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FIXED POINTS IN b-METRIC SPACES VIA SIMULATION FUNCTION

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Abstract. We introduce the concept of generalized α-η-Z-contraction mapping with respect to a simulation function ζ in b-metric spaces and study the existence of fixed points for such mappings in complete b-metric spaces. Further, we extend it to partially ordered complete b-metric spaces. We provide examples in support of our results. Our results extend the fixed point results of Olgun, Bicer and Alyildiz [15].

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1. Introduction

The famous Banach contraction principle introduced by Banach [5], ensures the existence and uniqueness of fixed points for a contraction mapping in complete metric spaces. Several researchers generalized and extended this principle by introducing various contractions in different ambient spaces. (see [1],[2], [4], [6], [8], [9], [10], [12], [13]).

In 1993, Stefan Czerwik [9] introduced the concept of a b-metric space as a generalization of a metric space.

Definition 1.1. [9] Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied;

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) there exists $s \geq 1$ such that $d(x, z) \leq s\big[d(x, y) + d(y, z)\big]$ for all $x, y, z \in X$.

In this case, the pair (X, d) is called a b-metric space with coefficient s.

Definition 1.2. [7] Let (X, d) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

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(iii) A b-metric space (X, d) is said to be a b-complete metric space if every b-Cauchy sequence in X is b-convergent.

(iv) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X$ then $z \in B$.

Theorem 1.3. [9] Let (X, d) be a complete b-metric space with coefficient $s = 2$. Let $T : X \to X$ satisfy

$$d(Tx, Ty) \leq \varphi(d(x, y)) \text{ for all } x, y \in X$$

where $\varphi : [0, \infty) \to [0, \infty)$ is an increasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Then T has exactly one fixed point u in X and $\lim_{n\to\infty} d(T^n(x), u) = 0$ for all $x \in X$.

Babu and Sailaja [4] proved the following lemma which plays an important role in proving the Cauchy part of an iterative sequence in metric spaces.

Lemma 1.4. [4] Suppose (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_k-1}) < \epsilon$ and
(i) $\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k+1}) = \epsilon$
(ii) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon$
(iii) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$ and (iv) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \epsilon$.

An analog of Lemma 1.4 in the setting of b-metric spaces is the following.

Lemma 1.5. [3] Suppose (X, d) is a b-metric space with coefficient $s \geq 1$ and let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_k-1}) < \epsilon$ and
(i) $\epsilon \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq s\epsilon$
(ii) $\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq s^2\epsilon$
(iii) $\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq s^2\epsilon$
(iv) $\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq s^3\epsilon$.

In 2012, Samet, Vetro and Vetro [17], introduced an α-admissible mapping as follows;

Definition 1.6. [17] Let $T : X \to X$ be a mapping and let $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is an α-admissible mapping if $x, y \in X, \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$.

Definition 1.7. [16] Let $T : X \to X$ be a mapping and let $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is an α-orbital admissible mapping if $x, y \in X, \alpha(x, Tx) \geq 1 \implies \alpha(Tx, T^2x) \geq 1$.

Definition 1.8. [16] Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that T is a triangular α-orbital admissible mapping if

(i) T is an α-orbital admissible mapping and

(ii) $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1 \implies \alpha(x, Ty) \geq 1, \quad x, y \in X$.

Remark 1.9. Every triangular α-admissible mapping is a triangular α-orbital admissible mapping. There exists a triangular α-orbital admissible mapping which is not a triangular α-admissible mapping. For more details see[16].

Definition 1.10. [8] Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, \infty)$. Then T is said to be an α-orbital admissible mapping with respect to η if $\alpha(x, Tx) \geq \eta(x, Tx)$ implies $\alpha(Tx, T^2x) \geq \eta(Tx, T^2x)$.

Definition 1.11. [8] Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, \infty)$. Then T is said to be a triangular α-orbital admissible mapping with respect to η if (i) α-orbital admissible mapping with respect to η

(ii) $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$ implies $\alpha(x, Ty) \geq \eta(x, Ty)$.

Lemma 1.12. [8] Let T be a triangular α-orbital admissible mapping with respect to η. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then $\alpha(x_m, x_n) \geq \eta(x_m, x_n)$ for all $m, n \in \mathbb{N}$ with $m < n$.

Definition 1.13. [11] Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be α-η-continuous if every sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ implies $Tx_n \to Tx$ as $n \to \infty$.

