FIXED POINTS IN $b$-METRIC SPACES VIA
SIMULATION FUNCTION

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Abstract. We introduce the concept of generalized $\alpha$-$\eta$-$Z$-contraction mapping with respect to a simulation function $\zeta$ in $b$-metric spaces and study the existence of fixed points for such mappings in complete $b$-metric spaces. Further, we extend it to partially ordered complete $b$-metric spaces. We provide examples in support of our results. Our results extend the fixed point results of Olgun, Bicer and Alyildiz [15].

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1. Introduction

The famous Banach contraction principle introduced by Banach [3], ensures the existence and uniqueness of fixed points for a contraction mapping in complete metric spaces. Several researchers generalized and extended this principle by introducing various contractions in different ambient spaces. (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]).

In 1993, Stefan Czerwik [9] introduced the concept of a $b$-metric space as a generalization of a metric space.

Definition 1.1. [9] Let $X$ be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a $b$-metric if the following conditions are satisfied:

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) there exists $s \geq 1$ such that $d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

In this case, the pair $(X, d)$ is called a $b$-metric space with coefficient $s$.

Definition 1.2. [9] Let $(X, d)$ be a $b$-metric space.

(i) A sequence $\{x_n\}$ in $X$ is called $b$-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n \to \infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in $X$ is called $b$-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

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Theorem 1.3. Let \((X, d)\) be a complete \(b\)-metric space with coefficient \(s = 2\). Let \(T : X \to X\) satisfy
\[
d(Tx, Ty) \leq \varphi(d(x, y))\]
for all \(x, y \in X\), where \(\varphi : [0, \infty) \to [0, \infty)\) is an increasing function such that \(\lim_{n \to \infty} \varphi^n(t) = 0\) for all \(t > 0\). Then \(T\) has exactly one fixed point \(u\) in \(X\) and \(\lim_{n \to \infty} d(T^n(x), u) = 0\) for all \(x \in X\).

Babu and Sailaja proved the following lemma which plays an important role in proving the Cauchy part of an iterative sequence in metric spaces.

Lemma 1.4. Suppose \((X, d)\) is a metric space. Let \(\{x_n\}\) be a sequence in \(X\) such that \(d(x_n, x_{n+1}) \to 0\) as \(n \to \infty\). If \(\{x_n\}\) is not a Cauchy sequence then there exists an \(\epsilon > 0\) and sequences of positive integers \(\{m_k\}\) and \(\{n_k\}\) with \(n_k > m_k \geq k\) such that \(d(x_{m_k}, x_{n_k}) \geq \epsilon\). For each \(k > 0\), corresponding to \(m_k\), we can choose \(n_k\) to be the smallest positive integer such that
\[
d(x_{m_k}, x_{n_k}) > \epsilon, d(x_{m_k}, x_{n_k-1}) < \epsilon\]
and
\[
\begin{align*}
  (i) & \lim_{k \to \infty} d(x_{n_k-1}, x_{m_{k+1}}) = \epsilon \\
  (ii) & \lim_{k \to \infty} d(x_{n_k}, x_{m_k}) = \epsilon \\
  (iii) & \lim_{k \to \infty} d(x_{m_{k-1}}, x_{n_k}) = \epsilon \quad \text{and} \quad (iv) \lim_{k \to \infty} d(x_{n_k}, x_{m_{k+1}}) = \epsilon.
\end{align*}
\]

An analog of Lemma 1.4 in the setting of \(b\)-metric spaces is the following.

Lemma 1.5. Suppose \((X, d)\) is a \(b\)-metric space with coefficient \(s \geq 1\) and let \(\{x_n\}\) be a sequence in \(X\) such that \(d(x_n, x_{n+1}) \to 0\) as \(n \to \infty\). If \(\{x_n\}\) is not a Cauchy sequence then there exist an \(\epsilon > 0\) and sequences of positive integers \(\{m_k\}\) and \(\{n_k\}\) with \(n_k > m_k \geq k\) such that \(d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_k-1}) < \epsilon\) and
\[
\begin{align*}
  (i) & \epsilon \leq \limsup_{k \to \infty} d(x_{m_k}, x_{n_k}) \leq s\epsilon \\
  (ii) & \frac{\epsilon}{s} \leq \limsup_{k \to \infty} d(x_{m_{k+1}}, x_{n_k}) \leq s^2\epsilon \\
  (iii) & \frac{\epsilon}{s} \leq \limsup_{k \to \infty} d(x_{m_k}, x_{n_{k+1}}) \leq s^2\epsilon \\
  (iv) & \frac{\epsilon}{s^2} \leq \liminf_{k \to \infty} d(x_{m_{k+1}}, x_{n_{k+1}}) \leq \limsup_{k \to \infty} d(x_{m_{k+1}}, x_{n_{k+1}}) \leq s^3\epsilon.
\end{align*}
\]

In 2012, Samet, Vetro and Vetro introduced an \(\alpha\)-admissible mapping as follows.

Definition 1.6. Let \(T : X \to X\) be a mapping and let \(\alpha : X \times X \to [0, \infty)\) be a function. We say that \(T\) is an \(\alpha\)-admissible mapping if \(x, y \in X, \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1\).
Definition 1.7. \([\text{[1]}]\) Let \(T : X \to X\) be a mapping and let \(\alpha : X \times X \to [0, \infty)\) be a function. We say that \(T\) is an \(\alpha\)-orbital admissible mapping if \(x, y \in X, \alpha(x, Tx) \geq 1 \implies \alpha(Tx, T^2x) \geq 1\).

Definition 1.8. \([\text{[2]}]\) Let \(T : X \to X\) and \(\alpha : X \times X \to [0, \infty)\). We say that \(T\) is a triangular \(\alpha\)-orbital admissible mapping if

(i) \(T\) is an \(\alpha\)-orbital admissible mapping and

(ii) \(\alpha(x, y) \geq 1\) and \(\alpha(y, Ty) \geq 1 \implies \alpha(x, Ty) \geq 1, \ x, y \in X\).

