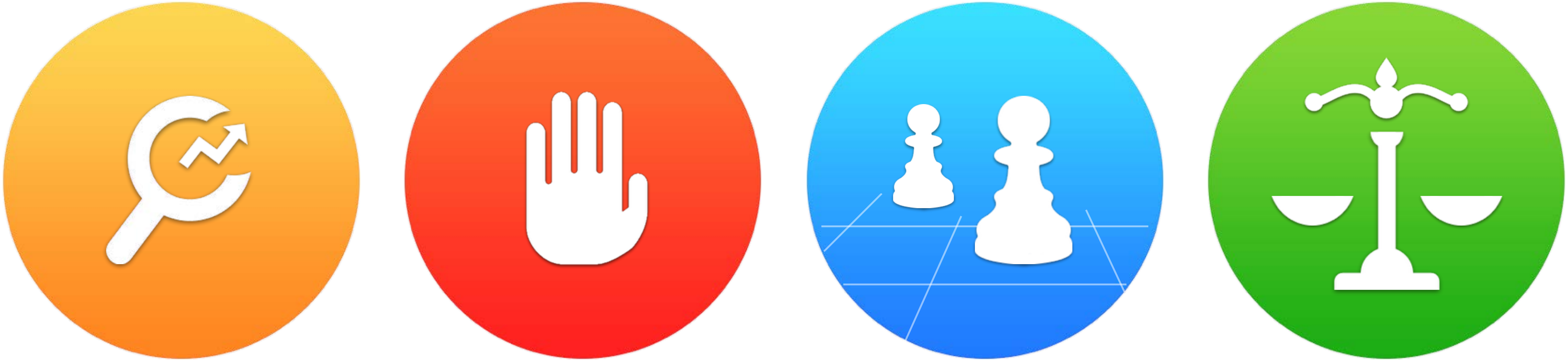
Quadratic Programming

Optimal stopping



Bimatrix games

Market equilibria



RETROSPECT

RETROSPECT

Matrix Theory

Def

For an arbitrary matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, its **comparison matrix** $\mathcal{M}(A) := [\mu_{ij}] \in \mathbb{R}^{n,n}$ is entry-wise defined as follows

$$\mu_{ij} := \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

Def

Matrix A is referred to as an **H-matrix** if and only if $\mathcal{M}(A)$ is an M-matrix, i.e. $\mathcal{M}(A)^{-1} \geq 0$.

H^+-matrix $\rightsquigarrow$ H-matrix with positive diagonal entries.

LCP solvability

Th

Given $A \in \mathbb{R}^{n,n}$ being an H^{+}-matrix, there **exists** a **unique** solution $z_{\star}$ of $\mathrm{LCP}(q, A)$.

! Assumption of A being H^{+} poses no restriction.

Solving $\mathrm{LCP}(q, A)$ numerically.

Bai, Z-Z., Evans, D.J.: Matrix multisplitting relaxation methods for linear complementarity problems, International Journal of Computer Mathematics, 63 (1997), 309-326.

Iteration methods for LCP

Basic concepts

Def

Let $A \in \mathbb{R}^{n,n}$. If $\exists M, N \in \mathbb{R}^{n,n}$ so that M is nonsingular and

$$A = M - N, \qquad ♪$$

then ♪ represents a **splitting** of matrix A .

Splitting is **H-compatible** if $\mathcal{M}(A) = \mathcal{M}(M) - |N|$, additionally.

Th

Let $A = M - N$ be a splitting of $A \in \mathbb{R}^{n,n}$,

$\Omega \geq 0$ $\cdots$ nonnegative diagonal matrix,

$\gamma > 0$ $\cdots$ positive arbitrary scalar.

1. If z is a solution of $\mathrm{LCP}(q, A)$, then $x = \frac{1}{2}\gamma(z - \Omega^{-1}r)$ satisfies IFPE

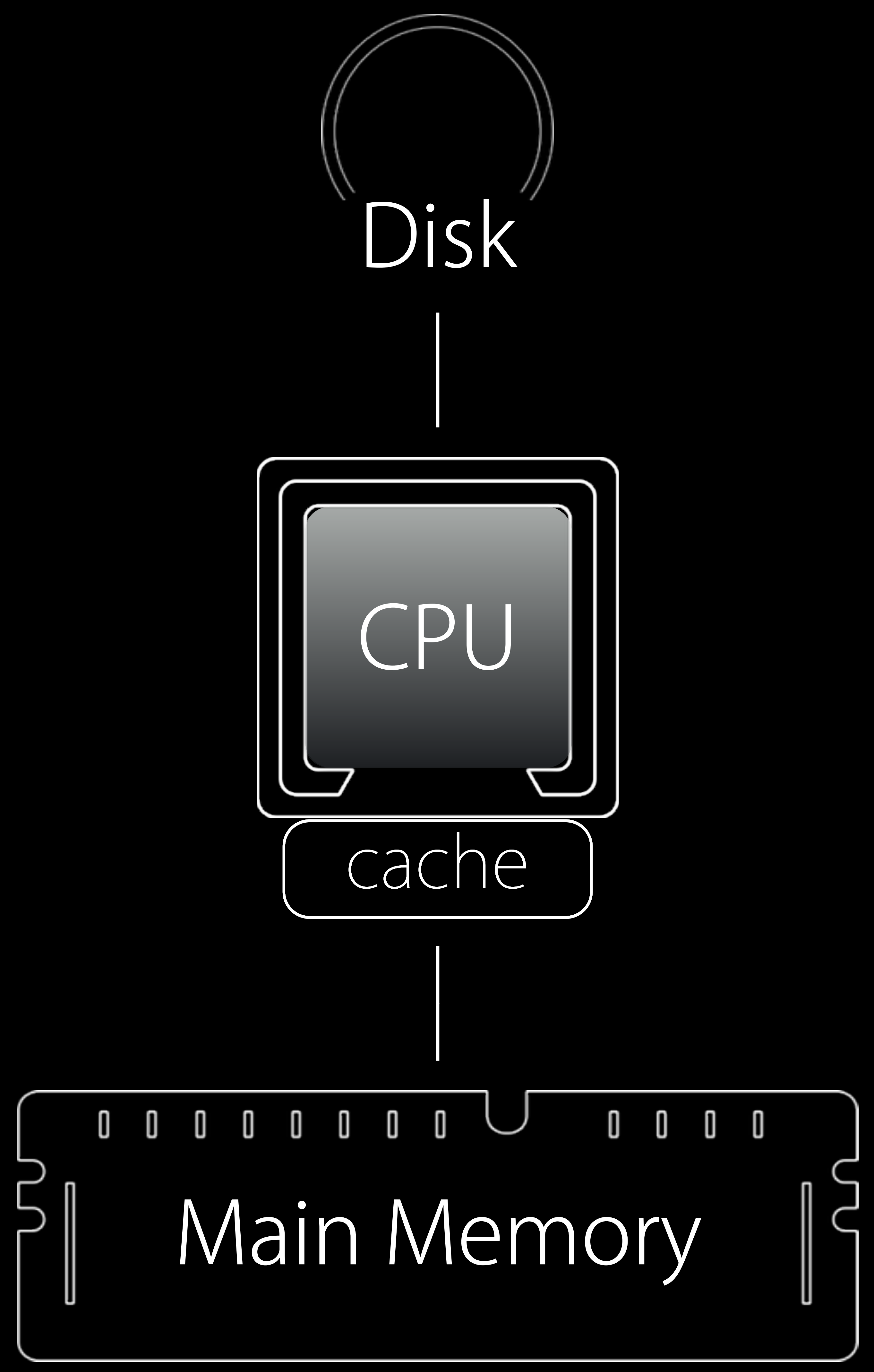
$$(\Omega + M)x = Nx + (\Omega - A)|x| - \gamma q, \qquad \spadesuit$$

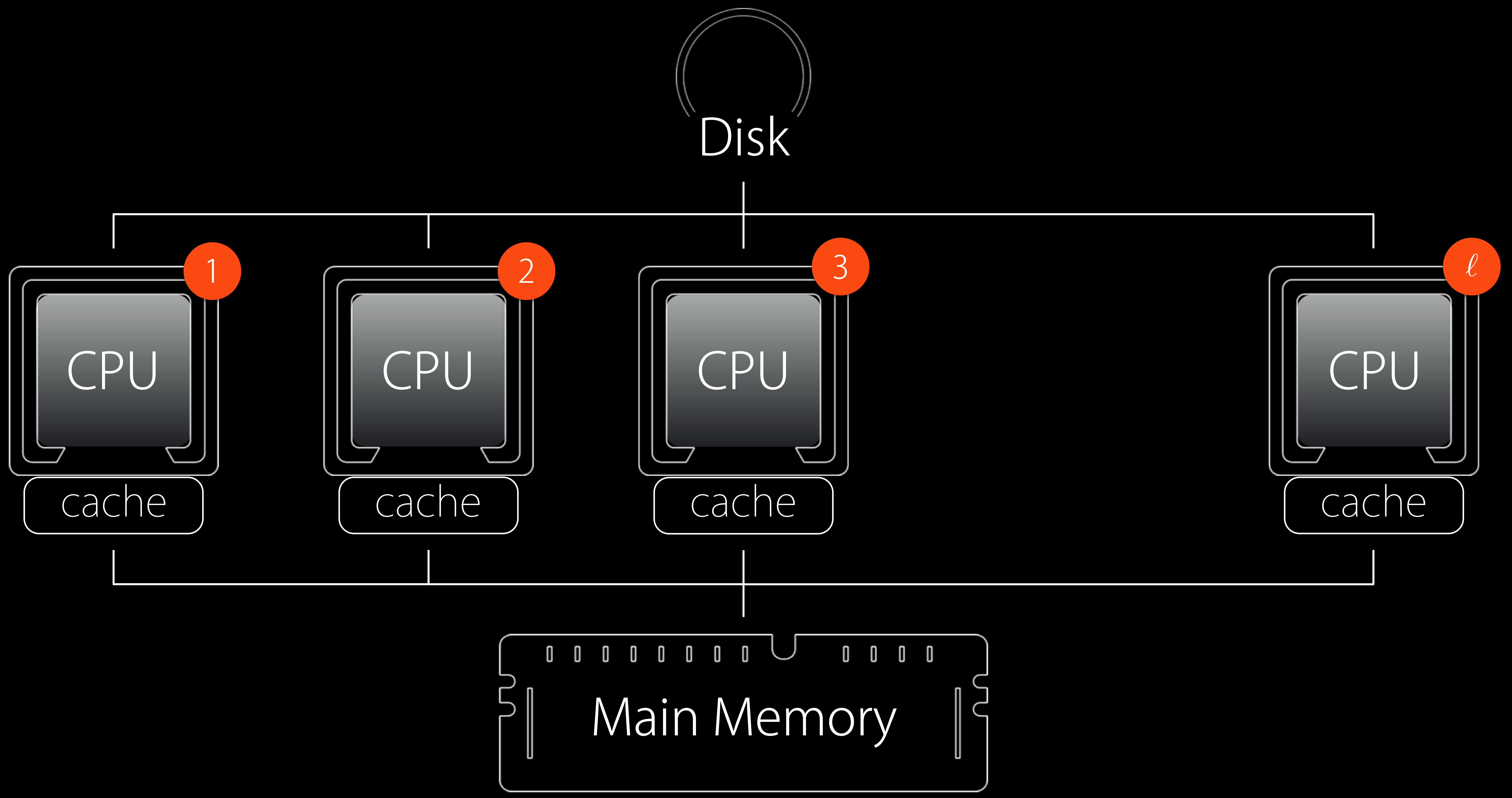
2. If x satisfies $\spadesuit$, then the solution of $\mathrm{LCP}(q, A)$ is given by

$$z = \gamma^{-1}(|x| + x) \quad \text{and} \quad r = \gamma^{-1}\Omega(|x| - x).$$

MPC

Multiprocessor parallel computers


Disk
CPU
cache
Main Memory



Disk
1
CPU
cache
2
CPU
cache
3
CPU
cache
ℓ
CPU
cache
Main Memory

Def

$\ell \cdots$ given integer - number of processors $(\ell \leq n)$

$A = M_p - N_p, p = 1, 2, \ldots, \ell \cdots$ splittings of A

$E_p \in \mathbb{R}^{n,n} \cdots$ nonnegative diagonal matrices: $\sum_{p=1}^{\ell} E_p = E$.

Collection of triples

$$(M_p, N_p, E_p)(p = 1, 2, \ldots, \ell)$$

represents a **multisplitting** of A.

Modulus synchronous multisplitting (MSM) iteration method for LCP

$(M_p, N_p, E_p)(p = 1, 2, \ldots, \ell)$ $\cdots$ multisplitting of A

$z^{(0)} \in \mathbb{R}^n_+$ $\cdots$ arbitrary initial vector

for $k \geq 0$ until convergence of $\{z^{(k)}\}_{k=0}^{\infty} \subset \mathbb{R}^n_+$,

$$(\Omega + M_p)\, x^{(k,p)} = N_p x^{(k)} + (\Omega - A)|x^{(k)}| - \gamma\, q \qquad p = 1, 2, \ldots, \ell,$$

with

$$x^{(0)} = \frac{1}{2}\gamma\, \Omega^{-1}((\Omega - A)z^{(0)} - q),$$

$$x^{(k+1)} = \sum_{p=1}^{\ell} E_p\, x^{(k,p)} \qquad z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)}).$$

$\Omega \geq 0$ $\cdots$ arbitrary square matrix of order n

$\gamma > 0$ $\cdots$ arbitrary constant

Def

$D = \mathrm{diag}(A)$ and for $p = 1, 2, \ldots, \ell$

L_p $\cdots$ strictly lower triangular

$U_p = D - L_p - A$, $\cdots$ zero-diagonal

$E_p \in \mathbb{R}^{n,n}$ $\cdots$ nonnegative diagonal matrices: $\sum_{p=1}^{\ell} E_p = E$.