Definition 1.14. Let (X, d) be a b-metric space and $\alpha, \eta : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be α-η-continuous if every sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ implies $Tx_n \to Tx$ as $n \to \infty$.

In 2015, Khojasteh, Shukla and Radenović [14] introduced simulation functions and defined Z-contraction with respect to a simulation function.

Definition 1.15. [14] A simulation function is a mapping

$$\zeta : [0, \infty) \times [0, \infty) \to (-\infty, \infty)$$

satisfying the following conditions:

(ζ_1) $\zeta(0, 0) = 0$;

(ζ_2) $\zeta(t, s) < s - t$, for all $s, t > 0$;

(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0, \infty)$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Remark 1.16. Let ζ be a simulation function, if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty)$, then $\limsup_{n\to\infty} \zeta(kt_n, s_n) < 0$ for any $k > 1$.

The following are examples of simulation functions.

Example 1.17. Let $\zeta : [0,\infty) \times [0,\infty) \to (-\infty,\infty)$, be defined by
(i) $\zeta(t,s) = \lambda s - t$ for all $t, s \in [0,\infty)$, where $\lambda \in [0,1)$.
(ii) $\zeta(s,t) = \frac{s}{1+s} - t$ for all $t, s \in [0,\infty)$.
(iii) $\zeta(t,s) = s - kt$ otherwise, where $k > 1$.
(iv) $\zeta(s,t) = \frac{s}{1+s} - te^t$ for all $t, s \in [0,\infty)$.

Definition 1.18. [14] Let (X,d) be a metric space and T be a selfmap of X. We say that T is a Z-contraction with respect to ζ, if there exists simulation function ζ such that

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0 \text{ for all } x, y \in X \tag{1.1}$$

Theorem 1.19. [14] Let (X,d) be a complete metric space and $T : X \to X$ be a Z-contraction with respect to a certain simulation function ζ, then T has a unique fixed point in X.

Moreover, for every $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.

Recently, Olgun, Bicer and Alyildiz [15] proved the following result.

Theorem 1.20. [15] Let (X,d) be a complete metric space and T be a selfmap on X. If there exists simulation function ζ such that

$$\zeta(d(Tx,Ty), M(x,y)) \geq 0 \text{ for all } x, y \in X, \tag{1.2}$$

where $M_T(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$, then T has a unique fixed point in X. Moreover, for every $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.

Motivated by the works of Olgun, Bicer and Alyildiz [15], we now introduce a generalized α-η-Z-contraction with respect to ζ in b-metric spaces.

Definition 1.21. Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $\alpha, \eta : X \times X \to [0,\infty)$ be mappings. A mapping $T : X \to X$ is said to be a generalized α-η-Z-contraction with respect to ζ if there exists a simulation mapping ζ such that for any $x, y \in X$ with $\alpha(x,y) \geq \eta(x,y)$ implies

$$\zeta(s^4 d(Tx,Ty), M_T(x,y)) \geq 0, \tag{1.3}$$

where $M_T(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$.

Example 1.22. Let $X = [0,\infty)$ and let $d : X \times X \to [0,\infty)$ be defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2|x-y| & \text{if } x, y \in [0,1) \\ \frac{1}{2}|x-y| & \text{othewise.} \end{cases}$$

Clearly (X,d) is a b-metric space with coefficient $s=4$.

Now, we define $T : X \to X$ by

$$Tx = \begin{cases} (\frac{x}{40})^2 & \text{if } \ x \in [0,1) \\ \frac{3x}{4} + \frac{1}{4} & \text{if } x \in [1,\infty), \end{cases}$$

and $\alpha, \eta : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 2+xy & \text{if } \ x,y \in [0,\frac{1}{2}] \\ 1 & \text{otherwise,} \end{cases} \quad \text{and } \eta(x,y) = \begin{cases} 0 & \text{if } \ x,y \in [0,\frac{1}{2}] \\ 4 & \text{otherwise.} \end{cases}$$

We now have $\alpha(x,y) \geq \eta(x,y) \iff x,y \in [0,\frac{1}{2}]$.

