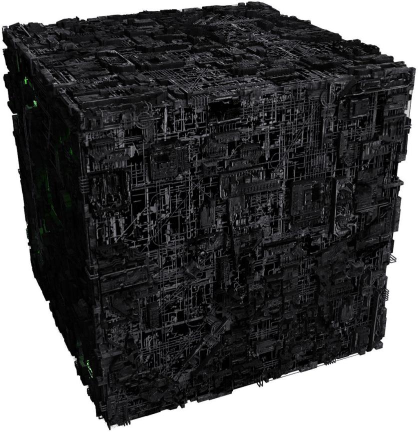
Linear sandwich semigroups

Igor Dolinka

sc:ala seminar, 10 December 2015

The bad news. . .



. . . all groups will be monoids with no identity.

The good news. . .

. . . all sandwiches are vegan, halal, kosher, nut free, gluten free. . .

Joint work of: yours truly...

. . . and James East (FBI)

. . . errr, I mean, James East (Western Sydney University)

Sandwiches?

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Linear sandwich semigroups (Lyapin, 1960; cf Brown 1955)

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Linear sandwich semigroups (Lyapin, 1960; cf Brown 1955)

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- *For* $X, Y \in \mathcal{M}_{mn}$, *define* $X \star Y = XAY$.

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- $\mathcal{M}_{mn}^{A} = (\mathcal{M}_{mn}, \star)$ *is a linear sandwich semigroup.*

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- *For $X, Y \in \mathcal{M}_{mn}$, define $X \star Y = XAY$.*
- *$\mathcal{M}_{mn}^A = (\mathcal{M}_{mn}, \star)$ is a linear sandwich semigroup.*

Example

If $m = n$ and $A = I$, then $\mathcal{M}_{mn}^A = \mathcal{M}_n$ is the full linear monoid.

Plan (non-linear)

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- *Green's relations*

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- *regular elements*

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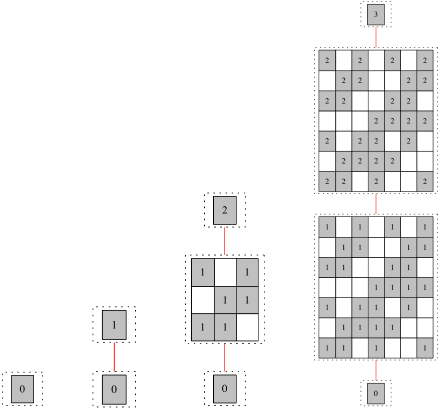
- *small (idempotent) generating sets*
- *bigger sandwiches?*

Egg-box diagrams for $\mathcal{M}_n$ $(F = \mathbb{Z}_2)$



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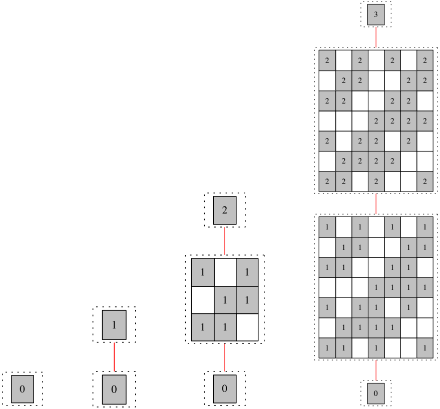
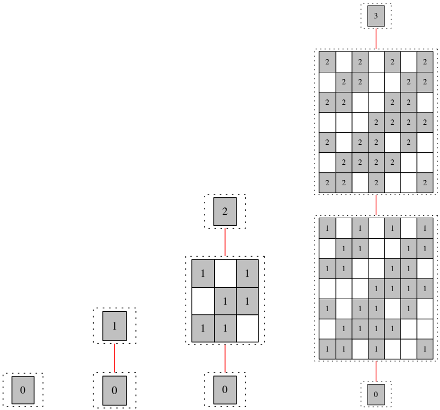


$$\text{Within a} \left\{ \begin{array}{c} \text{box} \\ \text{row} \\ \text{column} \end{array} \right\}, \text{ matrices have same} \left\{ \begin{array}{c} \text{rank} \\ \text{column space} \\ \text{row space} \end{array} \right\}.$$

Egg-box diagrams for $\mathcal{M}_n$ ($F = \mathbb{Z}_2$)



Within a $\left\{\begin{array}{c}\text{box}\\ \text{row}\\ \text{column}\end{array}\right\}$, matrices are $\left\{\begin{array}{c}\mathscr{D}\text{-related}\\ \mathscr{R}\text{-related}\\ \mathscr{L}\text{-related}\end{array}\right\}$.