Collection of triples

$$(D - L_p, U_p, E_p)(p = 1, 2, \ldots, \ell)$$

represents a **triangular multisplitting** of A.

Triangular multisplitting: $\text{MSM} \hookrightarrow \text{MSMAOR}$

Convergence results:

- **Bai**, Z.-Z., **Zhang**, L.-L.: Modulus-based synchronous multisplitting iteration methods for linear complementarity problems, Numerical Linear Algebra with Applications, 20 (2013), 425-439.

- **Cvetković**, Lj., **Kostić**, V.: A note on the convergence of the MSMAOR method for linear complementarity problems, Numerical Linear Algebra with Applications, 21 (2014), 534-539.

SEARCH


SEARCH

Def

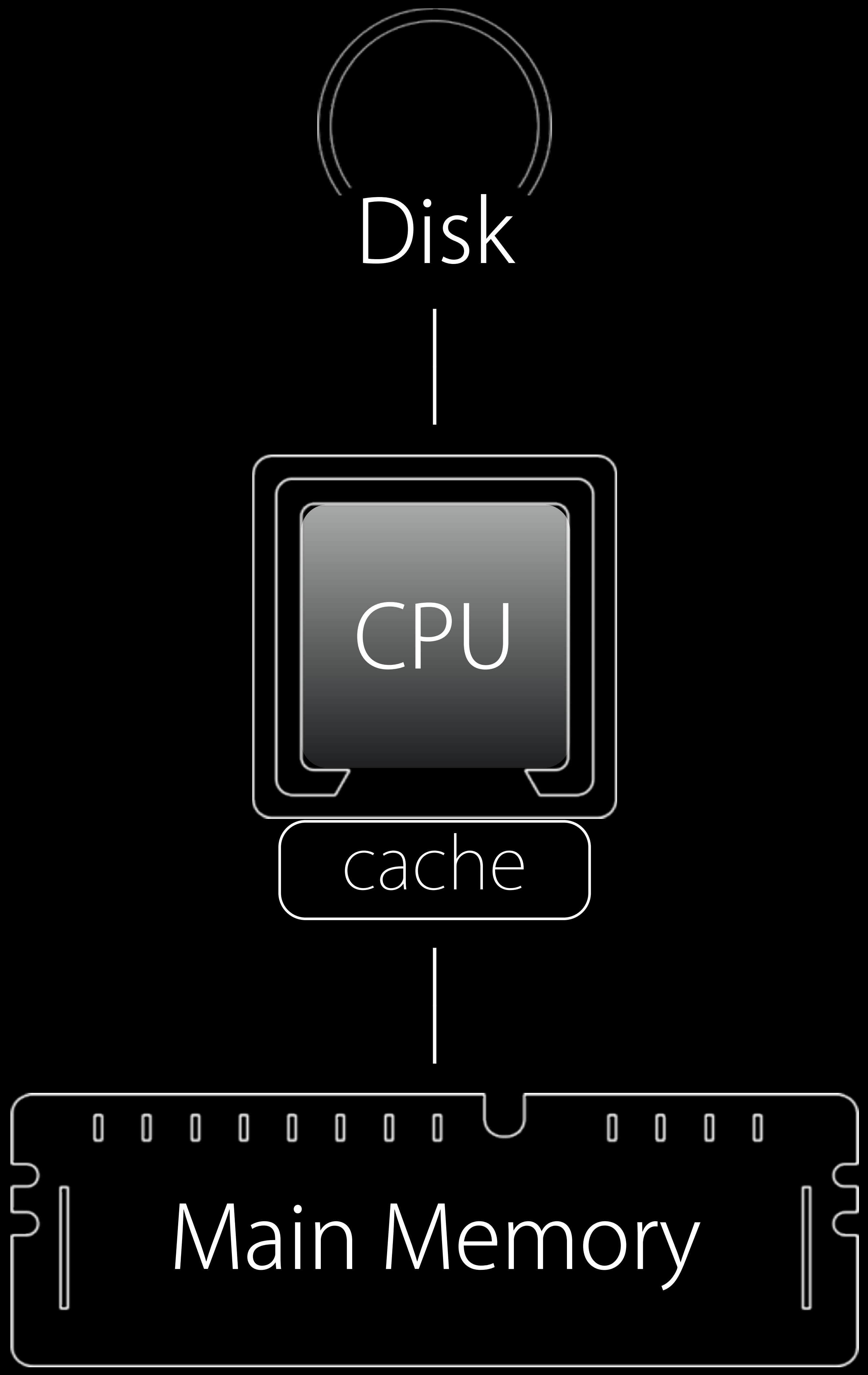
$(M_p, N_p, E_p)(p = 1, 2, \ldots, \ell)$ $\cdots$ multisplitting of A

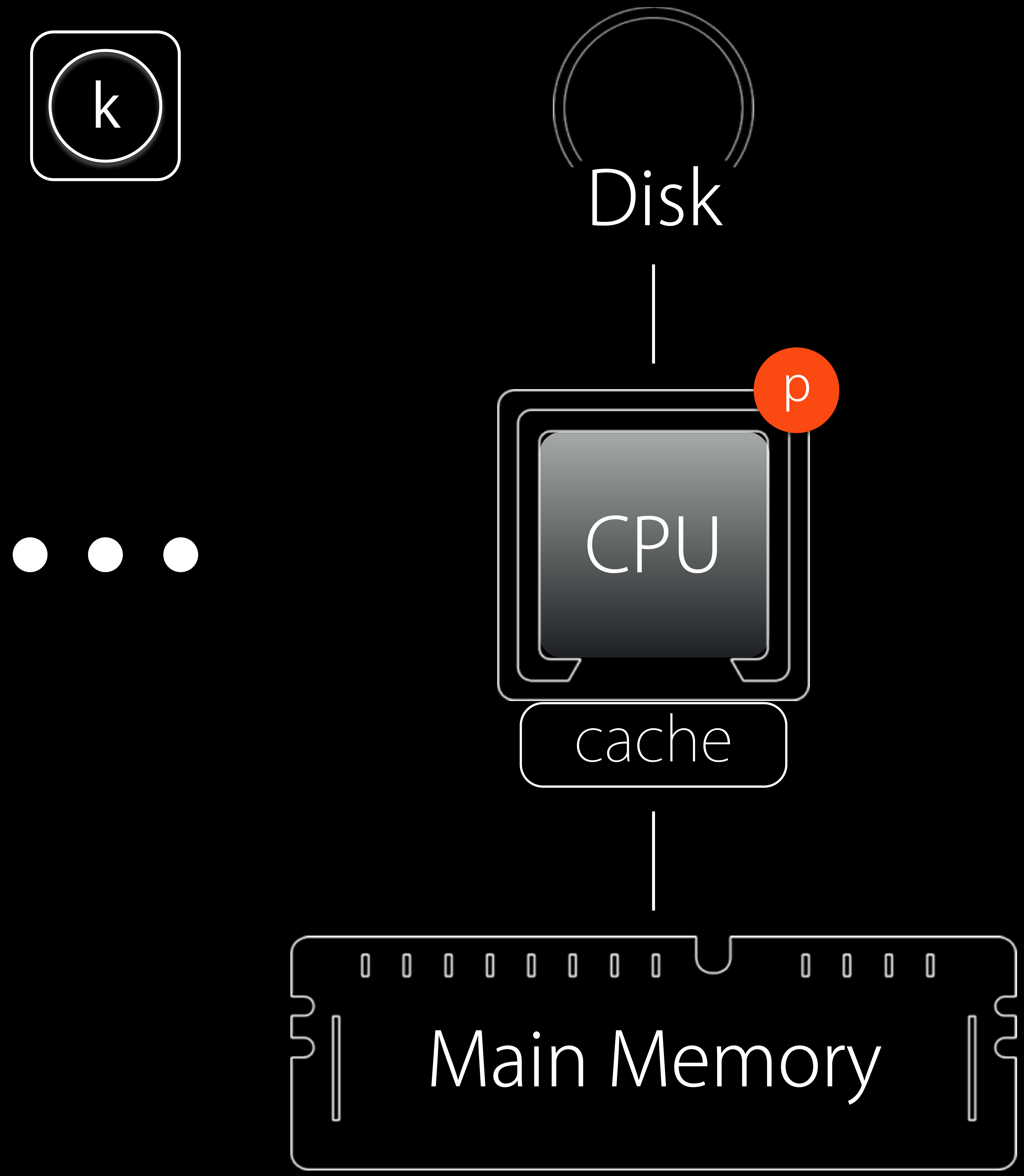
$M_p = F_p - G_p (p = 1, 2, \ldots, \ell)$ $\cdots$ splittings of M_p

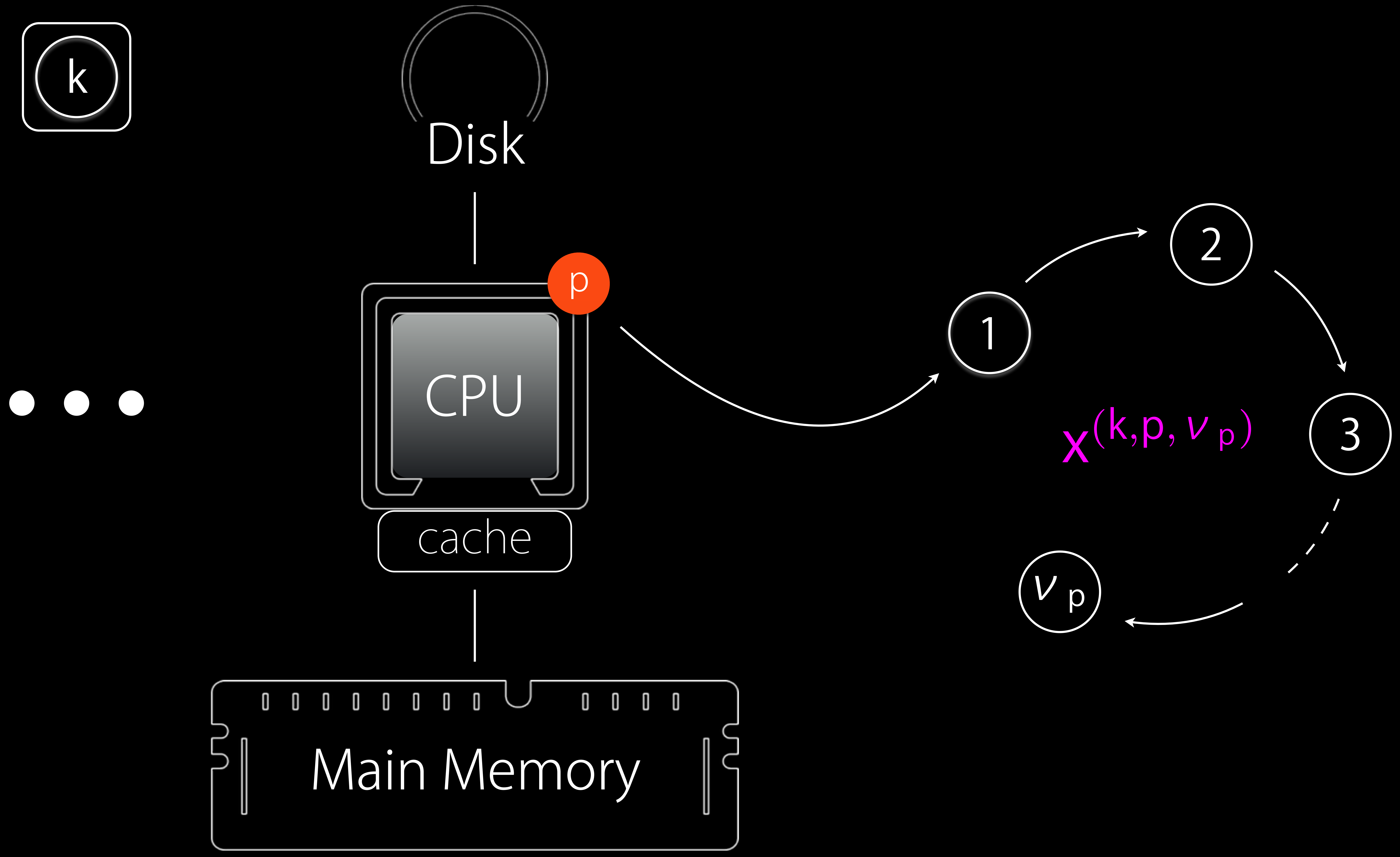
Collection of triples

$$(M_p : F_p, G_p; N_p; E_p)(p = 1, 2, \ldots, \ell)$$

represents a **2-stage multisplitting** of A .