Now, we verify the inequality (1.3) for $x,y \in [0,\frac{1}{2}]$. For this purpose we choose $\zeta(t,s) = \frac{5}{6}s - t$

For $x,y \in [0,\frac{1}{2}]$ we have $Tx = (\frac{x}{40})^2, Ty = (\frac{y}{40})^2$, and hence

$$\begin{aligned} \zeta(s^4 d(Tx,Ty), M_T(x,y)) &= \zeta(4^4 d(Tx,Ty), M_T(x,y)) \\ &= \frac{5}{6} M_T(x,y) - 256 d(Tx,Ty) \\ &\geq \frac{5}{6} d(x,y) - 256 d(Tx,Ty) \\ &= \frac{5}{3}|x-y| - \frac{256}{800}|x^2 - y^2| \\ &\geq \frac{5}{3}|x-y| - \frac{256}{800}|x-y| \geq 0. \end{aligned}$$

Hence T is a generalized α-η-Z-contraction with respect to ζ.

Here we observe that the b-metric d is not continuous. For,

$$\lim_{n\to\infty} d(1, 1-\frac{1}{n}) = \frac{1}{2} \lim_{n\to\infty} \frac{1}{n} = 0.$$

Hence the sequence $1 - \frac{1}{n} \to 1$ as $n \to \infty$. But

$$\lim_{n\to\infty} d(0, 1-\frac{1}{n}) = \lim_{n\to\infty} 2|1-\frac{1}{n}| = 2 \neq \frac{1}{2} = d(0,1).$$

In Section 2, we prove our main results in which we study the existence of fixed points of generalized α-η-Z-contraction mapping with respect to ζ in complete b-metric spaces. In Section 3, we extend the main results of Section 2 to partially ordered complete b-metric spaces. In Section 4, we provide corollaries and examples in support of our results.

2. Main results

Theorem 2.1. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0,\infty)$ be mappings.

Suppose that the following conditions are satisfied:

(i) T is a generalized α-η-Z-contraction with respect to ζ,

(ii) T is a triangular α-orbital admissible mapping with respect to η,
(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$, and
(iv) T is an α-η-continuous mapping.
Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be as in (iii), i.e., $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. We define a sequence $\{x_n\}$ in X by $x_{n+1} = T^n x_1 = Tx_n$ for all $n \in \mathbb{N}$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, we have $Tx_{n_0} = x_{n_0}$, so that x_{n_0} is a fixed point of T and we are through.

Hence, without loss of generality, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. By Lemma 1.12, we have $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. From (1.3), we have

(2.1)
$$\zeta(s^4 d(x_{n+1}, x_{n+2}), M_T(x_n, x_{n+1})) = \zeta(s^4 d(Tx_n, Tx_{n+1}), M_T(x_n, x_{n+1})) \geq 0,$$

where

$$\begin{aligned} M_T(x_n, x_{n+1}) = {} & \max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\ & \frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2s}\} \\ = {} & \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ & \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2s}\} \\ \leq {} & \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ & \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\} \\ = {} & \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

Hence $M_T(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$.

Suppose that $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N}$. Then we have

$$M_T(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2}).$$

Hence, from (2.1), we have

$$\begin{aligned} 0 &\leq \zeta(s^4 d(x_{n+1}, x_{n+2}), M_T(x_n, x_{n+1})) \\ &= \zeta(s^4 d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2})) \\ &< d(x_{n+1}, x_{n+2}) - s^4 d(x_{n+1}, x_{n+2}) \leq 0, \end{aligned}$$

a contradiction. Hence $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Therefore, $\{d(x_n, x_{n+1})\}$ is decreasing and bounded below. Thus there exist $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$.

Suppose that $r > 0$. Now, using condition (ζ_3), with $t_n = d(x_{n+1}, x_{n+2})$ and $s_n = d(x_n, x_{n+1})$, we have $0 \leq \limsup_{n\to\infty} \zeta(s^4 d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) < 0$, a contradiction. Therefore, $r = 0$ i.e.,

(2.2)
$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Now, we consider the following two cases

<u>*Case*</u> (i) : $s = 1$.

In this case (X, d) is a metric space. Then by Lemma 1.4 there exist $\epsilon > 0$ and sequence of positive integers $\{n_k\}$ and $\{m_k\}$ such that $n_k > m_k \geq k$ satisfying

$$d(x_{m_k}, x_{n_k}) \geq \epsilon. \tag{2.3}$$

Let us choose the smallest n_k satisfying (2.3), then we have $n_k > m_k \geq k$ with $d(x_{m_k}, x_{n_k}) \geq \epsilon$ and $d(x_{m_k}, x_{n_k-1}) < \epsilon$ satisfying (i)- (iv) of Lemma 1.4.