Remark 1.9. Every triangular \(\alpha\)-admissible mapping is a triangular \(\alpha\)-orbital admissible mapping. There exists a triangular \(\alpha\)-orbital admissible mapping which is not a triangular \(\alpha\)-admissible mapping. For more details see [\text{[3]}].

Definition 1.10. \([\text{[4]}]\) Let \(T : X \to X\) and \(\alpha, \eta : X \times X \to [0, \infty)\). Then \(T\) is said to be an \(\alpha\)-orbital admissible mapping with respect to \(\eta\) if \(\alpha(x, Tx) \geq \eta(x, Tx)\) implies \(\alpha(Tx, T^2x) \geq \eta(Tx, T^2x)\).

Definition 1.11. \([\text{[5]}]\) Let \(T : X \to X\) and \(\alpha, \eta : X \times X \to [0, \infty)\). Then \(T\) is said to be a triangular \(\alpha\)-orbital admissible mapping with respect to \(\eta\) if \(\alpha(x, y) \geq \eta(x, y)\) and \(\alpha(y, Ty) \geq \eta(y, Ty)\) implies \(\alpha(x, Ty) \geq \eta(x, Ty)\).

Lemma 1.12. \([\text{[6]}]\) Let \(T\) be a triangular \(\alpha\)-orbital admissible mapping with respect to \(\eta\). Assume that there exists \(x_0 \in X\) such that \(\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)\). We define a sequence \(\{x_n\}\) by \(x_{n+1} = Tx_n\). Then \(\alpha(x_m, x_n) \geq \eta(x_m, x_n)\) for all \(m, n \in \mathbb{N}\) with \(m < n\).

Definition 1.13. \([\text{[7]}]\) Let \((X, d)\) be a metric space and \(\alpha, \eta : X \times X \to [0, \infty)\). A mapping \(T : X \to X\) is said to be \(\alpha\)-\(\eta\)-continuous if every sequence \(\{x_n\}\) in \(X\) with \(\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})\) for all \(n \in \mathbb{N}\) and \(x_n \to x\) as \(n \to \infty\) implies \(Tx_n \to Tx\) as \(n \to \infty\).

Definition 1.14. Let \((X, d)\) be a \(b\)-metric space and \(\alpha, \eta : X \times X \to [0, \infty)\). A mapping \(T : X \to X\) is said to be \(\alpha\)-\(\eta\)-continuous if every sequence \(\{x_n\}\) in \(X\) with \(\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})\) for all \(n \in \mathbb{N}\) and \(x_n \to x\) as \(n \to \infty\) implies \(Tx_n \to Tx\) as \(n \to \infty\).

In 2015, Khojasteh, Shukla and Radenović \([\text{[8]}]\) introduced simulation functions and defined \(Z\)-contraction with respect to a simulation function.

Definition 1.15. \([\text{[9]}]\) A simulation function is a mapping \(\zeta : [0, \infty) \times [0, \infty) \to (-\infty, \infty)\) satisfying the following conditions:

(\(\zeta_1\)) \(\zeta(0, 0) = 0;\)

(\(\zeta_2\)) \(\zeta(t, s) < s - t, \text{ for all } s, t > 0;\)

(\(\zeta_3\)) if \(\{t_n\}, \{s_n\}\) are sequences in \((0, \infty)\) such that \(\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n = \ell \in (0, \infty),\) then \(\limsup_{n \to \infty} \zeta(t_n, s_n) < 0.\)
Remark 1.16. Let \( \zeta \) be a simulation function, if \( \{t_n\}, \{s_n\} \) are sequences in \((0, \infty)\) such that \( \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n = \ell \in (0, \infty) \), then \( \limsup_{n \to \infty} \zeta(kt_n, s_n) < 0 \) for any \( k > 1 \).

The following are examples of simulation functions.

Example 1.17. Let \( \zeta : [0, \infty) \times [0, \infty) \to (-\infty, \infty) \), be defined by

(i) \( \zeta(t, s) = 2s - t \) for all \( t, s \in [0, \infty) \), where \( \lambda \in [0, 1) \).

(ii) \( \zeta(s, t) = \frac{s}{1+s} - t \) for all \( t, s \in [0, \infty) \).

(iii) \( \zeta(t, s) = s - kt \) otherwise, where \( k > 1 \).

(iv) \( \zeta(s, t) = \frac{s}{1+s} - te^t \) for all \( t, s \in [0, \infty) \).

Definition 1.18. Let \((X, d)\) be a metric space and \( T \) be a selfmap of \( X \).

We say that \( T \) is a \( Z \)-contraction with respect to \( \zeta \), if there exists simulation function \( \zeta \) such that

\[
\zeta(d(Tx, Ty), d(x, y)) \geq 0 \quad \text{for all } x, y \in X
\]

Theorem 1.19. Let \((X, d)\) be a complete metric space and \( T : X \to X \) be a \( Z \)-contraction with respect to a certain simulation function \( \zeta \), then \( T \) has a unique fixed point in \( X \).

Moreover, for every \( x_0 \in X \), the Picard sequence \( \{T^n x_0\} \) converges to this fixed point.

Recently, Olgun, Bicer and Alyildiz proved the following result.

Theorem 1.20. Let \((X, d)\) be a complete metric space and \( T \) be a selfmap on \( X \). If there exists simulation function \( \zeta \) such that

\[
\zeta(d(Tx, Ty), M(x, y)) \geq 0 \quad \text{for all } x, y \in X,
\]

where \( M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\} \), then \( T \) has a unique fixed point in \( X \). Moreover, for every \( x_0 \in X \), the Picard sequence \( \{T^n x_0\} \) converges to this fixed point.

Motivated by the works of Olgun, Bicer and Alyildiz, we now introduce a generalized \( \alpha \)-\( \eta \)-\( Z \)-contraction with respect to \( \zeta \) in \( b \)-metric spaces.