Disk
CPU
cache
Main Memory


k
Disk
p
CPU
cache
Main Memory



k
Disk
p
CPU
cache
Main Memory
1
2
3
$x^{(k,p,\nu_p)}$
ν_p

Modulus synchronous 2-stage multisplitting (MSTM) iteration method

$(M_p : F_p, G_p; N_p; E_p)(p = 1, 2, \ldots, \ell)$ ⋯ 2-stage multisplitting of A

$z^{(0)} \in \mathbb{R}^n_+$ ⋯ arbitrary initial vector

$\nu_p (p = 1, 2, \ldots, \ell)$ ⋯ number of inner iterations

for $k \geq 0$ until convergence of $\{z^{(k)}\}_{k=0}^{\infty} \subset \mathbb{R}^n_+$,

$$\begin{cases} (\Omega + F_p)x^{(k,p,j+1)} = G_p x^{(k,p,j)} + b^{(k,p)} \\ p = 1, 2, \ldots, \ell \\ j = 0, 1, \ldots, \nu_p - 1 \end{cases}$$

$$b^{(k,p)} = N_p x^{(k)} + (\Omega - A)|x^{(k)}| - \gamma q, \quad x^{(k,p,0)} = x^{(k)},$$

↪

Modulus synchronous 2-stage multisplitting (MSTM) iteration method

$\looparrowright$

with

$$x^{(0)} = \frac{1}{2}\gamma\,\Omega^{-1}((\Omega - A)z^{(0)} - q),$$

$$x^{(k+1)} = \sum_{p=1}^{\ell} E_p\, x^{(k,p,\nu_p)}$$

$$z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)}).$$

Def

$D = \mathrm{diag}(A)$ and parts of $M_p (p = 1, 2, \ldots, \ell)$

$D_p^{(M)} = \mathrm{diag}(M_p) \cdots$ diagonal

$L_p^{(M)} \cdots$ strict lower triangle

$U_p^{(M)} \cdots$ zero-diagonal

so that $M_p = D_p^{(M)} - L_p^{(M)} - U_p^{(M)}$.

Collection of triples

$$(M_p : D_p^{(M)} - L_p^{(M)}, U_p^{(M)}; N_p; E_p)(p = 1, 2, \ldots, \ell)$$

represents a **2-stage triangular multisplitting** of A.

2-stage triangular multisplitting: MSTM $\hookrightarrow$ MSTMAOR

$$F_p = \frac{1}{\alpha}(D_p^{(M)} - \beta L_p^{(M)}),$$

$$G_p = \frac{1}{\alpha}\Big((1-\alpha)D_p^{(M)} + (\alpha-\beta)L_p^{(M)} + \alpha U_p^{(M)}\Big),$$

$\alpha, \beta \cdots$ relaxation parameters

Convergence for MSTMAOR iteration method (Bai, Z.-Z., Zhang, L.-L. 2013)

$A \in \mathbb{R}^{n,n}$ is H^+, $\quad D = \mathrm{diag}(A), \quad B = D - A, \quad \gamma > 0, \quad \Omega \geq D,$

and a 2-stage triangular multisplitting of A

$$(M_p : D_p^{(M)} - L_p^{(M)}, U_p^{(M)}; N_p; E_p)(p = 1, 2, \ldots, \ell).$$

If $A = M_p - N_p (p = 1, 2, \ldots, \ell)$ are H-compatible splittings and

$$\mathcal{M}(M_p) = D - |L_p^{(M)}| - |U_p^{(M)}| (p = 1, 2, \ldots, \ell), \quad D = \mathrm{diag}(M_p),$$

then $\lim_{k\to\infty} z^{(k)} = z_\star, \quad \forall z^{(0)} \in \mathbb{R}^n_+$ and arbitrary ν_p, under the condition that

$$0 < \beta \leq \alpha < \frac{1}{\rho\,(D^{-1}|B|)}.$$

Convergence for MSTMAOR iteration method (Bai, Z.-Z., Zhang, L.-L. 2013)

$A \in \mathbb{R}^{n,n}$ is H^+, $\quad D = \mathrm{diag}(A)$, $\quad B = D - A$, $\quad \gamma > 0$, $\quad \Omega \geq D$,

and a 2-stage triangular multisplitting of A

$$(M_p : D_p^{(M)} - L_p^{(M)}, U_p^{(M)}; N_p; E_p)(p = 1, 2, \ldots, \ell).$$

If $A = M_p - N_p (p = 1, 2, \ldots, \ell)$ are H-compatible splittings and

$$\mathcal{M}(M_p) = D - |L_p^{(M)}| - |U_p^{(M)}| (p = 1, 2, \ldots, \ell), \quad D = \mathrm{diag}(M_p),$$

then $\lim_{k \to \infty} z^{(k)} = z_\star$, $\quad \forall z^{(0)} \in \mathbb{R}^n_+$ and arbitrary ν_p, under the condition that

$$0 < \beta \leq \alpha < \frac{1}{\rho(D^{-1}|B|)}.$$

Expansion of the parameter area (Cvetković, Lj., Kostić, V., Šanca, E. 2015)

$$\beta \geq 0 \quad \text{and} \quad \left(\theta \max\{\alpha, \xi\beta\} + (1-\theta)\alpha\right)\rho(D^{-1}|B|) < \min\{1, \alpha\}$$

or, equivalently,

$$0 < \alpha < \frac{1}{\rho(D^{-1}|B|)}, \qquad 0 \leq \beta < \frac{\min\{1, \alpha\} - (1-\theta)\alpha\,\rho(D^{-1}|B|)}{\xi\,\theta\,\rho(D^{-1}|B|)}.$$

Expansion of the parameter area (Cvetković, Lj., Kostić, V., Šanca, E. 2015)

1st stage

$$A = M_p - N_p \quad \text{H-compatible splittings and} \quad \Longrightarrow M_p = \Theta_p \circ A,$$

$$\Theta_p = [\theta_{ij}^p] : \; 0 \le \theta_{ij}^p \le 1, \qquad 1 \le i, j \le n,$$

$$\theta = \max\{\theta_{ij}^p : p = 1, 2, \ldots, \ell, \quad j < i\},$$

2nd stage

$$\mathcal{M}(M_p) = D - |L_p^{(M)}| - |U_p^{(M)}| \;\Longrightarrow L_p^{(M)} = \Xi_p \circ (-M_p),$$

$$\Xi_p = [\xi_{ij}^p] : \; 0 \le \xi_{ij}^p \le 1 \text{ for } 1 \le j < i \le n \qquad \xi_{ij}^p = 0 \text{ otherwise}$$

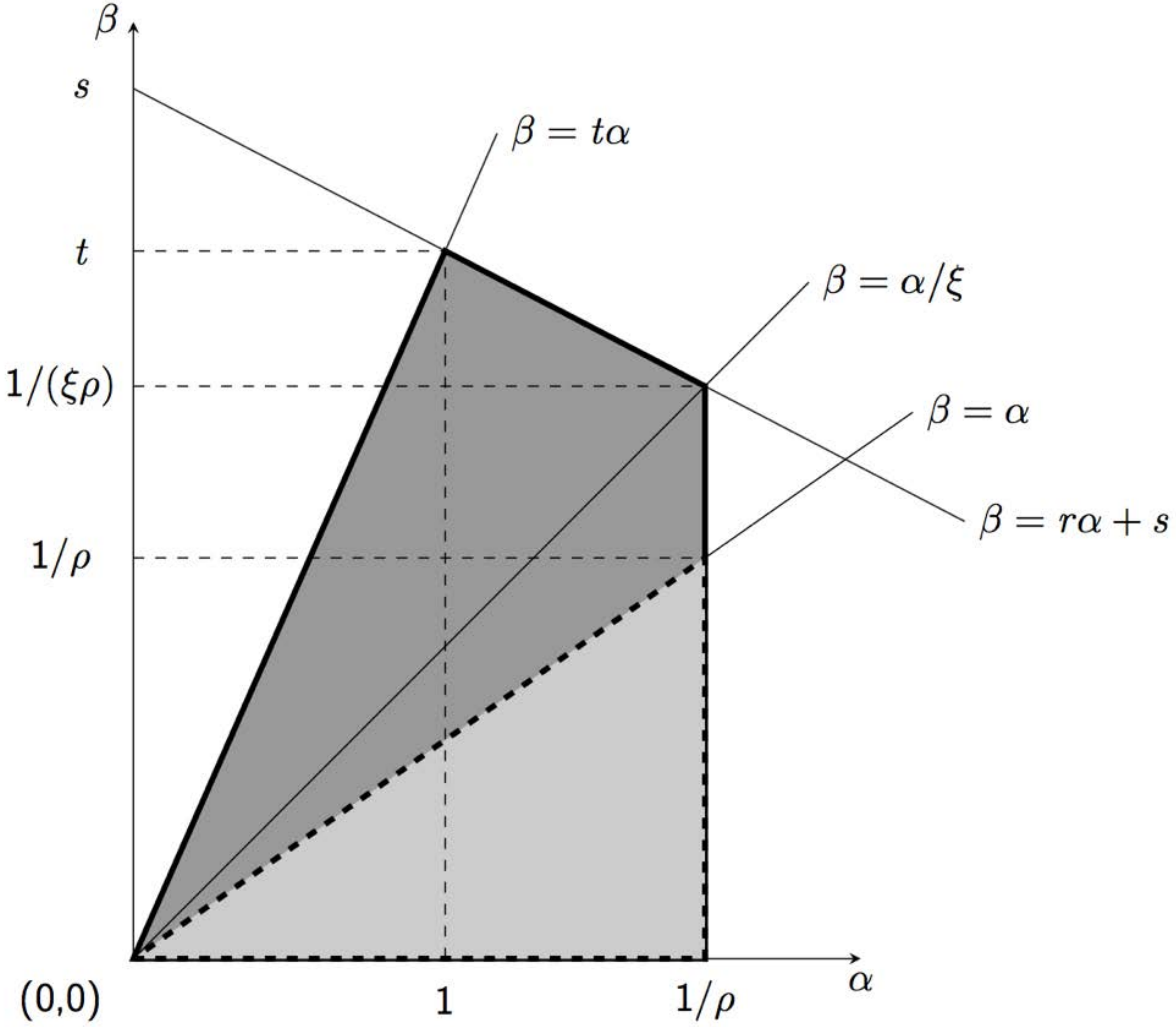
$$\xi = \max\{\xi_{ij}^p : p = 1, 2, \ldots, \ell, \quad j < i\}.$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{\mathsf{A}} = \underbrace{\begin{bmatrix} 2 & 0.5 & 0 \\ -0.7 & 1 & -0.3 \\ 0 & -0.6 & 2 \end{bmatrix}}_{\mathsf{M}} - \underbrace{\begin{bmatrix} 0 & -0.5 & 0 \\ 0.3 & -1 & 0.7 \\ 0 & 0.4 & 0 \end{bmatrix}}_{\mathsf{N}}, \quad \Theta = \begin{bmatrix} 1 & 0.5 & 1 \\ 0.7 & 0.5 & 0.3 \\ 0 & 0.6 & 1 \end{bmatrix},$$

$$\mathsf{M} = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{\mathsf{D}} - \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0.28 & 0 & 0 \\ 0 & 0.18 & 0 \end{bmatrix}}_{\mathsf{L}} - \underbrace{\begin{bmatrix} 0 & -0.5 & 0 \\ 0.42 & 0 & 0.3 \\ 0 & 0.42 & 0 \end{bmatrix}}_{\mathsf{U}}, \quad \Xi = \begin{bmatrix} 0 & 0 & 0 \\ 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}.$$