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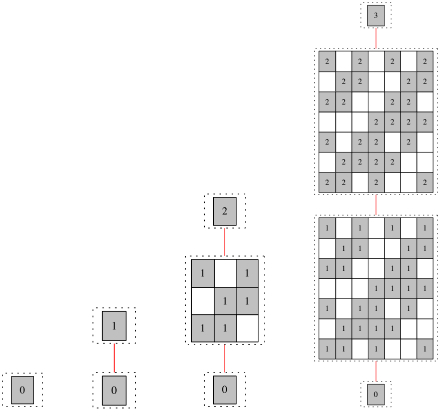
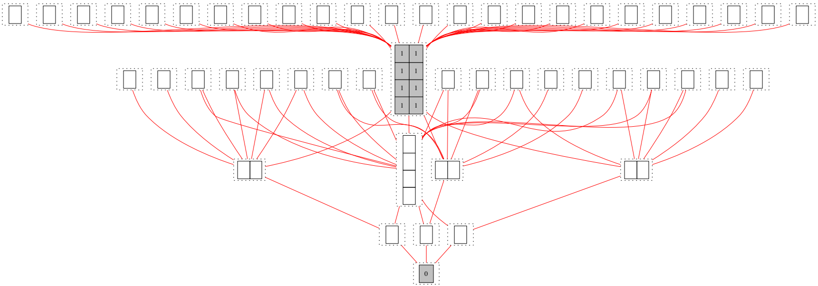
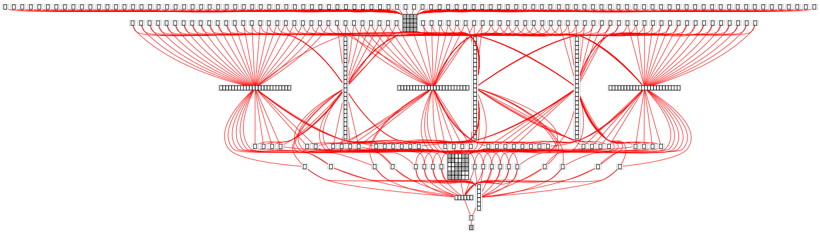


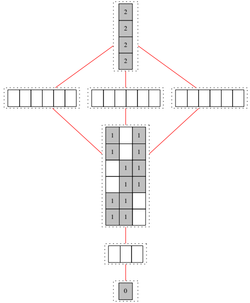
- Cells with an idempotent matrix are shaded.
- These are subgroups of $\mathcal{M}_n(F)$ isomorphic to $\mathrm{GL}(r, F)$.

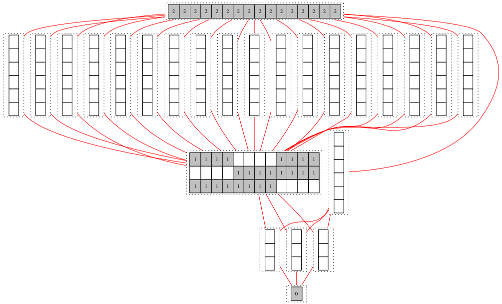
Egg sandwiches — $\mathcal{M}^{A}_{mn}$

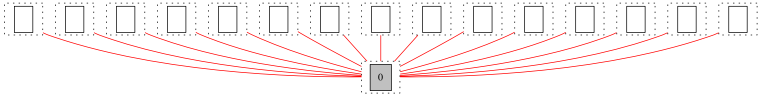




Egg sandwiches — $\mathcal{M}_{mn}^{A}$







Easy lemmas

Lemma

If $A, B \in \mathcal{M}_{nm}$ and $\text{rank}(A) = \text{rank}(B)$, then $\mathcal{M}_{mn}^{A} \cong \mathcal{M}_{mn}^{B}$.

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- So we study $\mathcal{M}_{mn}^{A}$, where

$$J = J_r = \left[\begin{smallmatrix} I_r & O \\ O & O \end{smallmatrix}\right] \in \mathcal{M}_{nm} \qquad (0 \leq r \leq \min(m, n) \text{ is fixed}).$$

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- Write elements of $\mathcal{M}_{mn}$ in 2×2 block form:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad \text{where } A \in \mathcal{M}_{rr},\ \ B \in \mathcal{M}_{r,n-r},\ \ \text{etc.}$$

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Lemma

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \star \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE & AF \\ CE & CF \end{bmatrix}.$$