Definition 1.21. Let \((X, d)\) be a \( b \)-metric space with coefficient \( s \geq 1 \) and \( \alpha, \eta : X \times X \to [0, \infty) \) be mappings. A mapping \( T : X \to X \) is said to be a generalized \( \alpha \)-\( \eta \)-\( Z \)-contraction with respect to \( \zeta \) if there exists a simulation mapping \( \zeta \) such that for any \( x, y \in X \) with \( \alpha(x, y) \geq \eta(x, y) \) implies

\[
\zeta(s^4d(Tx, Ty), M_T(x, y)) \geq 0,
\]

where \( M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\} \).

Example 1.22. Let \( X = [0, \infty) \) and let \( d : X \times X \to [0, \infty) \) be defined by

\[
d(x, y) = \begin{cases} 
0 & \text{if } x = y \\
\frac{1}{2}|x - y| & \text{if } x, y \in [0, 1) \\
\frac{1}{2}|x - y| & \text{otherwise.}
\end{cases}
\]
Clearly $(X, d)$ is a $b$-metric space with coefficient $s = 4$.

Now, we define $T : X \to X$ by

$$Tx = \begin{cases} \left(\frac{x}{40}\right)^2 & \text{if } x \in [0, 1) \\ \frac{3x}{4} + \frac{1}{4} & \text{if } x \in [1, \infty), \end{cases}$$

and $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 + xy & \text{if } x, y \in [0, \frac{1}{2}] \\ 1 & \text{otherwise}, \end{cases} \quad \text{and} \quad \eta(x, y) = \begin{cases} 0 & \text{if } x, y \in [0, \frac{1}{2}] \\ 4 & \text{otherwise}. \end{cases}$$

We now have $\alpha(x, y) \geq \eta(x, y) \iff x, y \in [0, \frac{1}{2}]$.

Now, we verify the inequality (1.3) for $x, y \in [0, \frac{1}{2}]$. For this purpose we choose $\zeta(t, s) = \frac{5}{2}s - t$.

For $x, y \in [0, \frac{1}{2}]$ we have $Tx = \left(\frac{x}{40}\right)^2, Ty = \left(\frac{y}{40}\right)^2$, and hence

$$\zeta(s^4d(Tx, Ty), M_T(x, y)) = \zeta(4^4d(Tx, Ty), M_T(x, y))$$

$$= \frac{5}{6}M_T(x, y) - 256d(Tx, Ty)$$

$$\geq \frac{5}{6}d(x, y) - 256d(Tx, Ty)$$

$$= \frac{5}{3}|x - y| - \frac{256}{800}|x^2 - y^2|$$

$$\geq \frac{5}{3}|x - y| - \frac{256}{800}|x - y| \geq 0.$$ 

Hence $T$ is a generalized $\alpha$-$\eta$-$Z$-contraction with respect to $\zeta$.

Here we observe that the $b$-metric $d$ is not continuous. For,

$$\lim_{n \to \infty} d(1, 1 - \frac{1}{n}) = \frac{1}{2} \lim_{n \to \infty} \frac{1}{n} = 0.$$

Hence the sequence $1 - \frac{1}{n} \to 1$ as $n \to \infty$. But

$$\lim_{n \to \infty} d(0, 1 - \frac{1}{n}) = \lim_{n \to \infty} 2|1 - \frac{1}{n}| = 2 \neq \frac{1}{2} = d(0, 1).$$

In Section 2, we prove our main results in which we study the existence of fixed points of generalized $\alpha$-$\eta$-$Z$-contraction mapping with respect to $\zeta$ in complete $b$-metric spaces. In Section 3, we extend the main results of Section 2 to partially ordered complete $b$-metric spaces. In Section 4, we provide corollaries and examples in support of our results.

2. Main results

**Theorem 2.1.** Let $(X, d)$ be a complete $b$-metric space with coefficient $s \geq 1$. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, \infty)$ be mappings.

Suppose that the following conditions are satisfied:

(i) $T$ is a generalized $\alpha$-$\eta$-$Z$-contraction with respect to $\zeta$,
(ii) $T$ is a triangular $\alpha$-orbital admissible mapping with respect to $\eta$, 
(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$, and 
(iv) $T$ is an $\alpha$-$\eta$-continuous mapping.

Then $T$ has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to $x^*$.

**Proof.** Let $x_1 \in X$ be as in (iii), i.e., $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. We define a sequence $\{x_n\}$ in $X$ by $x_{n+1} = T^n x_1 = Tx_n$ for all $n \in \mathbb{N}$. Suppose that $x_{n_0} = x_{n_0 + 1}$ for some $n_0 \in \mathbb{N}$, we have $Tx_{n_0} = x_{n_0}$, so that $x_{n_0}$ is a fixed point of $T$ and we are through.

Hence, without loss of generality, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. By Lemma 1.12, we have $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. From (1.3), we have

$$\zeta(s^4 d(x_{n+1}, x_{n+2}), M_T(x_n, x_{n+1})) = \zeta(s^4 d(Tx_n, Tx_{n+1}), M_T(x_n, x_{n+1})) \geq 0,$$

where

$$M_T(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_n), d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)\} \frac{2s}{2s} = \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2})\} \frac{2s}{2s} \leq \max\{d(x_n, x_{n+1}), d(x_n, x_{n+2}), d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\} \frac{2}{2} = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.$$ 

Hence $M_T(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$. Suppose that $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N}$. Then we have 

$$M_T(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2}).$$

Hence, from (2.1), we have

$$0 \leq \zeta(s^4 d(x_{n+1}, x_{n+2}), M_T(x_n, x_{n+1})) = \zeta(s^4 d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2})) < d(x_{n+1}, x_{n+2}) - s^4 d(x_{n+1}, x_{n+2}) \leq 0,$$

a contradiction. Hence $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Therefore, 
$
{d(x_n, x_{n+1})}$ is decreasing and bounded below. Thus there exist $r \geq 0$ such that 
$
\lim_{n \to \infty} d(x_n, x_{n+1}) = r.$