$$\theta = \max\{0.7, 0.6\} = 0.7$$

$$\xi = \max\{0.4, 0.3\} = 0.4$$

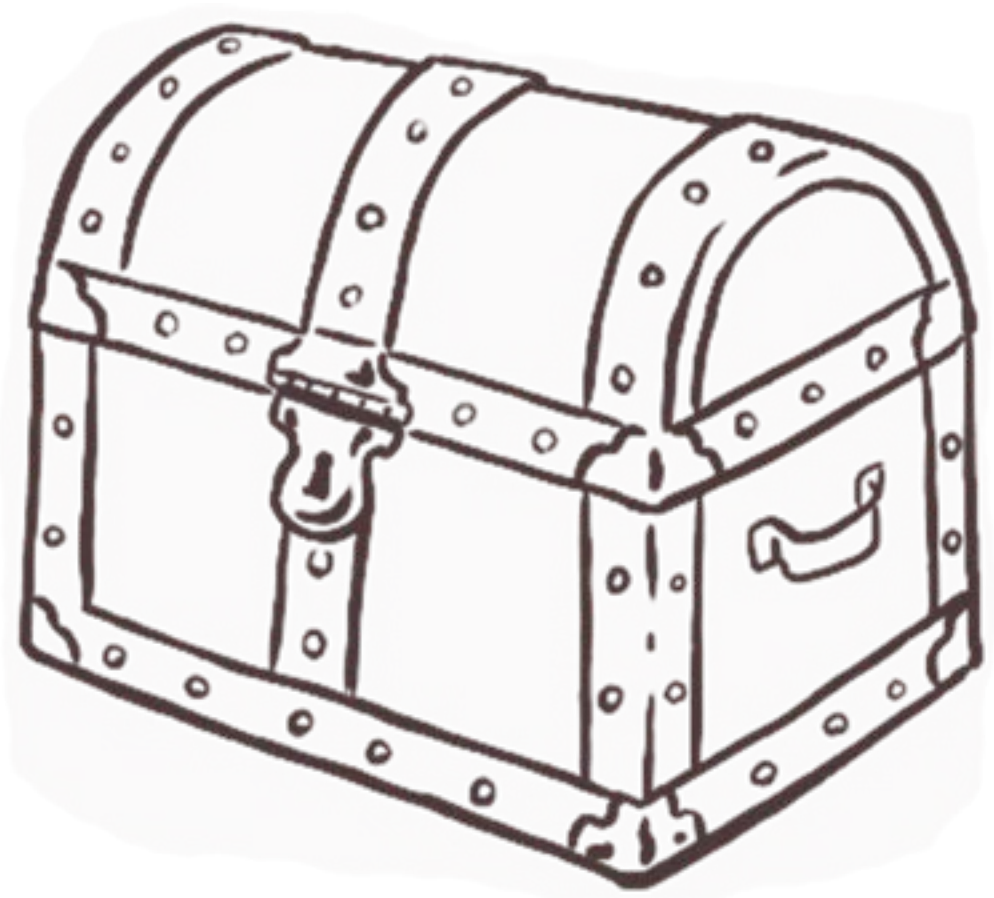


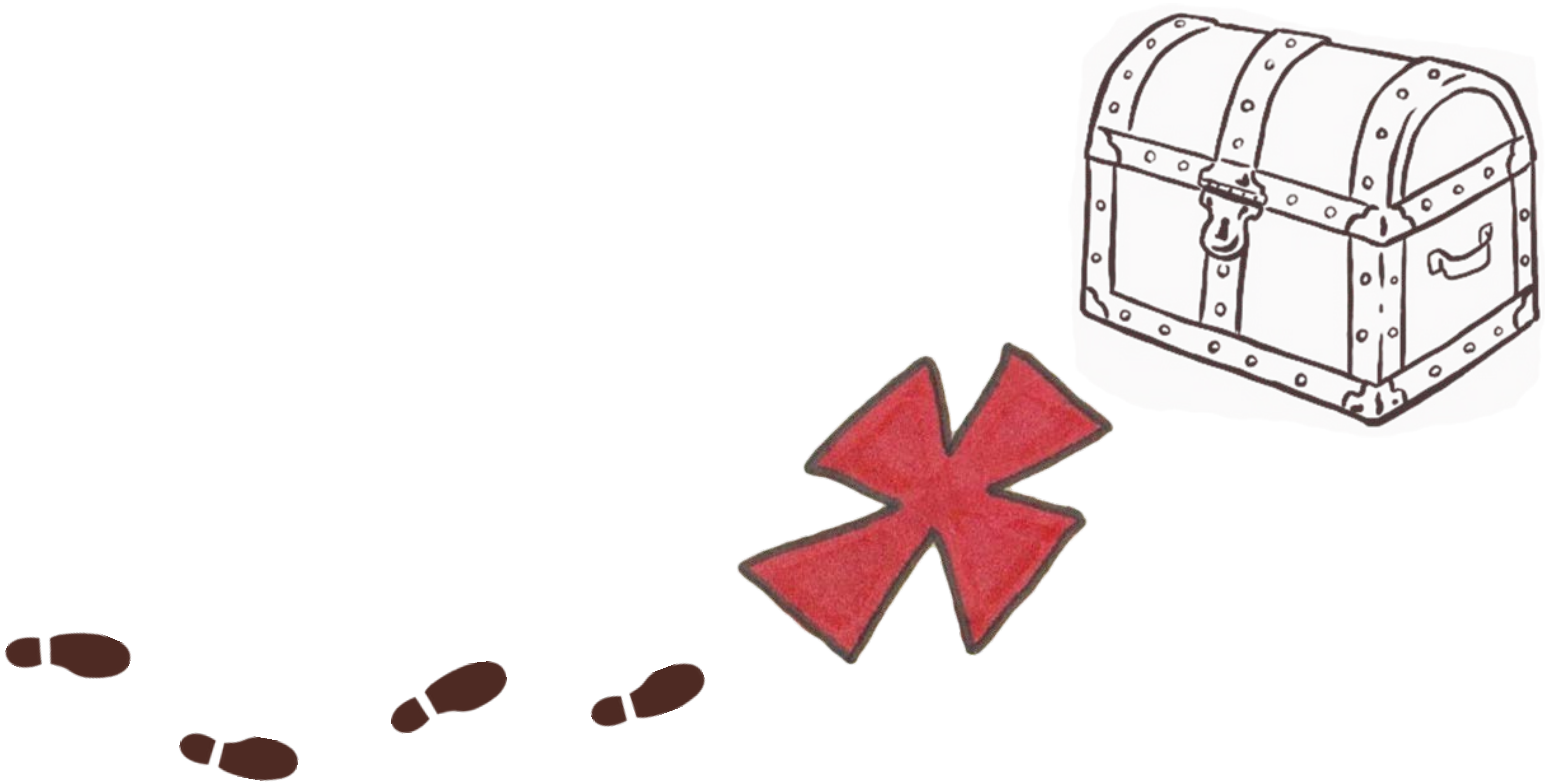
$$\rho \;:=\; \rho\,(\mathsf{D}^{-1}|\mathsf{B}|),$$

$$\mathsf{r} := -\frac{1-\theta}{\xi\,\theta},$$

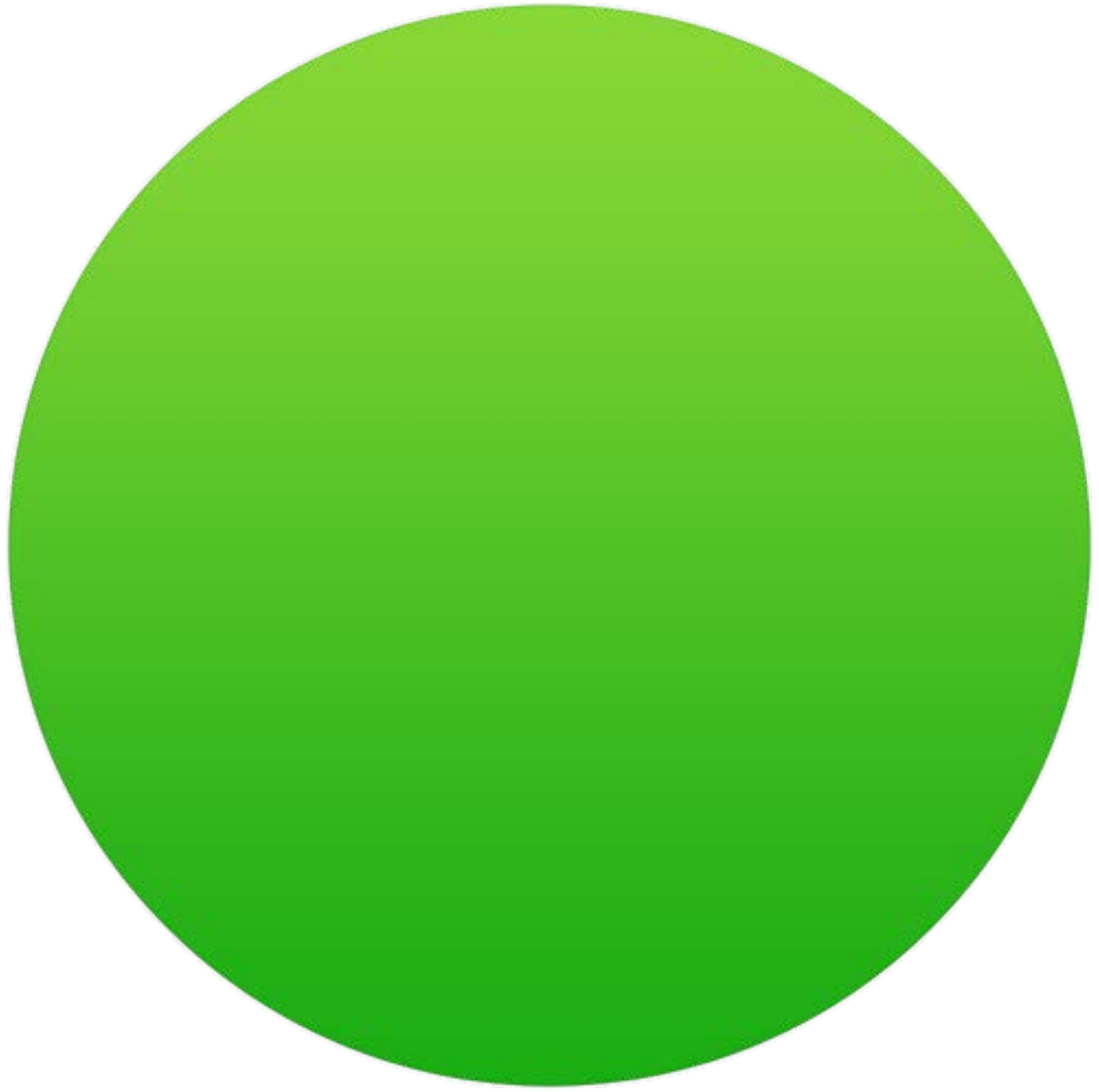
$$\mathsf{s} := \frac{1}{\xi\,\theta\,\rho},$$

$$\mathsf{t} := \frac{1-(1-\theta)\,\rho}{\xi\,\theta\,\rho}.$$





Numerical examples



Matrix catalogue

Selected collection

assumptions

$\ell = 1$ ⋯ single processor $\qquad \nu = 2 \qquad \Omega = D$

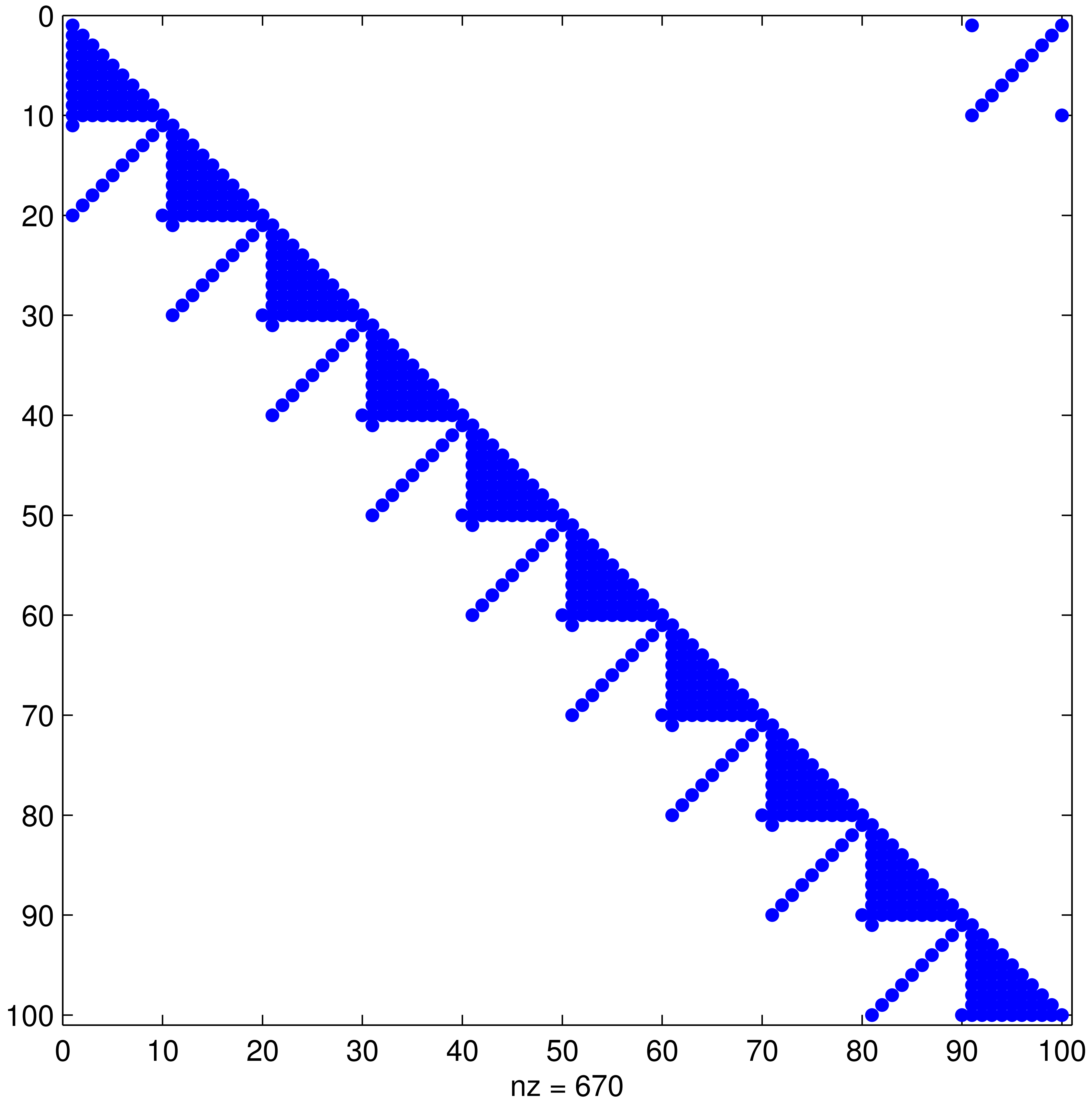
(1st stage) $\quad M = D - \theta B \quad (A = D - B, \quad D = \mathrm{diag}(A))$