Hence we have

$$\begin{aligned} &M_s(x_{m_k}, x_{n_k}) \\ &= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{n_k}), \frac{d(x_{m_k}, Tx_{n_k}) + d(x_{n_k}, Tx_{m_k})}{2}\}. \end{aligned}$$

On taking limit as $k \to \infty$ we have $\lim\limits_{k\to\infty} M_s(x_{m_k}, x_{n_k}) = \epsilon$.

Using condition (ζ_3) with $t_k = d(x_{m_k+1}, x_{n_k+1})$ and $s_k = M(x_{m_k}, x_{n_k})$, we have $0 \leq \limsup\limits_{k\to\infty} \zeta(d(x_{m_k+1}, x_{n_k+1}), M_s(x_{m_k}, x_{n_k})) < 0$, a contradiction. Thus $\{x_n\}$ is a Cauchy sequence.

<u>*Case*</u> (ii) : $s > 1$.

Then by Lemma 1.5 there exist $\epsilon > 0$ and sequence of positive integers $\{n_k\}$ and $\{m_k\}$ such that $n_k > m_k \geq k$ satisfying

$$d(x_{m_k}, x_{n_k}) \geq \epsilon. \tag{2.4}$$

Let us choose the smallest n_k satisfying (2.4), then we have $n_k > m_k \geq k$ with $d(x_{m_k}, x_{n_k}) \geq \epsilon$ and $d(x_{m_k}, x_{n_k-1}) < \epsilon$ satisfying (i)- (iv) of Lemma 1.5.

(2.5)

$$\begin{aligned} \epsilon \leq d(x_{m_k}, x_{n_k}) &\leq M_s(x_{m_k}, x_{n_k}) \\ &= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{n_k}), \\ &\qquad \frac{d(Tx_{m_k}, x_{n_k}) + d(x_{m_k}, Tx_{n_k})}{2s}\}. \end{aligned}$$

Letting $n \to \infty$ in (2.5) and using (i) - (iv) of Lemma 1.5, we have

$$\epsilon \leq \limsup_{k\to\infty} M_s(x_{m_k}, x_{n_k}) \leq \max\{s\epsilon, 0, \frac{s^2\epsilon + s^2\epsilon}{2s}\} = s\epsilon. \tag{2.6}$$

By Lemma 1.12 we have $\alpha(x_{m_k}, x_{n_k}) \geq \eta(x_{m_k}, x_{n_k})$. Hence from (1.3) we have $0 \leq \zeta(s^4 d(Tx_{m_k}, Tx_{n_k}), M_T(x_{m_k}, x_{n_k}))$.

Now we have

(2.7)

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \zeta(s^4 d(Tx_{m_k}, Tx_{n_k}), M_T(x_{m_k}, x_{n_k})) \\ &\leq \limsup_{k\to\infty} [M_T(x_{m_k}, x_{n_k}) - s^4 d(Tx_{m_k}, Tx_{n_k})] \\ &\leq \limsup_{k\to\infty} M_T(x_{m_k}, x_{n_k}) - s^4 \liminf_{k\to\infty} d(Tx_{m_k}, Tx_{n_k}) \leq s\epsilon - s^4(\frac{\epsilon}{s^2}) < 0, \end{aligned}$$

a contradiction. So we conclude that $\{x_n\}$ is a Cauchy sequence in (X,d).

Since X is a complete b-metric space then, there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$. Since T is α-η-continuous and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, we have $x^* = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} Tx_n = T \lim\limits_{n\to\infty} x_n = Tx^*$. Hence T has a fixed point. $\square$

In the following theorem, we replace the α-η-continuity of T by another condition.

Theorem 2.2. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0,\infty)$ be mappings.

Suppose that the following conditions are satisfied:

(i) T is a generalized α-η-Z-contraction with respect to ζ,

(ii) T is a triangular α-orbital admissible mapping with respect to η,

(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$, and

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq \eta(x_{n_k}, x^*)$ for all $k \in \mathbb{N}$.

Then $\{T^n x_1\}$ converges to an element x^* of X and x^* is a fixed point of T.