Suppose that $r > 0$. Now, using condition (C3), with $t_n = d(x_{n+1}, x_{n+2})$ and 
$s_n = d(x_n, x_{n+1})$, we have $0 \leq \limsup_{n \to \infty} \zeta(s^4 d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) < 0$, a 
contradiction. Therefore, $r = 0$ i.e.,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.$$
Now, we show that \( \{ x_n \} \) is a Cauchy sequence. Suppose that \( \{ x_n \} \) is not a Cauchy sequence. Now, we consider the following two cases

**Case (i):** \( s = 1 \).

In this case \((X, d)\) is a metric space. Then by Lemma 1.3 there exist \( \epsilon > 0 \) and sequence of positive integers \( \{ n_k \} \) and \( \{ m_k \} \) such that \( n_k > m_k \geq k \) satisfying

\[
(2.3) \quad d(x_{m_k}, x_{n_k}) \geq \epsilon.
\]

Let us choose the smallest \( n_k \) satisfying (2.3), then we have \( n_k > m_k \geq k \) with \( d(x_{m_k}, x_{n_k}) \geq \epsilon \) and \( d(x_{m_k}, x_{n_k-1}) < \epsilon \) satisfying (i)- (iv) of Lemma 1.3.

Hence we have

\[
M_s(x_{m_k}, x_{n_k})
= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{n_k}), \frac{d(x_{m_k}, Tx_{n_k}) + d(x_{n_k}, Tx_{m_k})}{2}\}.
\]

On taking limit as \( k \to \infty \) we have \( \lim_{k \to \infty} M_s(x_{m_k}, x_{n_k}) = \epsilon \).

Using condition (i.3) with \( t_k = d(x_{m_k+1}, x_{n_k+1}) \) and \( s_k = M(x_{m_k}, x_{n_k}) \), we have \( 0 \leq \lim_{k \to \infty} \sup \zeta(d(x_{m_k+1}, x_{n_k+1}), M_s(x_{m_k}, x_{n_k})) < 0 \), a contradiction. Thus \( \{ x_n \} \) is a Cauchy sequence.

**Case (ii):** \( s > 1 \).

Then by Lemma 1.3 there exist \( \epsilon > 0 \) and sequence of positive integers \( \{ n_k \} \) and \( \{ m_k \} \) such that \( n_k > m_k \geq k \) satisfying

\[
(2.4) \quad d(x_{m_k}, x_{n_k}) \geq \epsilon.
\]

Let us choose the smallest \( n_k \) satisfying (2.3), then we have \( n_k > m_k \geq k \) with \( d(x_{m_k}, x_{n_k}) \geq \epsilon \) and \( d(x_{m_k}, x_{n_k-1}) < \epsilon \) satisfying (i)- (iv) of Lemma 1.3.

\[
(2.5) \quad \epsilon \leq d(x_{m_k}, x_{n_k}) \leq M_s(x_{m_k}, x_{n_k})
= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{n_k}), \frac{d(Tx_{m_k}, x_{n_k}) + d(x_{m_k}, Tx_{n_k})}{2}\}.
\]

Letting \( n \to \infty \) in (2.3) and using (i) - (iv) of Lemma 1.3, we have

\[
(2.6) \quad \epsilon \leq \lim_{k \to \infty} \sup M_s(x_{m_k}, x_{n_k}) \leq \max\{s\epsilon, 0, \frac{s^2\epsilon + s^2\epsilon}{2s}\} = s\epsilon.
\]

By Lemma 1.1 we have \( \alpha(x_{m_k}, x_{n_k}) = s\eta(x_{m_k}, x_{n_k}) \). Hence from (1.3) we have \( 0 \leq \zeta(s^4d(Tx_{m_k}, Tx_{n_k}), MT(x_{m_k}, x_{n_k})) \).

Now we have

\[
(2.7) \quad 0 \leq \lim_{k \to \infty} \sup \zeta(s^4d(Tx_{m_k}, Tx_{n_k}), MT(x_{m_k}, x_{n_k}))
\leq \lim_{k \to \infty} \sup [MT(x_{m_k}, x_{n_k}) - s^4d(Tx_{m_k}, Tx_{n_k})]
\leq \lim_{k \to \infty} MT(x_{m_k}, x_{n_k}) - s^4 \liminf_{k \to \infty} d(Tx_{m_k}, Tx_{n_k}) \leq s\epsilon - s^4(\frac{\epsilon}{s^2}) < 0,
\]
a contradiction. So we conclude that \( \{x_n\} \) is a Cauchy sequence in \((X, d)\).

Since \( X \) is a complete \( b \)-metric space then, there exists \( x^* \in X \) such that \( \lim_{n \to \infty} x_n = x^* \). Since \( T \) is \( \alpha \)-\( \eta \)-continuous and \( \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \) for all \( n \in \mathbb{N} \), we have \( x^* = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_n = Tx^* \). Hence \( T \) has a fixed point.

In the following theorem, we replace the \( \alpha \)-\( \eta \)-continuity of \( T \) by another condition.

**Theorem 2.2.** Let \((X, d)\) be a complete \( b \)-metric space with coefficient \( s \geq 1 \). Let \( T : X \to X \) and \( \alpha, \eta : X \times X \to [0, \infty) \) be mappings.

Suppose that the following conditions are satisfied:

- (i) \( T \) is a generalized \( \alpha \)-\( \eta \)-\( Z \)-contraction with respect to \( \zeta \).
- (ii) \( T \) is a triangular \( \alpha \)-orbital admissible mapping with respect to \( \eta \).
- (iii) there exists \( x_1 \in X \) such that \( \alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1) \), and
- (iv) if \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \) for all \( n \in \mathbb{N} \) and \( x_n \to x^* \in X \) as \( n \to \infty \), then there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n_k}, x^*) \geq \eta(x_{n_k}, x^*) \) for all \( k \in \mathbb{N} \).

Then \( \{T^n x_1\} \) converges to an element \( x^* \) of \( X \) and \( x^* \) is a fixed point of \( T \).