(2nd stage) $\quad L^{(M)} = \xi \theta L_A \quad (A = D_A - L_A - U_A)$

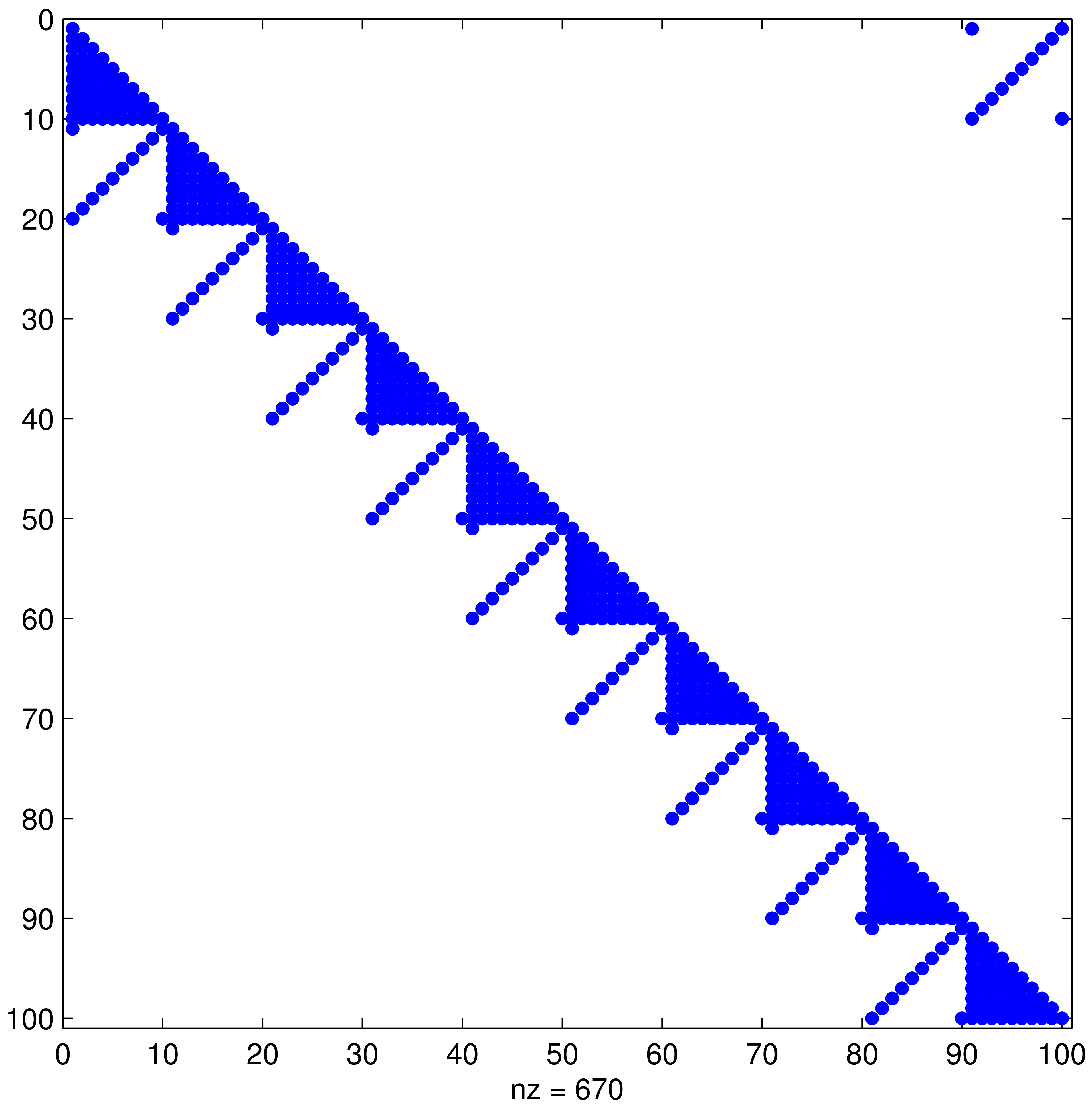
$$\xi_{ij} = \xi, \quad \theta_{ij} = \theta, \quad \forall i, j.$$

aim

plotting level curves of $\rho(\mathcal{L}_{MSTMAOR}(\alpha, \beta))$ in $\alpha - \beta$ plane