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Proposition

- $R_X = \{Y \in \mathcal{M}_{mn} : \mathrm{Col}(X) = \mathrm{Col}(Y)\}$,
- $L_X = \{Y \in \mathcal{M}_{mn} : \mathrm{Row}(X) = \mathrm{Row}(Y)\}$,
- $J_X = D_X = \{Y \in \mathcal{M}_{mn} : \mathrm{rank}(X) = \mathrm{rank}(Y)\}$

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Let $X, Y \in \mathcal{M}_{mn}$. Write

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$$= \mathrm{Reg}(\mathcal{M}_{mn}^A) \leq \mathcal{M}_{mn}^A.$$

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Proposition

For $X \in \mathcal{M}_{mn}$,

- $R_X^J = \begin{cases} R_X \cap P_1 & \text{if } X \in P_1 \\ \{X\} & \text{if } X \in \mathcal{M}_{mn} \setminus P_1, \end{cases}$

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- $H_X^J = \begin{cases} H_X & \text{if } X \in P \\ \{X\} & \text{if } X \in \mathcal{M}_{mn} \setminus P, \end{cases}$

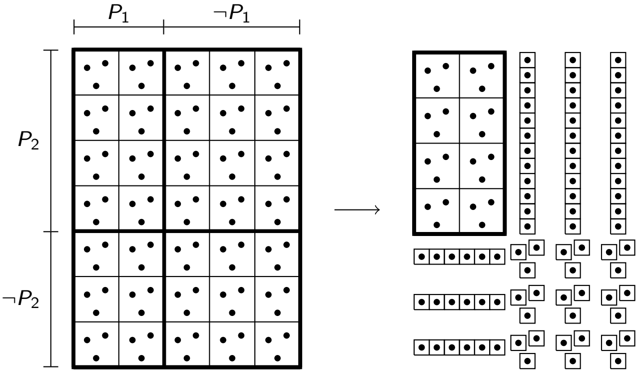
Green's relations

Proposition (continued)

For $X \in \mathcal{M}_{mn}$,

- $$D_X^J = \begin{cases} D_X \cap P & \text{if } X \in P \\ L_X \cap P_2 & \text{if } X \in P_2 \setminus P_1 \\ R_X \cap P_1 & \text{if } X \in P_1 \setminus P_2 \\ \{X\} & \text{if } X \in \mathcal{M}_{mn} \setminus (P_1 \cup P_2). \end{cases}$$

High energy semigroup theory — from $\mathcal{M}_{mn}$ to $\mathcal{M}_{mn}^{A}$



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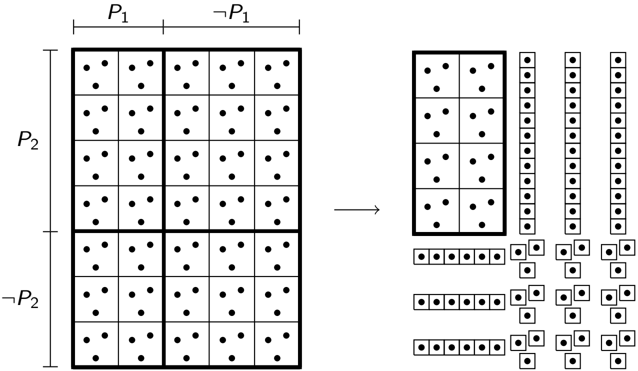

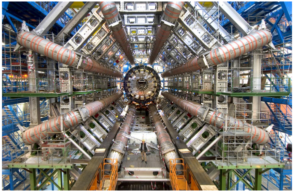

Small generating sets

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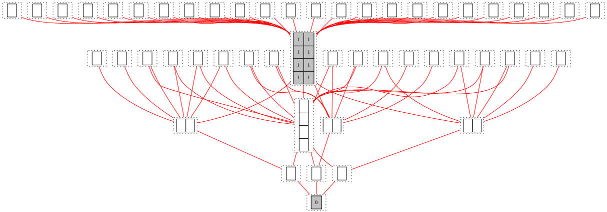
Theorem

Suppose $r = \text{rank}(A) < \min(m, n)$.