**Proof.** By using similar arguments as in the proof of Theorem 2.1, we obtain that the sequence \( \{x_n\} \) defined by \( x_{n+1} = Tx_n \) converges to \( x^* \in X \) and \( \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \) for all \( n \in \mathbb{N} \).

By (iv), there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n_k}, x^*) \geq \eta(x_{n_k}, x^*) \) for all \( k \in \mathbb{N} \). Hence from (1.3) we have

\[
0 \leq \zeta(s^4d(Tx_{n_k}, Tx^*), MT(x_{n_k}, x^*)) = \zeta(s^4d(x_{n_k+1}, Tx^*), MT(x_{n_k}, x^*)) \\
< MT(x_{n_k}, x^*) - s^4d(x_{n_k+1}, Tx^*),
\]

which implies that \( s^4d(x_{n_k+1}, Tx^*) < MT(x_{n_k}, x^*) \).

Now, we have

\[
sd(x_{n_k+1}, Tx^*) \leq s^4d(x_{n_k+1}, Tx^*) < MT(x_{n_k}, x^*) \text{ and}
\]

\[
d(x^*, Tx^*) \leq MT(x_{n_k}, x^*) = \max\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*) \} \\
= \max\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*) \} \\
\leq \max\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*) \} \\
\leq \left( \frac{d(x_{n_k}, x^*) + d(x^*, Tx^*)}{2} + d(Tx_{n_k}, x^*) \right) \leq \left( \frac{d(x_{n_k}, x^*) + d(x^*, Tx^*) + d(Tx_{n_k}, x^*)}{2} \right),
\]

On taking limits as \( n \to \infty \) we have

\[
d(x^*, Tx^*) \leq \lim_{k \to \infty} MT(x_{n_k}, x^*) \leq d(x^*, Tx^*).
\]

Therefore \( \lim_{k \to \infty} MT(x_{n_k}, x^*) = d(x^*, Tx^*) \).
From (2.9) we now have
\[ d(x^*, T x^*) \leq s d(x^*, T x_{n_k}) + s d(T x_{n_k}, T x^*) \leq s d(x^*, T x_{n_k}) + M_T(x_{n_k}, x^*) \]

On taking limit as \( k \to \infty \) on (2.10), we have
\[ d(x^*, T x^*) \leq s \lim_{k \to \infty} d(x_{n_k+1}, T x^*) \leq d(x^*, T x^*). \]

Hence we have
\[ \lim_{k \to \infty} d(x_{n_k+1}, T x^*) = \frac{1}{s} d(x^*, T x^*). \]

Suppose \( x^* \neq T x^* \). Now by choosing \( t_k = s d(x_{n_k+1}, T x^*) \) and \( s_k = M_T(x_{n_k}, x^*) \)
from property (\( \zeta_3 \)), it follows that
\[ 0 \leq \lim_{k \to \infty} \zeta(s d(T x_{n_k}, T x^*), M_T(x_{n_k}, x^*)) < 0, \]
a contradiction. Hence \( T x^* = x^* \). Therefore \( T \) has a fixed point. \( \square \)

**Theorem 2.3.** In addition to the hypotheses of Theorem 2.1 (Theorem 2.2) assume the following.

Condition (H): for all \( x \neq y \in X \), there exists \( v \in X \) such that
\[ \alpha(x, v) \geq \eta(x, v), \alpha(y, v) \geq \eta(y, v) \text{ and } \alpha(v, T v) \geq \eta(v, T v). \]

Then \( T \) has a unique fixed point.

**Proof.** Suppose that \( z^* \) and \( y^* \) are two fixed points of \( T \) with \( z^* \neq y^* \). Then by our assumption, there exists \( v \in X \) such that \( \alpha(z^*, v) \geq \eta(z^*, v), \alpha(y^*, v) \geq \eta(y^*, v) \) and \( \alpha(v, T v) \geq \eta(v, T v) \).

Then \( T \) has a unique fixed point.

Now, since \( d(z^*, T^n v) \leq s \{ d(z^*, x^*) + d(x^*, T^n v) \} \), we have the sequence \( \{d(z^*, T^n v)\} \) is bounded. Therefore there exists a subsequence \( \{d(z^*, T^n v)\} \) such that \( \lim_{n \to \infty} d(z^*, T^n v) = \ell \), for some nonnegative real \( \ell \).

Now, we have
\[ d(z^*, T^k v) \leq M_T(z^*, T^k v) \]
\[ = \max \{d(z^*, T^k v), d(z^*, T z^*), d(T^k v, T z^*), d(T^k v, T z^* + 1) \}, \]
\[ \frac{d(z^*, T z^* + 1) + d(T z^*, T z^* + 1)}{2s} \]
\[ \leq \max \{d(z^*, T^k v), d(T^k v, T z^*), d(T^k v, T z^* + 1) \}, \]
\[ \frac{d(z^*, T z^* + 1) + d(z^*, T z^* + 1)}{2s} \]
\[ \leq \max \{d(z^*, T^k v), d(T^k v, T z^*), d(T^k v, T z^* + 1) \}, \]
\[ \frac{d(z^*, T z^* + 1) + d(z^*, T z^* + 1)}{2s} \].
On taking limits as \( k \to \infty \) we have \( \lim_{n \to \infty} M_T(z^*, T^{n_k}v) = \ell \).

We now show that \( \ell = 0 \). Suppose \( \ell > 0 \).

Since \( T \) is triangular \( \alpha \)-orbital admissible with respect to \( \eta \), we have \( \alpha(v, T^nv) \geq \eta(v, T^nv) \) and hence \( \alpha(z^*, T^nv) \geq \eta(z^*, T^nv) \) and \( \alpha(y^*, T^nv) \geq \eta(y^*, T^nv) \) for all \( n \in \mathbb{N} \).

Now, from (3.3) we have \( \zeta(s^4d(z^*, T^{n_k+1}v), M_T(z^*, T^{n_k}v)) \geq 0 \).