A1

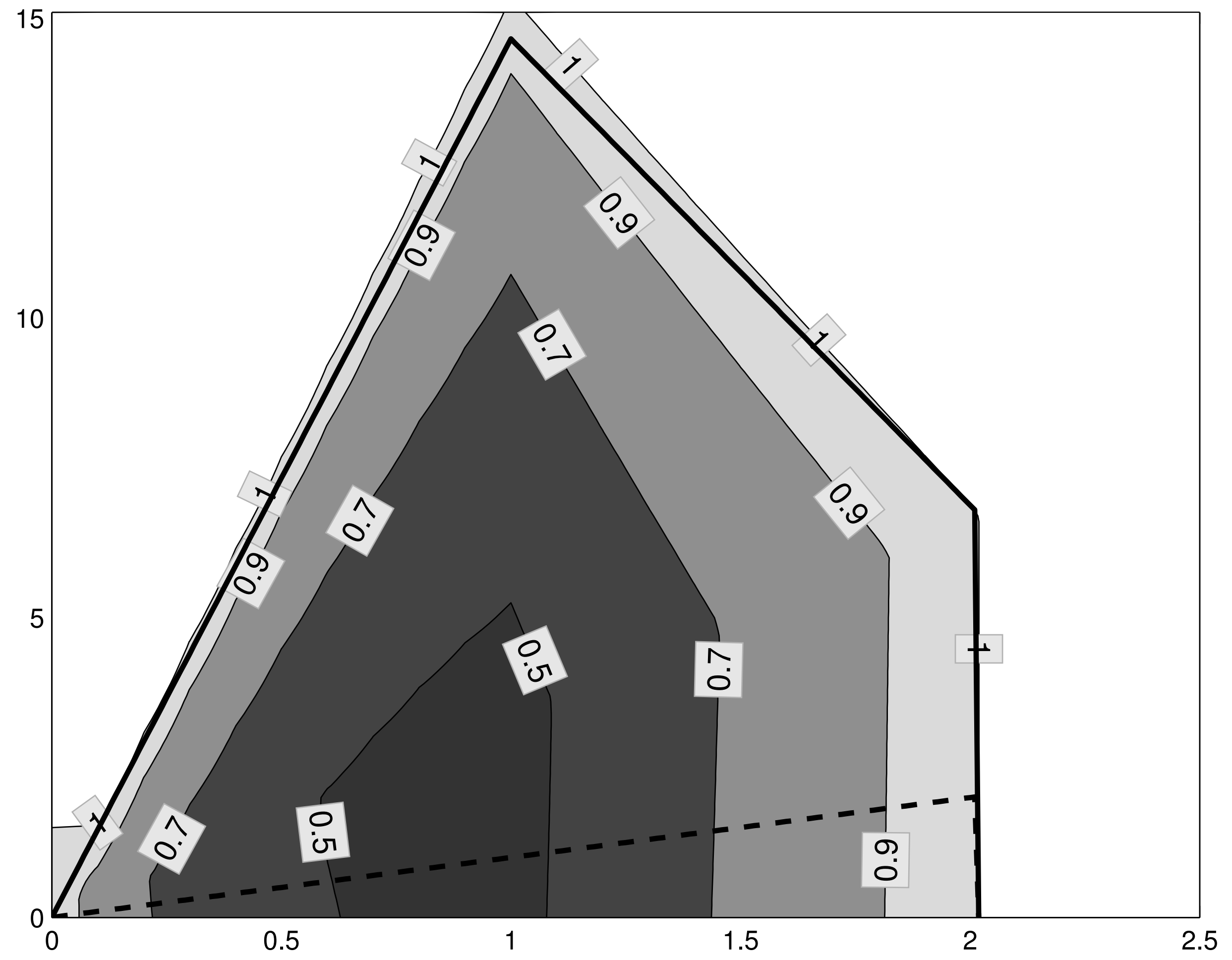


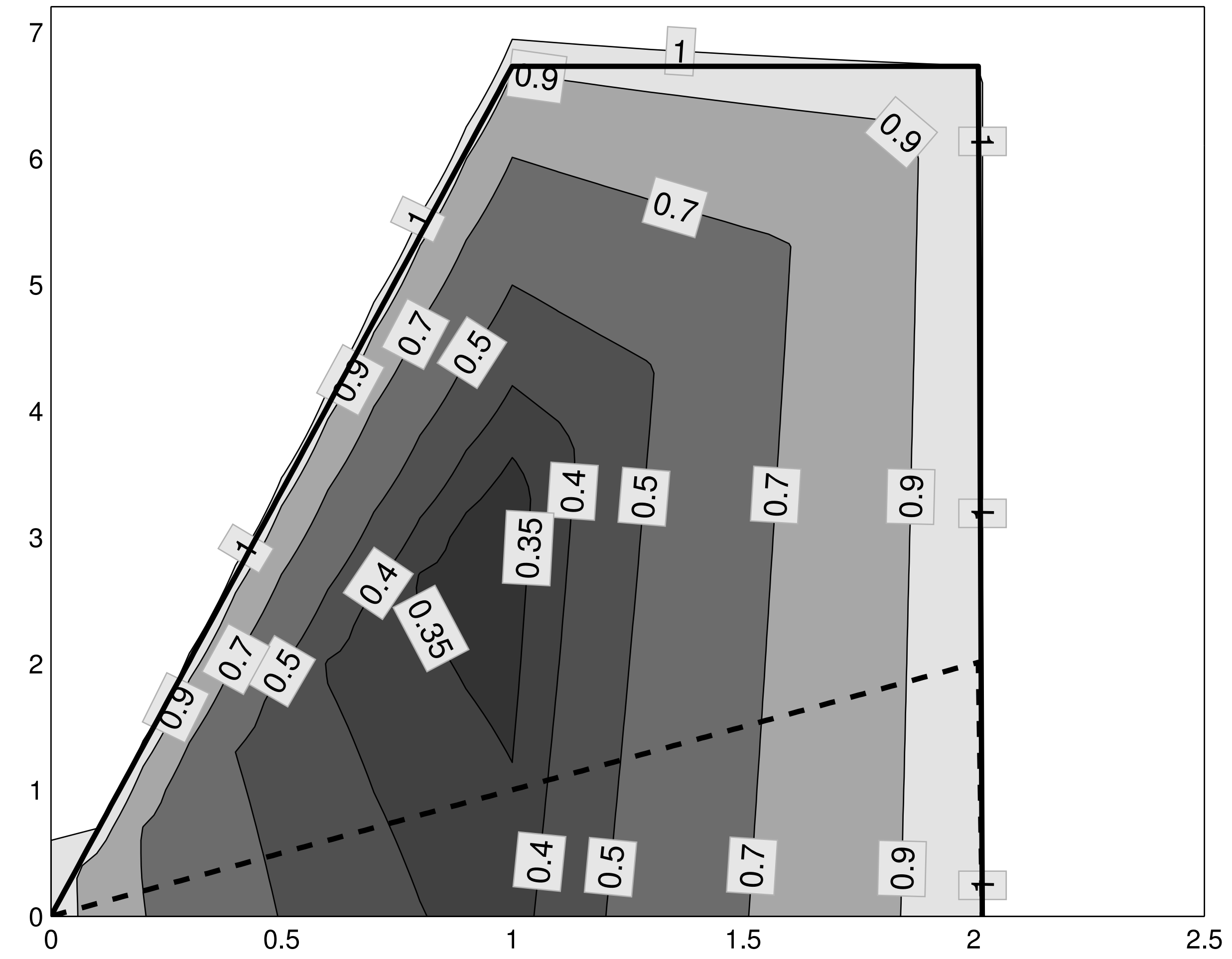
r/c : 100 / 100

nz : 670

MATLAB - coded

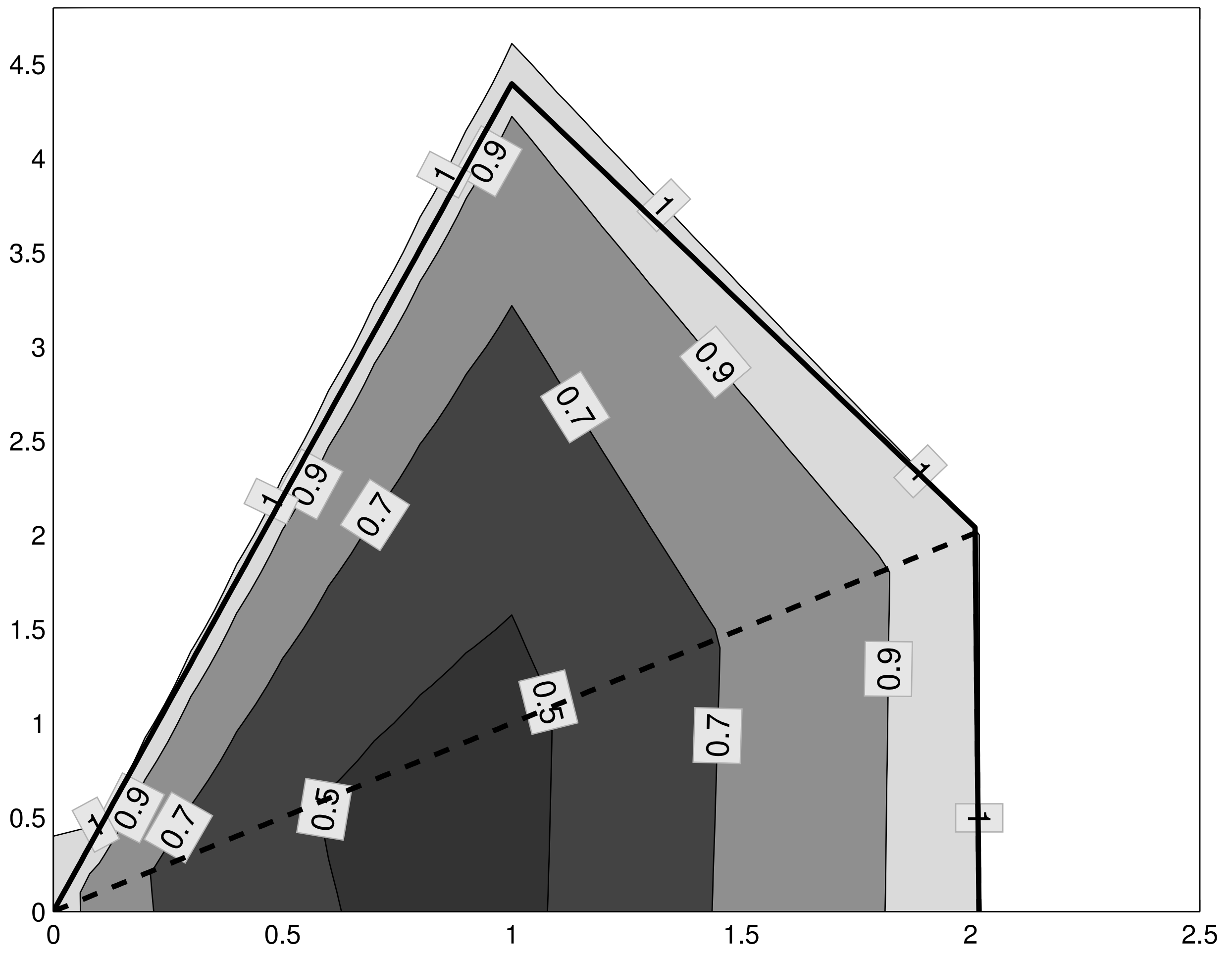
$\xi = 0.3$
$\theta = 0.3$

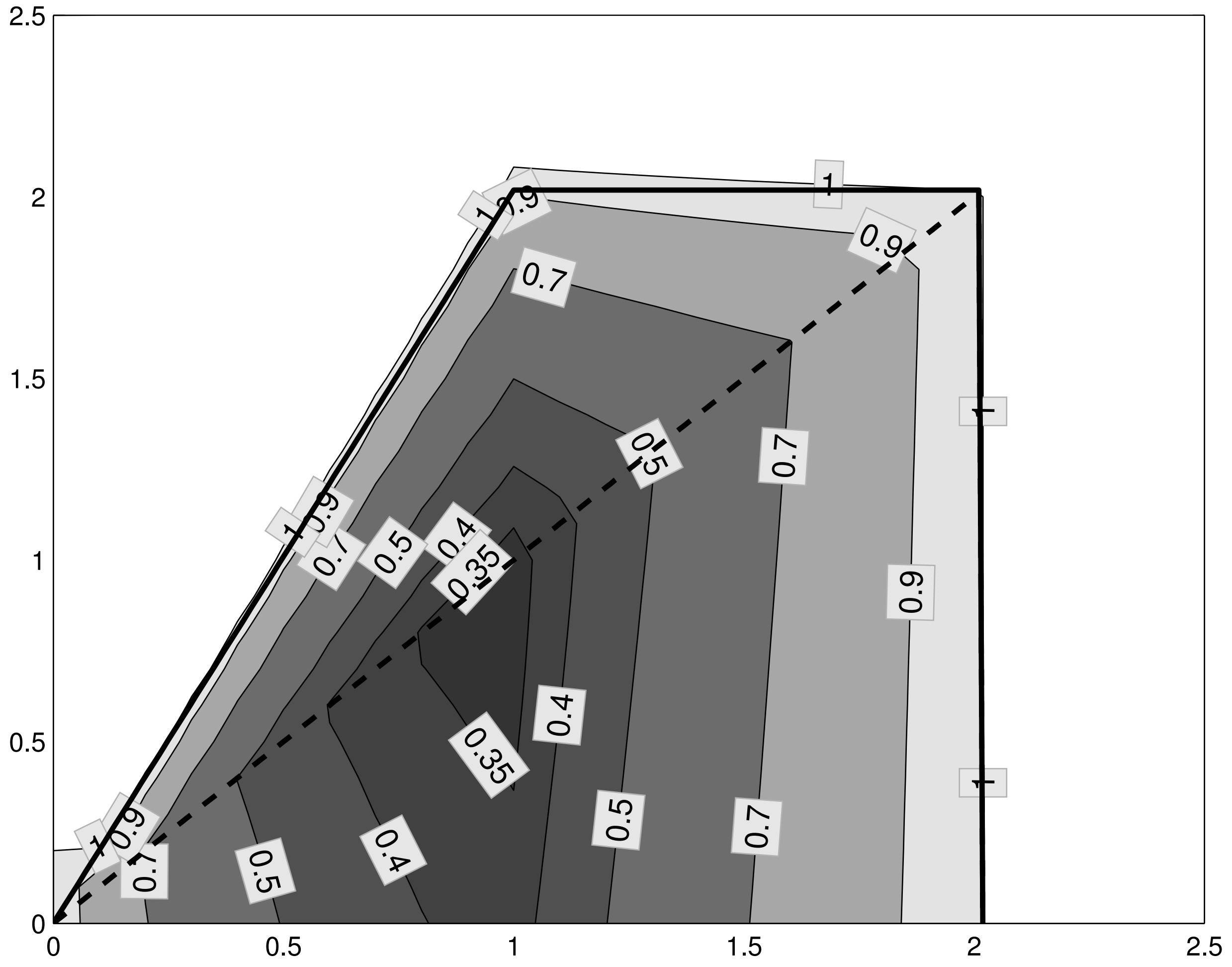




$\xi = 0.3$
$\theta = 1$

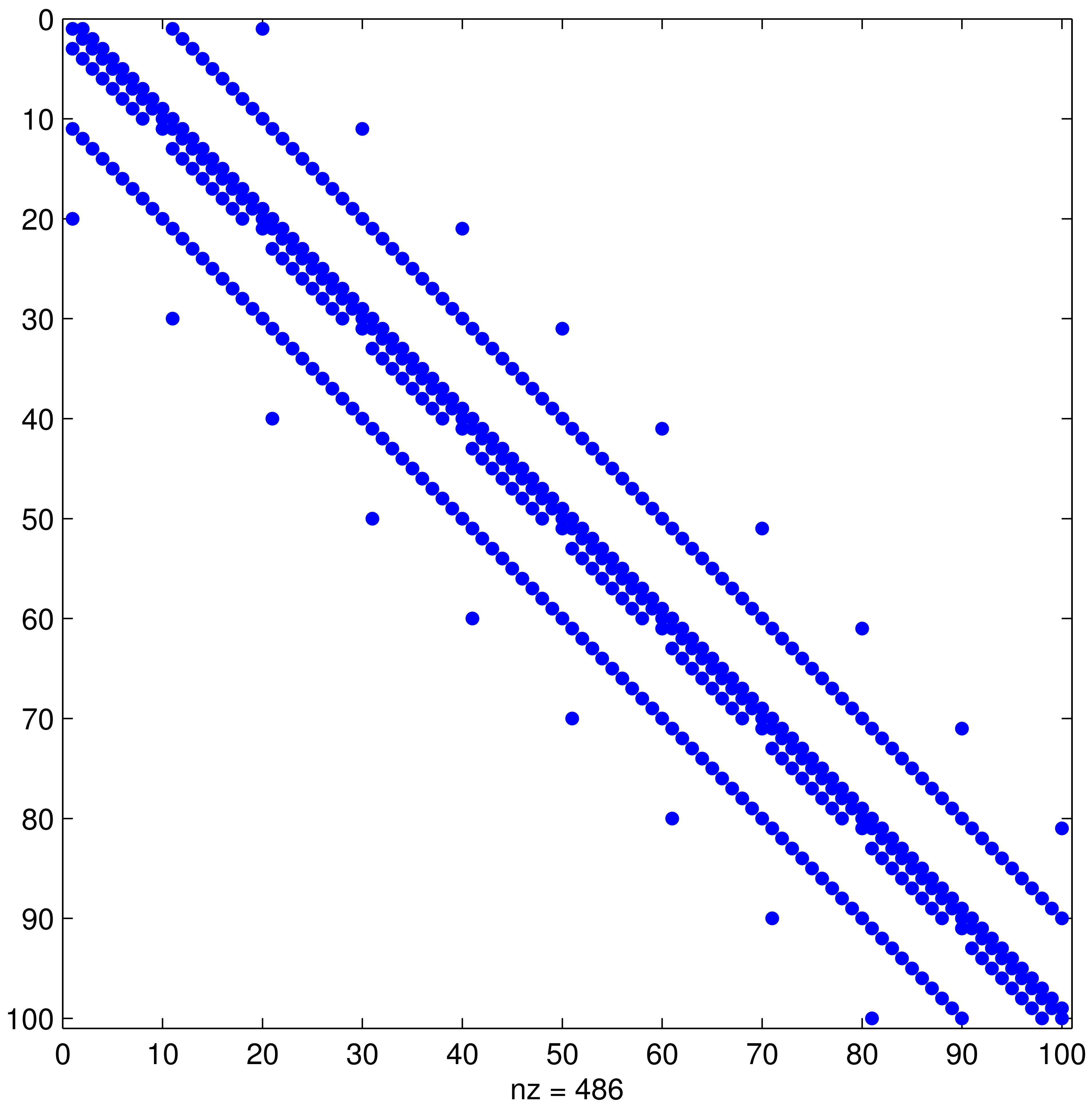
$\xi = 1$
$\theta = 0.3$



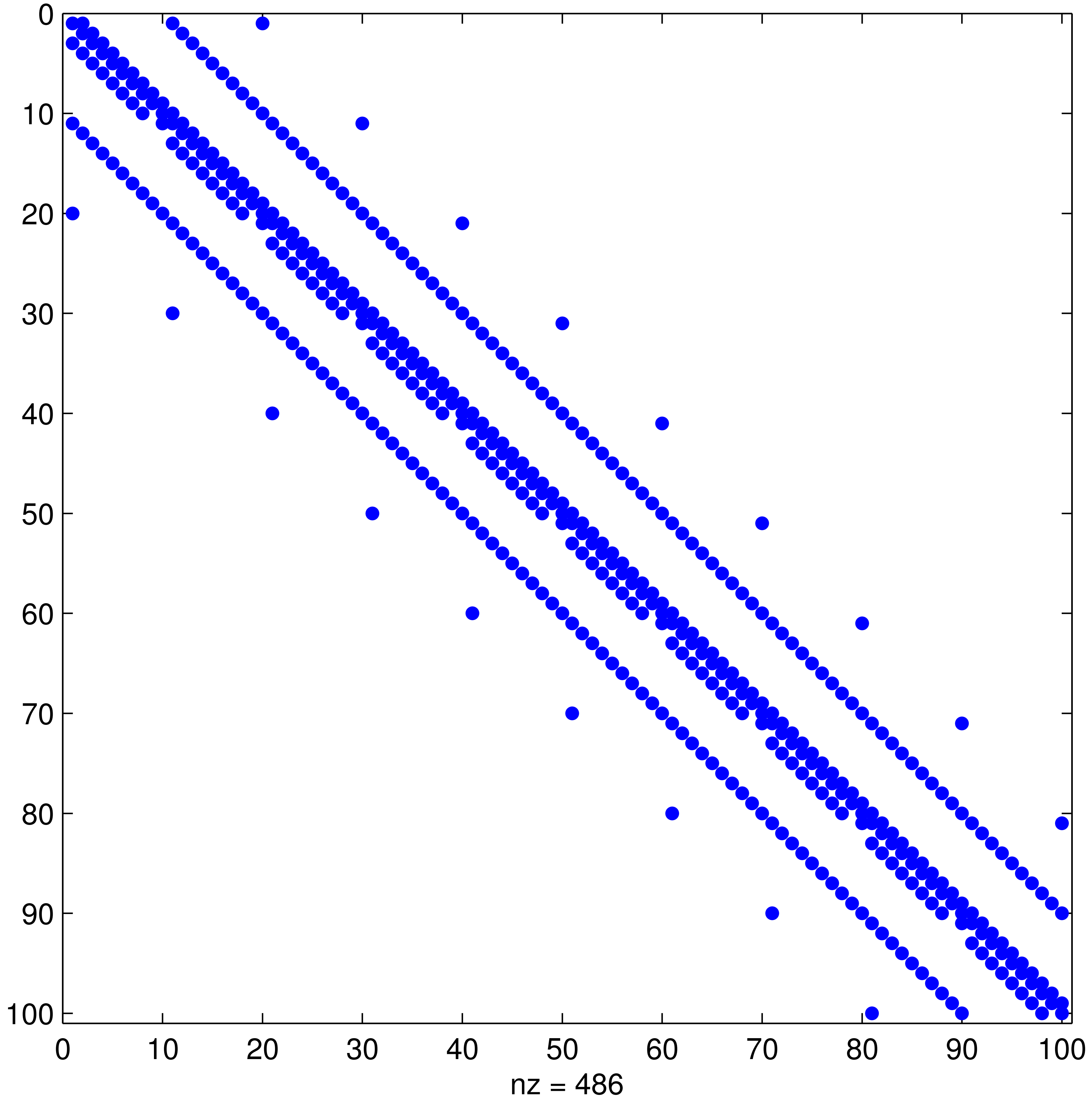


$\xi = 1$
$\theta = 1$

A1	$\xi = 0.3$ $\theta = 0.3$	$\xi = 0.3$ $\theta = 1$	$\xi = 1$ $\theta = 0.3$	$\xi = 1$ $\theta = 1$
BZ_val	0.4564	0.3534	0.4548	0.3292
CKS_val	0.45**4**8	0.3**2**92	0.4548	0.3292