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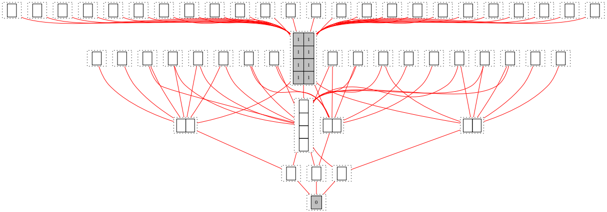

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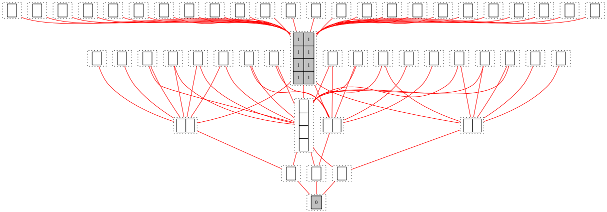

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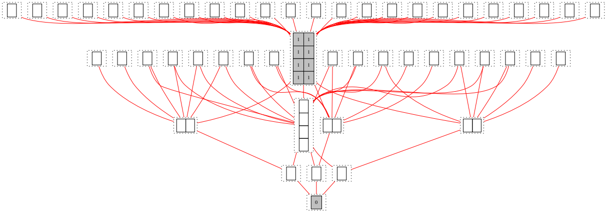
- The $\mathscr{D}$-maximal elements generate $\mathcal{M}_{mn}^A$.
- Any generating set must contain all $\mathscr{D}$-maximal elements.
- $\text{rank}(\mathcal{M}_{mn}^A) = \displaystyle\sum_{s=r+1}^{\min(m,n)} \left[{m \atop s} \right]_q \left[{n \atop s} \right]_q q^{\binom{s}{2}} (q-1)^s [s]_q!.$



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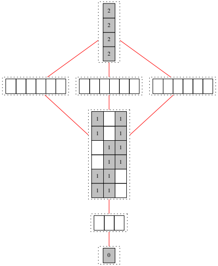

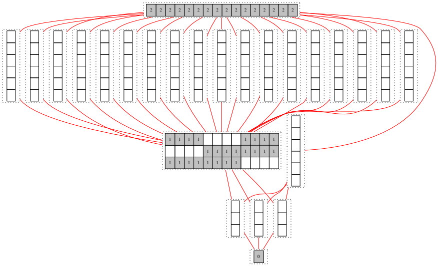

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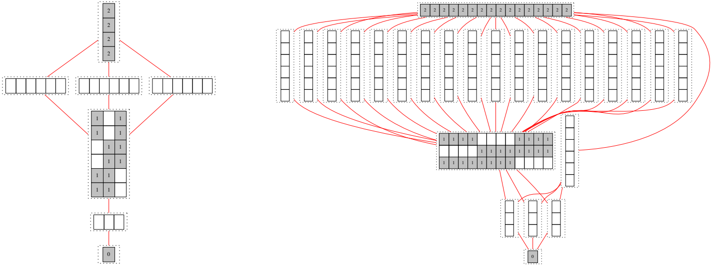

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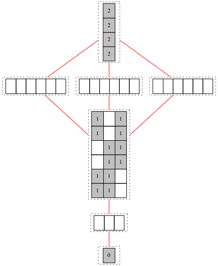
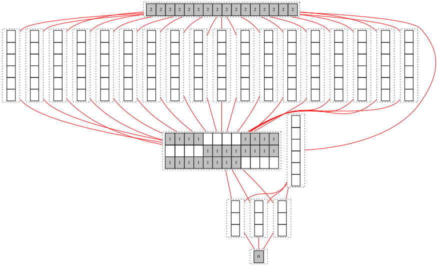
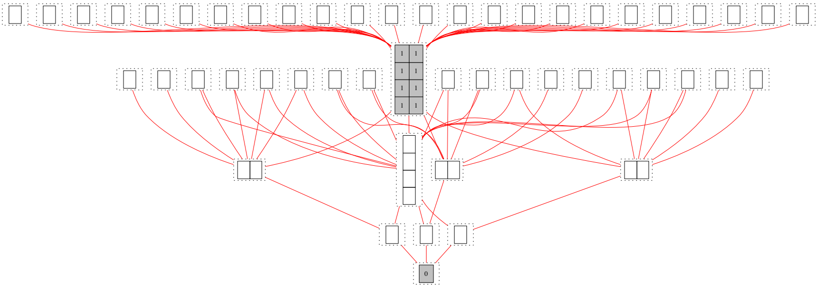
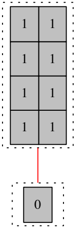