Hence, we have \( s^4d(z^*, T^{n_k+1}v) \leq M_T(z^*, T^{n_k}v) \) which implies that 
\[
sd(z^*, T^{n_k+1}v) \leq s^3d(z^*, T^{n_k+1}v) \leq M_T(z^*, T^{n_k}v).
\]

Now, we have
\[
d(z^*, T^{n_k}v) \leq sd(z^*, T^{n_k+1}v) + sd(T^{n_k+1}v, T^{n_k}v) \leq M_T(z^*, T^{n_k}v) + sd(z^*, T^{n_k}v).
\]

On taking limits as \( k \to \infty \) we have
\[
\lim_{n \to \infty} sd(z^*, T^{n_k+1}v) = \ell.
\]

Now, by choosing \( t_k = sd(z^*, T^{n_k+1}v) \) and \( s_k = M_T(z^*, T^{n_k}v) \), from property \((\zeta_3)\), it follows that
\[
0 \leq \lim_{k \to \infty} \sup_{k \to \infty} \zeta(s^4d(z^*, T^{n_k+1}v), M_T(z^*, T^{n_k}v)) < 0,
\]
a contradiction. Hence \( \ell = 0 \). Hence \( T^{n_k}v \to z^* \) as \( n \to \infty \). Therefore \( z^* = x^* \).

Similarly we can prove that \( y^* = x^* \).

Thus it follows that \( z^* = y^* \), a contradiction. Hence \( T \) has a unique fixed point. \( \square \)

3. A fixed point result in partially ordered \( b \)-metric spaces

**Definition 3.1.** Let \((X, \preceq)\) be a partially ordered set. If there exists a \( b \)-metric \( d \) on \( X \) with coefficient \( s \geq 1 \), such that \((X, d)\) is complete, then we say that \((X, \preceq, d)\) is a partially ordered complete \( b \)-metric space with coefficient \( s \geq 1 \).

**Theorem 3.2.** Let \((X, \preceq, d)\) be a partially ordered complete \( b \)-metric space with coefficient \( s \geq 1 \). Let \( T : X \to X \) be a selfmap of \( X \). Assume that the following conditions are satisfied:

(i) there exists a simulation mapping \( \zeta \) such that
\[
\zeta(s^4d(Tx, Ty), M_T(x, y)) \geq 0, \quad \text{for any } x, y \in X \text{ with } x \preceq y,
\]
where \( M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\} \),
(ii) \( T \) is a nondecreasing,
(iii) there exists an \( x_1 \in X \) such that \( x_1 \preceq Tx_1 \),
(iv) either \( T \) is continuous or if \( \{x_n\} \) is a decreasing sequence with \( x_n \to x^* \) as \( n \to \infty \), then there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \preceq x^* \) for all \( k \in \mathbb{N} \).

Then \( \{T^nx_1\} \) converges to an element \( x^* \) of \( X \) and \( x^* \) is a fixed point of \( T \).

Further, if for all \( x \neq y \in X \), there exists \( v \in X \) such that \( x \preceq v, y \preceq v \) and \( v \preceq Tv \), then \( T \) has a unique fixed point.
Proof. We define functions $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 
3 & \text{if } x \leq y \\
0 & \text{otherwise}, \\
1 & \text{if } x \leq y \\
4 & \text{otherwise}.
\end{cases} \quad \text{and } \eta(x, y) = \begin{cases} 
1 & \text{if } x \leq y \\
0 & \text{otherwise}.
\end{cases}$$

Now, for any $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ if and only if $x \leq y$. By (i), we have $\zeta(s^4d(Tx, Ty), M_T(x, y)) \geq 0$. Suppose that $\alpha(x, Tx) \geq \eta(x, Tx)$, then we have $x \leq Tx$. Since $T$ is nondecreasing, we have $Tx \leq TTx$ which implies that $\alpha(Tx, TTx) \geq \eta(Tx, TTx)$, hence $T$ is $\alpha$-orbital admissible with respect to $\eta$.

Further, suppose that $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$, so that we have $x \leq y$ and $y \leq Ty$. It follows that $x \leq Ty$ and hence $\alpha(x, Ty) \geq \eta(x, Ty)$. Thus $T$ is triangular $\alpha$-orbital admissible with respect to $\eta$. Hence $T$ satisfies all the hypotheses of Theorem 2.3 (Theorem 2.4) and $T$ has a fixed point.

Moreover, if for all $x \neq y \in X$, there exists a $v \in X$ such that $x \leq v, y \leq v$ and $v \leq Tv$, then we have $\alpha(x, v) \geq \eta(x, v), \alpha(y, v) \geq \eta(y, v)$ and $\alpha(v, Tv) \geq \eta(v, Tv)$. Hence by Theorem 2.4, $T$ has a unique fixed point. □

4. Corollaries and examples

Corollary 4.1. Let $(X, d)$ be a complete metric space. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, \infty)$ be mappings. Suppose that the following conditions are satisfied:

(i) there exists a simulation mapping $\zeta$ such that for any $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies $\zeta(d(Tx, Ty), M(x, y)) \geq 0$, where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}$,

(ii) $T$ is a triangular $\alpha$-orbital admissible mapping with respect to $\eta$,

(iii) there exists an $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$, and

(iv) $T$ is an $\alpha$-$\eta$-continuous mapping, or if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq \eta(x_{n_k}, x^*)$ for all $k \in \mathbb{N}$.

Then $T$ has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to $x^*$.

Moreover, if for all $x \neq y \in X$, there exists $v \in X$ such that $\alpha(x, v) \geq \eta(x, v), \alpha(y, v) \geq \eta(y, v)$ and $\alpha(v, Tv) \geq \eta(v, Tv)$, then $T$ has a unique fixed point.

Proof. Follows from Theorem 2.3 by taking $s = 1$. □

Remark 4.2. Theorem 2.3 follows as a corollary to Corollary 2.4 by choosing $\alpha(x, y) = \eta(x, y) = 1$ for all $x, y \in X$, which in turn Theorem 2.4 follows as a corollary to Theorem 2.4.