A2

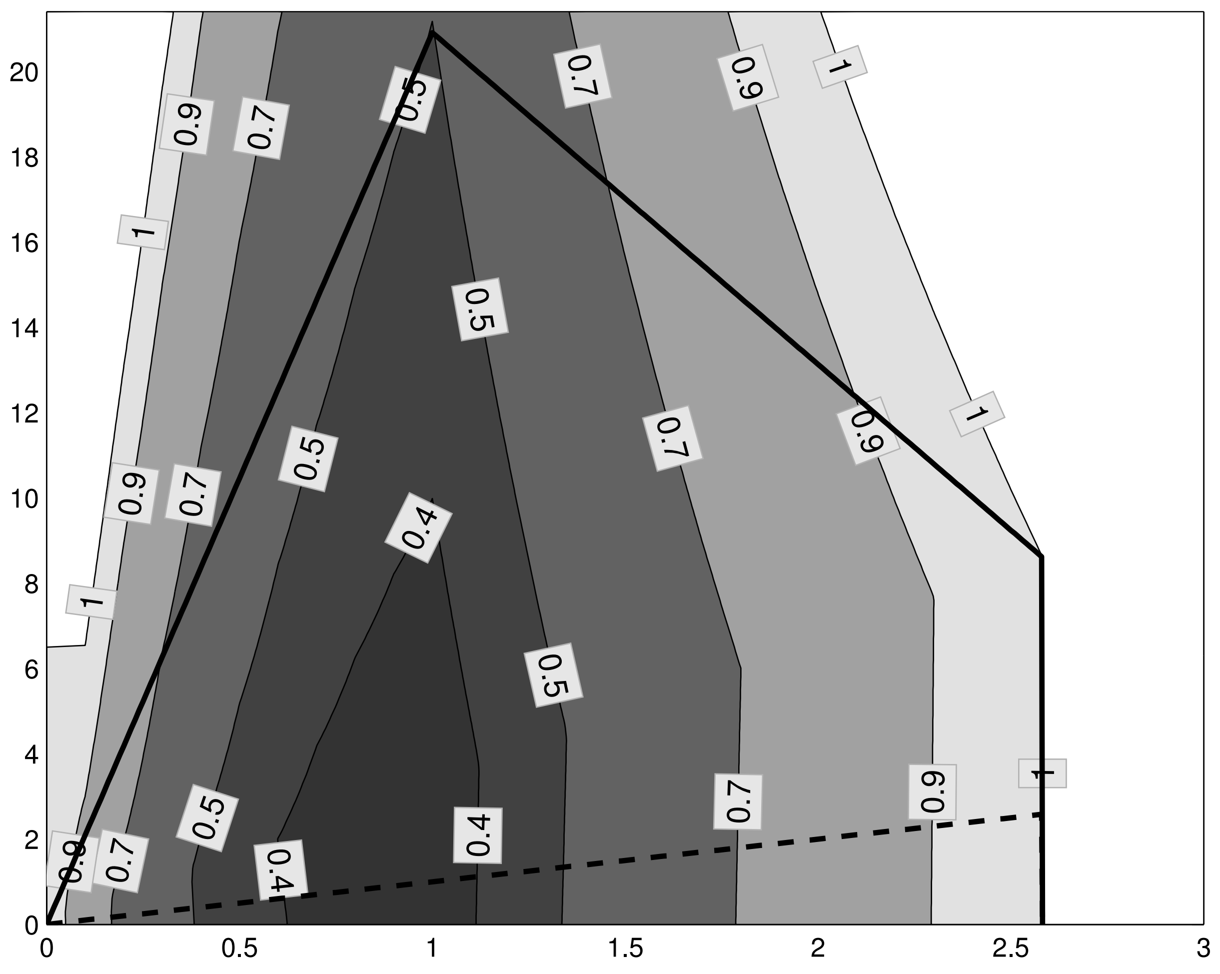


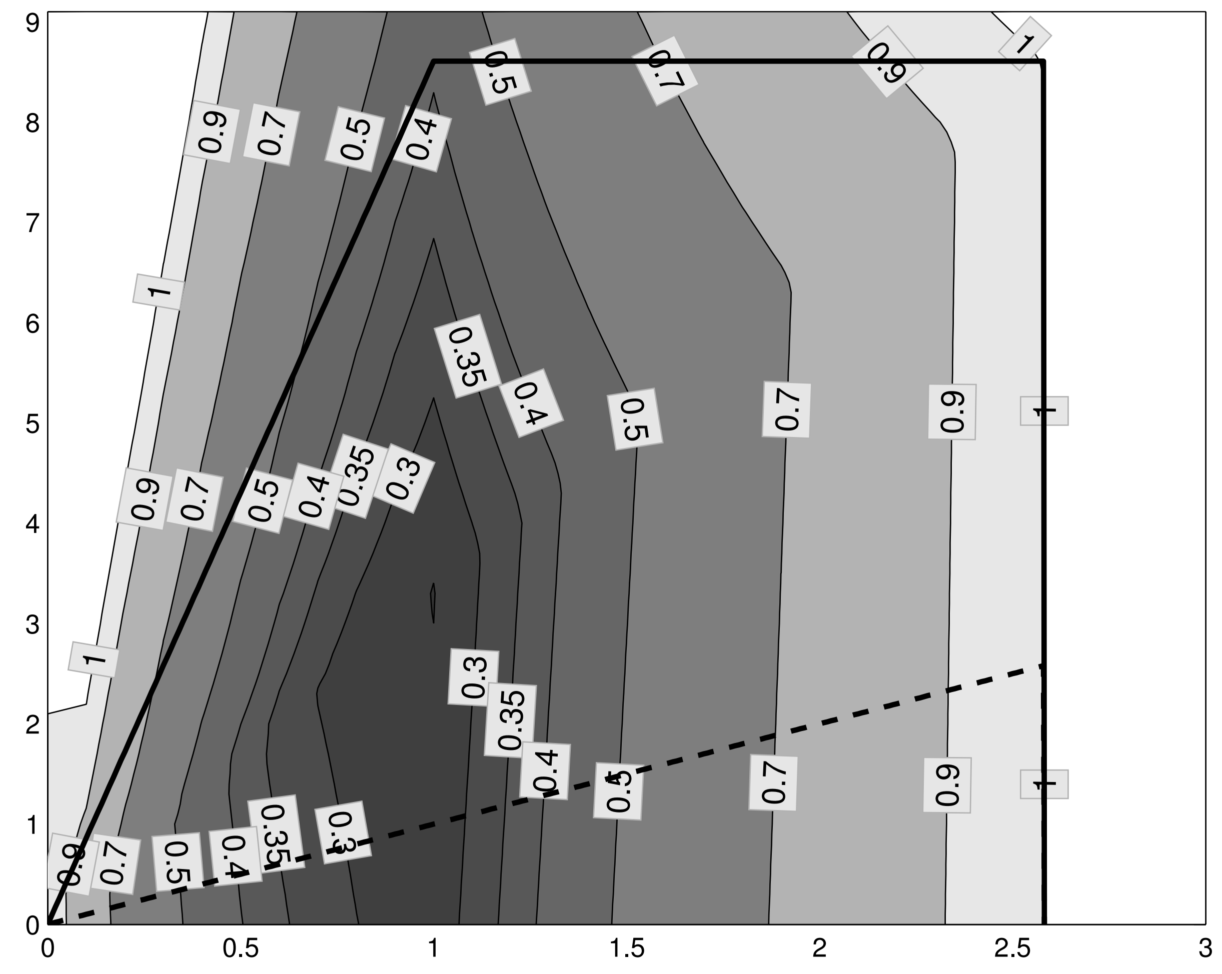
r/c : 100 / 100

nz : 486

MATLAB - coded

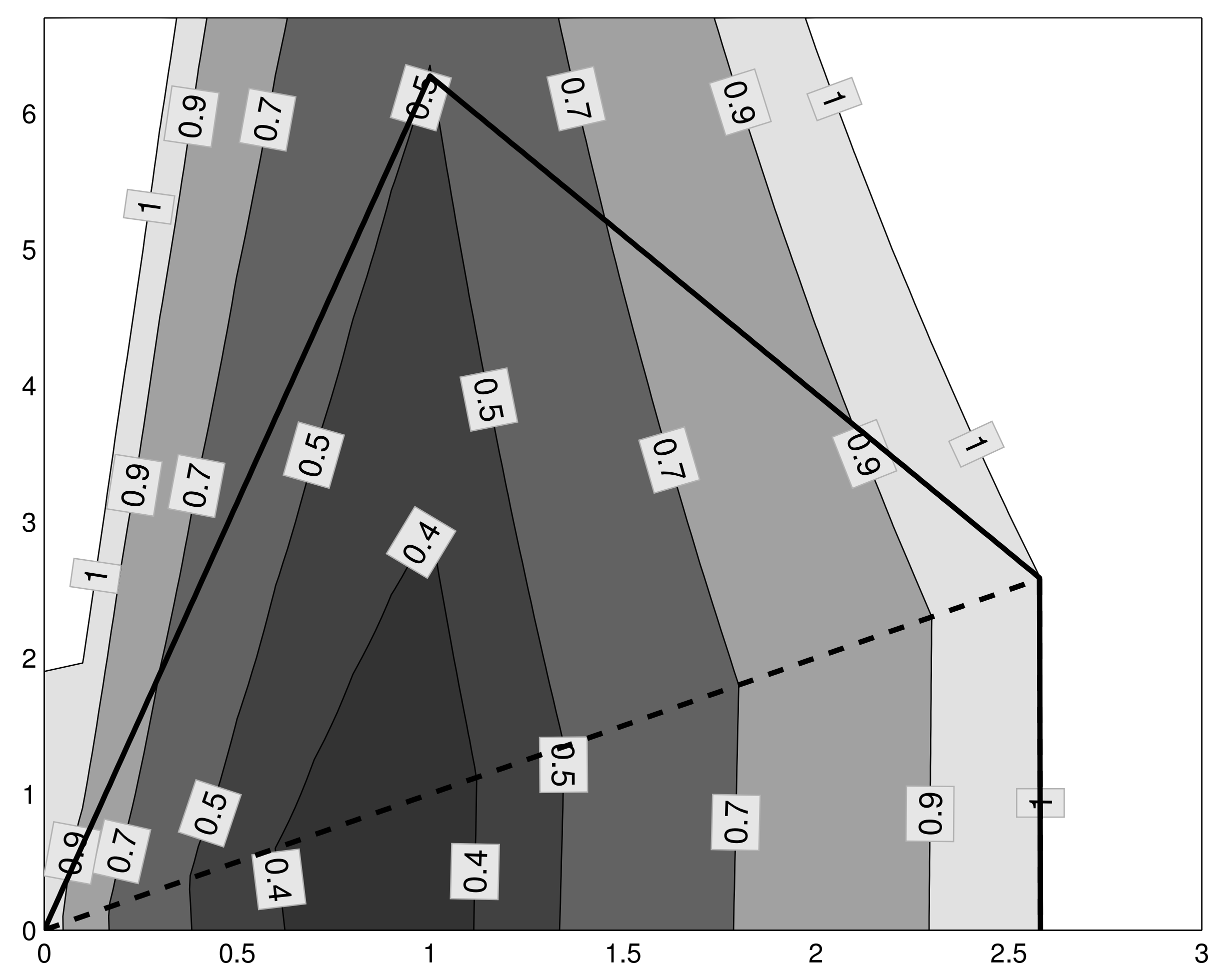
$\xi = 0.3$
$\theta = 0.3$

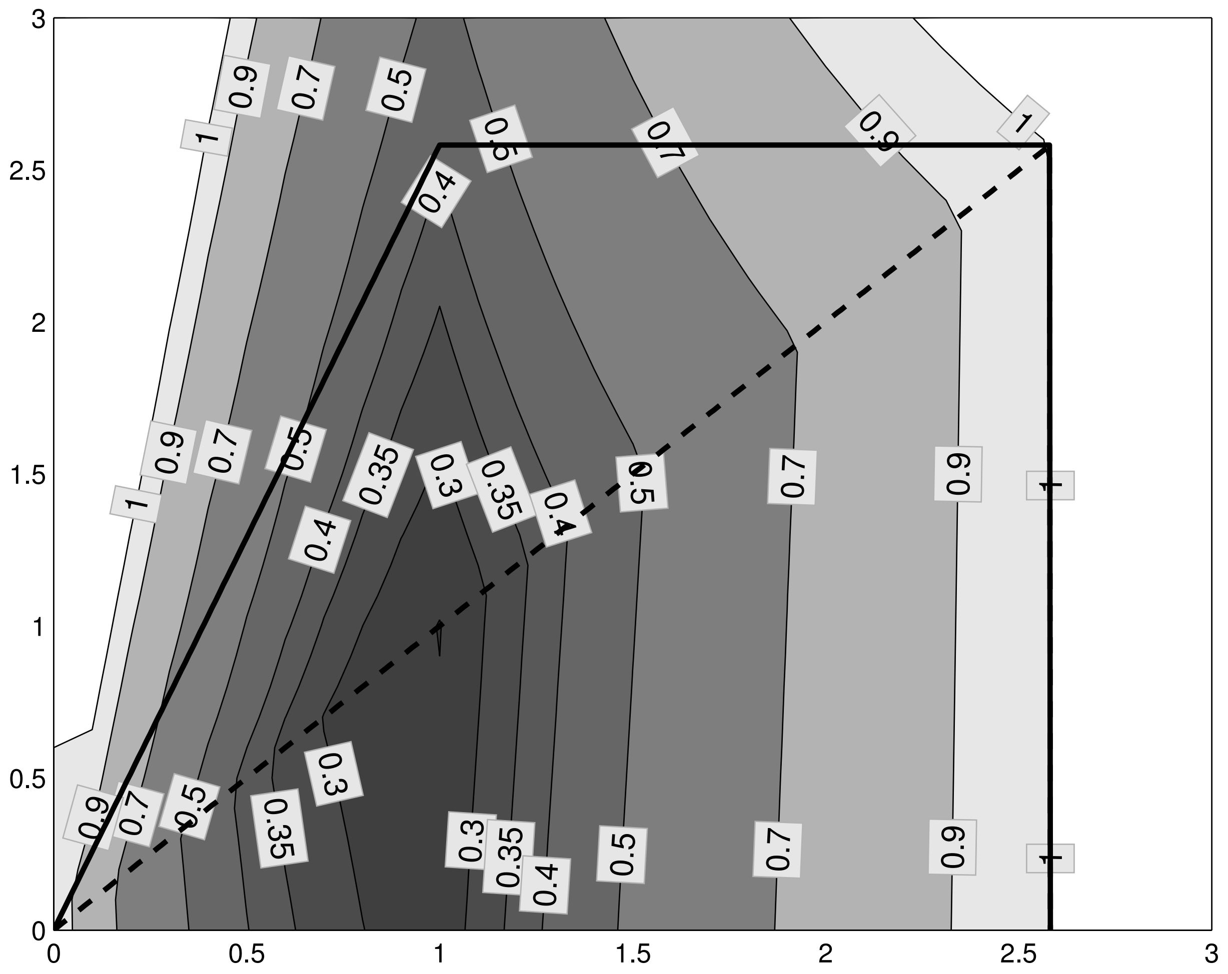




$\xi = 0.3$
$\theta = 1$

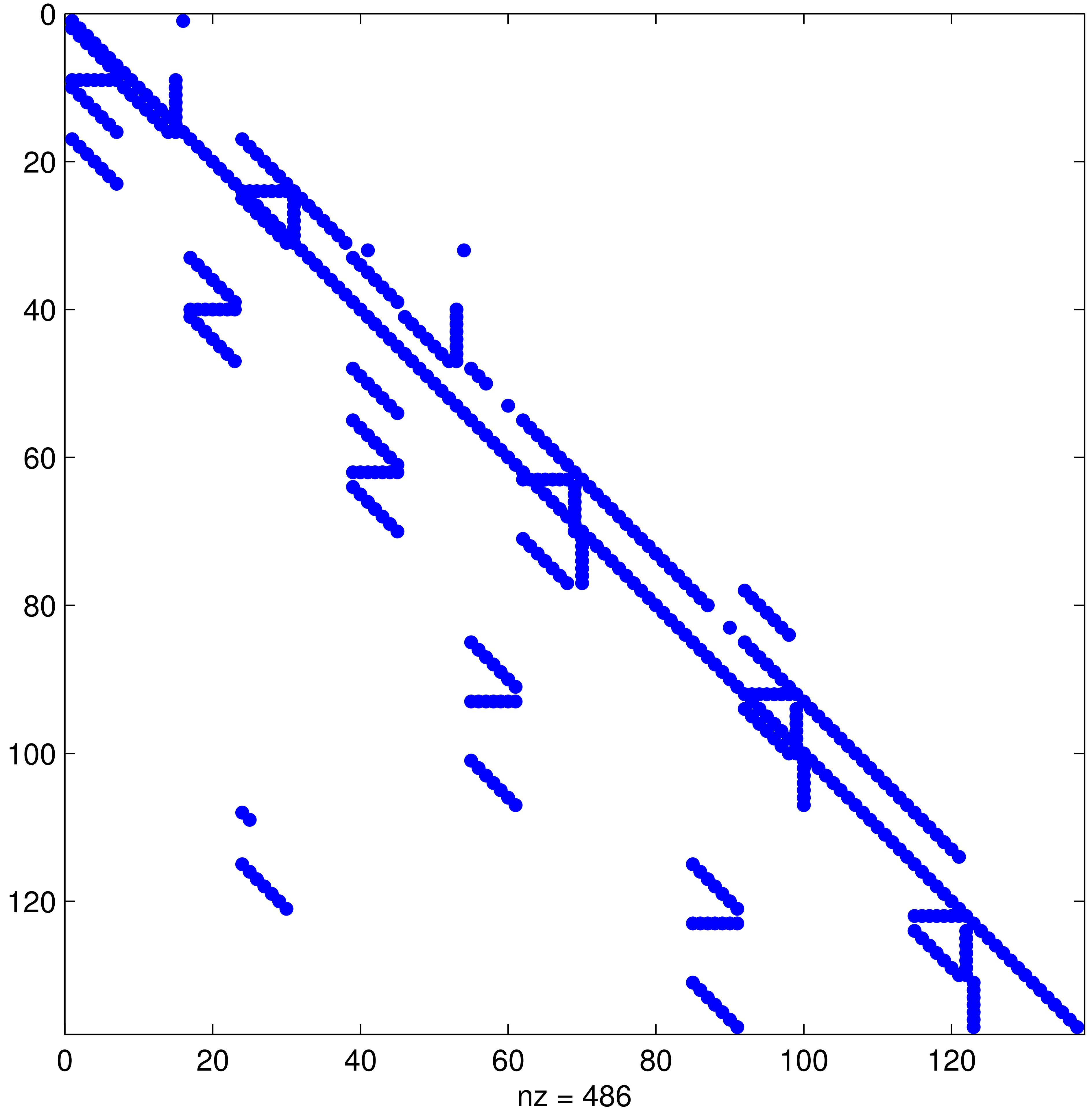
$\xi = 1$
$\theta = 0.3$



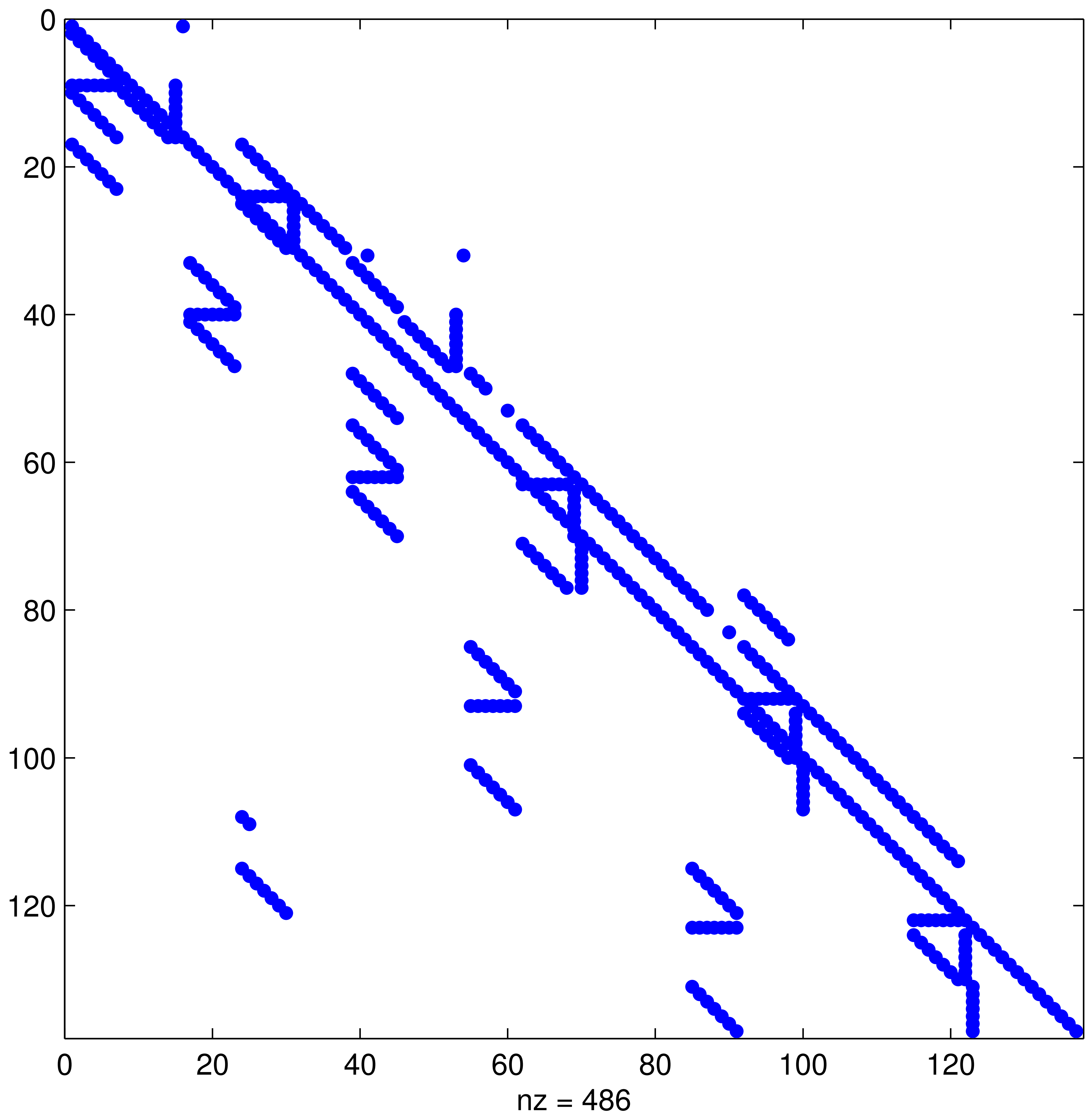


$\xi = 1$
$\theta = 1$

A2	$\xi = 0.3$ $\theta = 0.3$	$\xi = 0.3$ $\theta = 1$	$\xi = 1$ $\theta = 0.3$	$\xi = 1$ $\theta = 1$
BZ_val	0.3511	0.2620	0.3501	0.2483
CKS_val	0.35**0**1	0.2**4**85	0.3501	0.2483


0
20
40
60
80
100
120