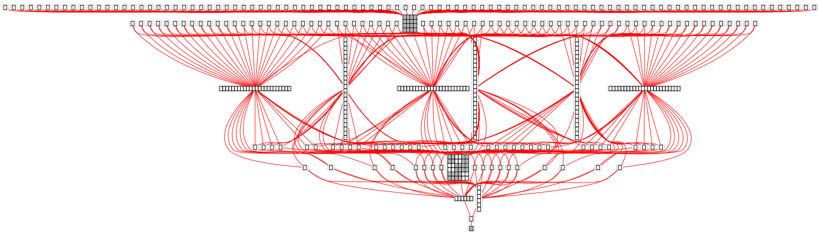


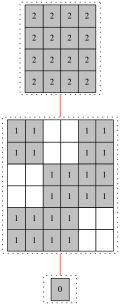
Unscrambling the egg — regular elements of $\mathcal{M}_{32}^{A_1}$



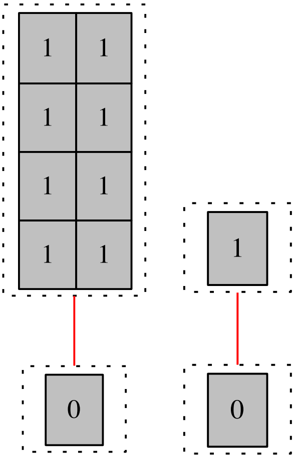


Unscrambling the egg — regular elements of $\mathcal{M}_{33}^{A_2}$

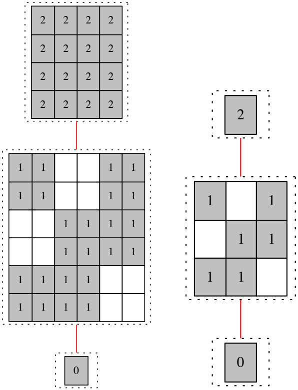




Look familiar?

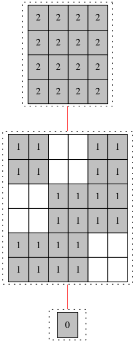


$\mathrm{Reg}(\mathcal{M}_{32}^{A_1})$ $\mathcal{M}_1$



$\mathrm{Reg}(\mathcal{M}_{33}^{A_2})$ $\mathcal{M}_2$

Look familiar?



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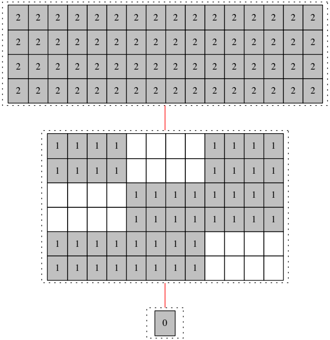

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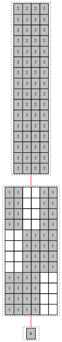
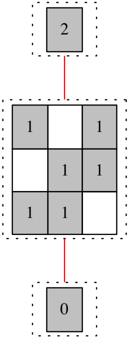


$\mathrm{Reg}(\mathcal{M}_{43}^{A_2})$



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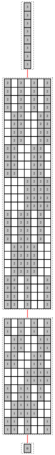

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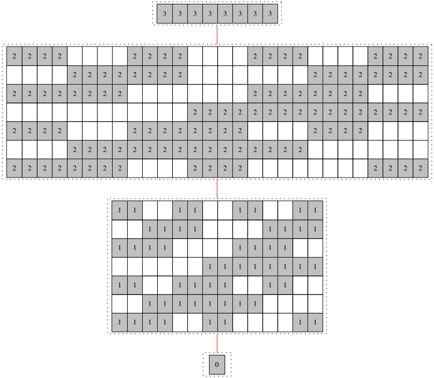

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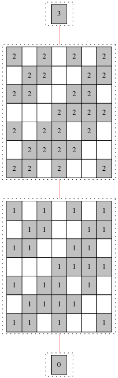

$\mathcal{M}_3$

Big bang — structure of $\mathrm{Reg}(\mathcal{M}_{mn}^A)$

Theorem

- $P = \mathrm{Reg}(\mathcal{M}_{mn}^A) \leq \mathcal{M}_{mn}^A$ is an "inflated $\mathcal{M}_r$" ($r = \mathrm{rank}(A)$).

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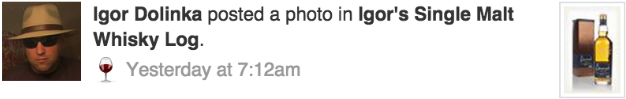

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- $\mathcal{M}_{mn}^{A}$ arises when $\mathscr{C}$ is a linear category.
- Work is under way for diagram categories and more...

Thank you!

- *Variants of finite full transformation semigroups*
  - Dolinka and East — `http://arxiv.org/abs/1410.5253`
- *Semigroups of rectangular matrices under a sandwich operation*
  - Dolinka and East — `http://arxiv.org/abs/1503.03139`