Corollary 4.3. Let $(X, d)$ be a complete $b$-metric space with coefficient $s \geq 1$. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, \infty)$ be mappings. Assume that there exist two continuous function $\psi, \varphi : [0, \infty) \to [0, \infty)$ with $\psi(t) < t \leq \varphi(t)$ for all $t > 0$ and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ such that for any $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y)$ implies

$$\varphi(s^4d(Tx, Ty)) \leq \psi(M_T(x, y)). \quad (4.1)$$
where \( M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\} \).

Suppose that the following conditions are satisfied:

(i) \( T \) is a triangular \( \alpha \)-orbital admissible mapping;

(ii) there exists \( x_1 \in X \) such that \( \alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1) \); and

(iii) either \( T \) is an \( \alpha \)-\( \eta \)-continuous mapping, or if \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \) for all \( n \in \mathbb{N} \) and \( x_n \to x^* \in X \) as \( n \to \infty \), then there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n_k}, x^*) \geq \eta(x_{n_k}, x^*) \) for all \( k \in \mathbb{N} \).

Then \( \{T^n x_1\} \) converges to an element \( x^* \) of \( X \) and \( x^* \) is a fixed point of \( T \).

**Proof.** The conclusion of this corollary follows from Theorem 2.1 (Theorem 2.2) by taking \( \zeta(t, s) = \psi(s) - \varphi(t) \) for all \( t, s \geq 0 \).

---

**Corollary 4.4.** Let \( (X, d) \) be a complete \( b \)-metric space with coefficient \( s \geq 1 \). Let \( T : X \to X \) and \( \alpha, \eta : X \times X \to [0, \infty) \) be mappings.

Suppose that the following conditions are satisfied:

(i) there exists a simulation mapping \( \zeta \) such that for any \( x, y \in X \) with \( \alpha(x, y) \geq 1 \) implies \( \zeta(s^4 d(Tx, Ty), M_T(x, y)) \geq 0 \), where \( M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\} \).

(ii) \( T \) is a triangular \( \alpha \)-orbital admissible mapping,

(iii) there exists an \( x_1 \in X \) such that \( \alpha(x_1, Tx_1) \geq 1 \), and

(iv) \( T \) is an \( \alpha \)-continuous mapping, or if \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \), and \( x_n \to x^* \in X \) as \( n \to \infty \), then there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n_k}, x^*) \geq 1 \) for all \( k \in \mathbb{N} \).

Then \( T \) has a fixed point \( x^* \in X \) and \( \{T^n x_1\} \) converges to \( x^* \).

Moreover, if for all \( x \neq y \in X \), there exists \( v \in X \) such that \( \alpha(x, v) \geq 1 \), \( \alpha(y, v) \geq 1 \) and \( \alpha(v, Tv) \geq 1 \), then \( T \) has a unique fixed point.

**Proof.** Follows from Theorem 2.1 (Theorem 2.2) and Theorem 2.2 by taking \( \eta(x, y) = 1 \) for all \( x, y \in X \).

---

**Example 4.5.** Let \( X = [0, \infty) \) and let \( d : X \times X \to [0, \infty) \) be defined by \( d(x, y) = |x - y|^2 \). Clearly \( (X, d) \) is a \( b \)-metric space with coefficient \( s = 2 \). We define \( T : X \to X \) by

\[
Tx = \begin{cases} 
1 - \frac{x}{6} & \text{if } x \in [0, 1] \\
2x - 2 & \text{if } x \in (1, \infty)
\end{cases}
\]

and \( \alpha, \eta : X \times X \to [0, \infty) \) by

\[
\alpha(x, y) = \begin{cases} 
2 + xy & \text{if } x, y \in [0, 1] \\
0 & \text{otherwise},
\end{cases} \quad \text{and} \quad \eta(x, y) = \begin{cases} 
1 + xy & \text{if } x, y \in [0, 1] \\
4 & \text{otherwise}.
\end{cases}
\]

We now have \( \alpha(x, y) \geq \eta(x, y) \iff x, y \in [0, 1] \). Let \( \alpha(Tx, Ty) \geq \eta(x, Tx) \), then \( x, Tx \in [0, 1] \) and hence \( Tx, TTx \in [0, 1] \), since for any \( x \in [0, 1] \) we have \( Tx \in [0, 1] \). Therefore \( \alpha(Tx, TTx) \geq \eta(Tx, TTx) \). Hence \( T \) is \( \alpha \)-orbital admissible with respect to \( \eta \).
Suppose that \( \alpha(x, y) \geq \eta(x, y) \) and \( \alpha(y, Ty) \geq \eta(y, Ty) \), then \( x, y, Ty \in [0, 1] \) which implies that \( \alpha(x, Ty) \geq \eta(x, Ty) \). Hence \( T \) is triangular \( \alpha \)-orbital admissible with respect to \( \eta \).

Let \( \{x_n\} \) be a sequence such that \( x_n \to x \) as \( n \to \infty \) and \( \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \) for all \( n \in \mathbb{N} \). Then \( \{x_n\} \subseteq [0, 1] \) for all \( n \in \mathbb{N} \). Then we have
\[
\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} (1 - x_n/6) = 1 - \lim_{n \to \infty} x_n/6 = 1 - x/6 = Tx .
\]
Hence \( T \) is \( \alpha, \eta \)-continuous.

Now, we verify the inequality (1.3) for \( x, y \in [0, 1] \). For \( x = y \) the inequality holds trivially, hence we verify for \( x \neq y \).