0
20
40
60
80
100
120
nz = 486


A3



University of Florida Sparse Matrix Collection

r/c : 137 / 137

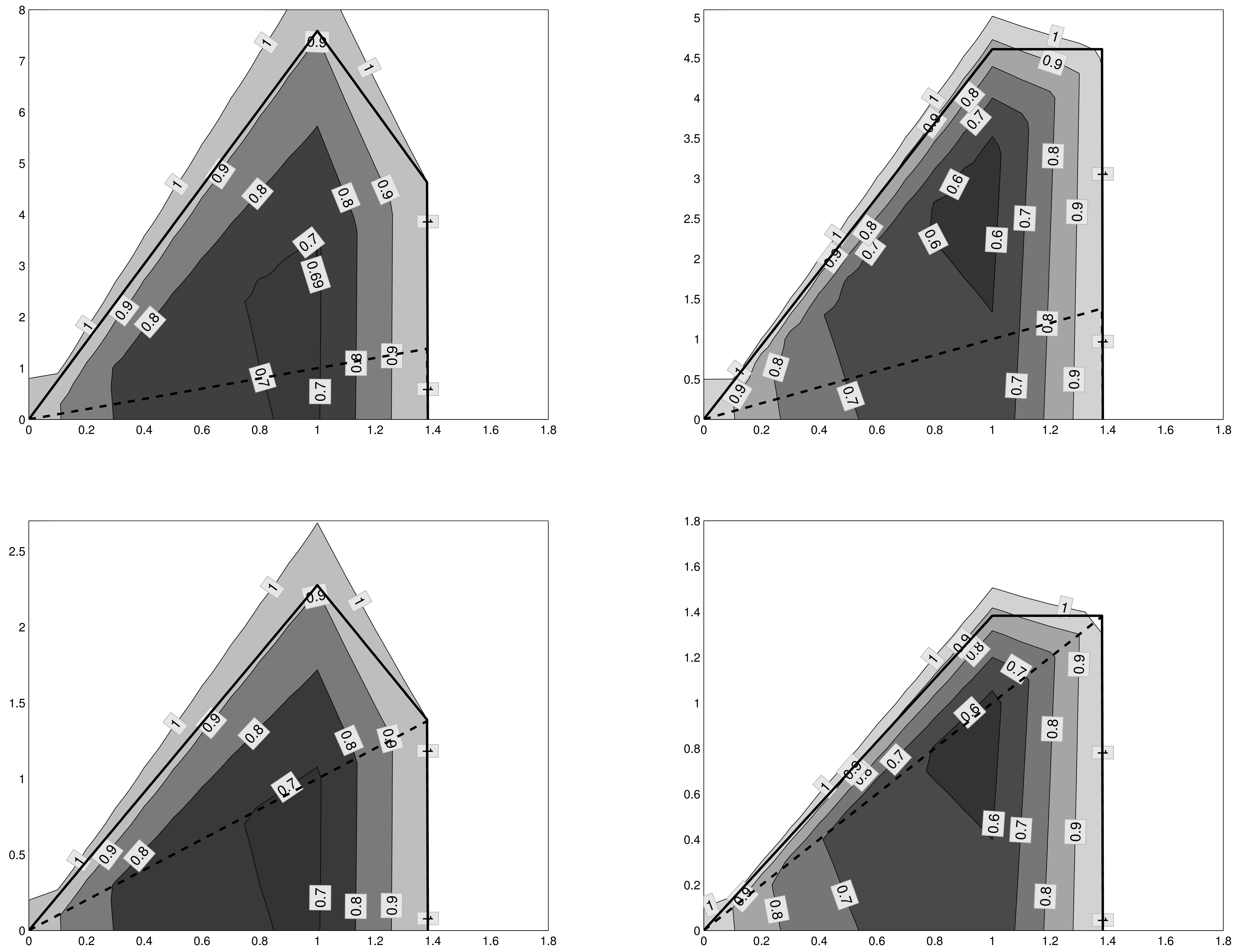
nz : 486

impcol_c (*modified)

chemical process simulation problem

$\xi = 0.3$
$\theta = 0.3$

$\xi = 0.3$
$\theta = 1$



$\xi = 1$
$\theta = 0.3$

$\xi = 1$
$\theta = 1$

A3	$\xi = 0.3$ $\theta = 0.3$	$\xi = 0.3$ $\theta = 1$	$\xi = 1$ $\theta = 0.3$	$\xi = 1$ $\theta = 1$
BZ_val	0.6917	0.6058	0.6895	0.5671
CKS_val	0.6**8**95	0.**5**676	0.6895	0.5671

С п а с и б о!

Obrigado!

Hvala!

Grazie!

Merci!

謝 謝

Thank you!

Gracias!

Danke schön!

Ε υ χ α ρ ι σ τ ώ