We define \( \zeta : [0, \infty) \times [0, \infty) \to [0, \infty) \) by
\[
\zeta(t, s) = \frac{\sqrt{s^2} - t}{1 + \sqrt{s^2}} - t.
\]
Since \( \alpha(x, y) \geq \eta(x, y) \) if and only if \( x, y \in [0, 1] \), we have \( Tx = 1 - x/5 \) and \( Ty = 1 - y/5 \). Hence
\[
\zeta(s^4d(Tx, Ty), M_T(x, y)) = \frac{M_T(x, y)}{1 + M_T(x, y)} - s^4d(Tx, Ty) \geq \frac{d(x, y)}{1 + d(x, y)} - 16d(Tx, Ty) = \frac{|x - y|^2}{1 + |x - y|^2} - \frac{16}{36}|x - y|^2 \geq \frac{16}{36}|x - y|^2 - \frac{16}{36}|x - y|^2 = 0.
\]
Hence \( T \) satisfies all the hypothesis of Theorem 7.1 with \( x = \frac{6}{7} \) and \( x = 2 \) are fixed points of \( T \).

Here we observe that ‘Condition (H)’ of Theorem 7.4 fails to hold. For, choose \( x = 5 \) and \( x = 6 \), then there is no \( v \in X \) such that \( \alpha(5, v) \geq \eta(5, v) \) and \( \alpha(6, v) \geq \eta(6, v) \).

Remark 4.6. In the usual metric, the inequality (1.2) fails. For, by choosing \( x = 3 \) and \( y = 4 \), we have \( M_T(3, 4) = 2 \) and \( d(T3, T4) = 2 \) and hence we have \( \zeta(d(T3, T4), M_T(3, 4)) = \zeta(2, 2) < 0 \), for any simulation function \( \zeta \).

Hence Theorem 1.20 is not applicable.

Example 4.7. Let \( X = [0, \infty) \) and let \( d : X \times X \to [0, \infty) \) be defined by \( d(x, y) = |x - y|^2 \). Hence \((X, d)\) is a complete \( b \)-metric space with coefficient \( s = 2 \). We define \( T : X \to X \) by
\[
Tx = \begin{cases} 
\frac{2}{11}x & \text{if } x \in [0, 6] \\
\frac{x}{6} - 1 & \text{if } x \in (6, \infty),
\end{cases}
\]
and \( \alpha, \eta : X \times X \to [0, \infty) \) by
\[
\alpha(x, y) = \begin{cases} 
2 & \text{if } x, y \in [0, 6], \\
3 & \text{if } x \in (6, \infty), y = 0, \\
1 + xy & \text{otherwise},
\end{cases}
\]
and
\[
\eta(x, y) = \begin{cases} 
0 & \text{if } x, y \in [0, 6], \\
1 & \text{if } x \in (6, \infty), y = 0, \\
4 + xy & \text{otherwise}.
\end{cases}
\]

We now have \(\alpha(x, y) \geq \eta(x, y) \iff x, y \in [0, 6]\) and \(x \in (6, \infty), y = 0\).

Suppose that \(\alpha(x, Tx) \geq \eta(x, Tx)\), then we have \(x \in [0, 6]\) and hence \(\alpha(Tx, TTx) \geq \eta(Tx, TTx)\). Therefore \(T\) is \(\alpha\)-orbital admissible with respect to \(\eta\).

Suppose that \(\alpha(x, y) \geq \eta(x, y)\) and \(\alpha(y, Ty) \geq \eta(y, Ty)\), then we have \(x, y \in [0, 6]\), or \(x \in (6, \infty)\) and \(y = 0\), which implies that \(x, Ty \in [0, 6]\), or \(x \in (6, \infty)\) and \(Ty = y = 0\) and hence \(\alpha(x, Ty) \geq \eta(x, Ty)\). Therefore \(T\) is triangular \(\alpha\)-orbital admissible with respect to \(\eta\).

We now verify the inequality (1.3). For this purpose we define \(\zeta : [0, \infty) \times [0, \infty) \to (-\infty, \infty)\) by \(\zeta(t, s) = \frac{3}{4}s - t\).

Now we have the following cases.

**Case (i) :** \(x, y \in [0, 6]\)

In this case \(Tx = \frac{2}{11}x, Ty = \frac{2}{11}y\), then we have
\[
\zeta\left(2^d(Tx, Ty); M_T(0, y)\right) = \zeta(16d(Tx, Ty), M_T(x, y))
\]
\[
= \frac{3}{4}M_T(x, y) - 16d(Tx, Ty)
\]
\[
\geq \frac{3}{4}d(x, y) - 16\left(\frac{4}{121}|x - y|^2\right)
\]
\[
= \frac{3}{4}|x - y|^2 - 16\left(\frac{4}{121}|x - y|^2\right) \geq 0.
\]

**Case (ii) :** \(x \in (6, \infty), y = 0\)

In this case \(Tx = \frac{3}{6} - 1, T0 = 0\), then we have
\[
\zeta(16d(Tx, T0), M_T(x, 0)) = \frac{3}{4}M_T(x, 0) - 16\left(\frac{1}{6}y - 1\right)^2
\]
\[
= \frac{3}{4}M_T(x, 0) - 16\left(\frac{1}{36}|y - 6|^2\right)
\]
\[
\geq \frac{3}{4}y^2 - \frac{16}{36}|y - 6|^2 \geq 0.
\]

Hence \(T\) satisfies the inequality (1.3). Also, since for any \(x \neq y \in X\) we have \(\alpha(x, 0) \geq \eta(x, 0), \alpha(y, 0) \geq \eta(y, 0)\) and \(\alpha(0, T0) \geq \eta(0, T0)\), \(T\) satisfies ‘Condition (H)’. Hence \(T\) satisfies all the hypotheses of Theorem 2.3 and \(x = 0\) is the unique fixed point of \(T\).

**Example 4.8.** Let \(X = [0, \infty)\) and a \(b\)-metric be as defined in Example 1.22. Then clearly \(T\) satisfies all the hypotheses of Theorem 2.3 and \(x = 0\) and \(x = 1\) are two fixed points.

**References**

Fixed points in b-metric spaces via ...


[3] Babu, G.V.R., Dula, T.M., Fixed points of almost generalized $(\alpha, \beta) - (\psi, \varphi)$ - contractive mappings in $b$-metric spaces. (Communicated).


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