Proof. By using similar arguments as in the proof of Theorem 2.1, we obtain that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ converges to $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

By (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq \eta(x_{n_k}, x^*)$ for all $k \in \mathbb{N}$. Hence from (1.3) we have

(2.8)
$$\begin{aligned} 0 \leq \zeta(s^4 d(Tx_{n_k}, Tx^*), M_T(x_{n_k}, x^*)) &= \zeta(s^4 d(x_{n_k+1}, Tx^*), M_T(x_{n_k}, x^*)) \\ &< M_T(x_{n_k}, x^*) - s^4 d(x_{n_k+1}, Tx^*), \end{aligned}$$

which implies that $s^4 d(x_{n_k+1}, Tx^*) < M_T(x_{n_k}, x^*)$.

Now, we have

$$sd(x_{n_k+1}, Tx^*) \leq s^4 d(x_{n_k+1}, Tx^*) < M_T(x_{n_k}, x^*) \text{ and} \tag{2.9}$$

$$\begin{aligned} d(x^*, Tx^*) \leq M_T(x_{n_k}, x^*) &= \max\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \\ &\qquad \frac{d(x_{n_k}, Tx^*) + d(Tx_{n_k}, x^*)}{2}\} \\ &\leq \max\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \\ &\qquad \frac{d(x_{n_k}, x^*) + d(x^*, Tx^*) + d(Tx_{n_k}, x^*)}{2}\}, \end{aligned}$$

On taking limits as $n \to \infty$ we have

$$d(x^*, Tx^*) \leq \lim_{k\to\infty} M_T(x_{n_k}, x^*) \leq d(x^*, Tx^*).$$

Therefore $\lim\limits_{k\to\infty} M_T(x_{n_k}, x^*) = d(x^*, Tx^*)$.

From (2.9) we now have

(2.10)
$$d(x^*, Tx^*) \leq sd(x^*, Tx_{n_k}) + sd(Tx_{n_k}, Tx^*) \leq sd(x^*, Tx_{n_k}) + M_T(x_{n_k}, x^*)$$

On taking limit as $k \to \infty$ on (2.10), we have

$$d(x^*, Tx^*) \leq s \lim_{k\to\infty} d(x_{n_k+1}, Tx^*) \leq d(x^*, Tx^*). \tag{2.11}$$

Hence we have

$$\lim_{k\to\infty} d(x_{n_k+1}, Tx^*) = \frac{1}{s} d(x^*, Tx^*). \tag{2.12}$$

Suppose $x^* \neq Tx^*$. Now by choosing $t_k = sd(x_{n_k+1}, Tx^*)$ and $s_k = M_T(x_{n_k}, x^*)$ from property (ζ_3), it follows that

$$0 \leq \limsup_{k\to\infty} \zeta(s^4 d(Tx_{n_k}, Tx^*), M_T(x_{n_k}, x^*)) < 0,$$

a contradiction. Hence $Tx^* = x^*$. Therefore T has a fixed point. □

Theorem 2.3. In addition to the hypotheses of Theorem 2.1 (Theorem 2.2) assume the following.

Condition (H): for all $x \neq y \in X$, there exists $v \in X$ such that $\alpha(x, v) \geq \eta(x, v), \alpha(y, v) \geq \eta(y, v)$ and $\alpha(v, Tv) \geq \eta(v, Tv)$.

Then T has a unique fixed point.

Proof. Suppose that z^* and y^* are two fixed points of T with $z^* \neq y^*$. Then by our assumption, there exists a $v \in X$ such that $\alpha(z^*, v) \geq \eta(z^*, v), \alpha(y^*, v) \geq \eta(y^*, v)$ and $\alpha(v, Tv) \geq \eta(v, Tv)$ so that condition (iii) of Theorem 2.1 (Theorem 2.2) holds with $x_1 = v$, also. Now, by applying Theorem 2.1 (Theorem 2.2), we deduce that $\{T^n v\}$ converges to a fixed point x^* (say) of T and hence the sequence is $\{d(x^*, T^n v)\}$ is bounded.

Now, since $d(z^*, T^n v) \leq s[d(z^*, x^*) + d(x^*, T^n v)]$, we have the sequence $\{d(z^*, T^n v)\}$ is bounded. Therefore there exists a subsequence $\{d(z^*, T^{n_k} v)\}$ of $\{d(z^*, T^n v)\}$ such that $\lim_{n\to\infty} d(z^*, T^{n_k} v) = \ell$, for some nonnegative real ℓ.

Now, we have

$$\begin{aligned} d(z^*, T^{n_k} v) &\leq M_T(z^*, T^{n_k} v) \\ &= \max\{d(z^*, T^{n_k} v), d(z^*, Tz^*), d(T^{n_k} v, T^{n_k+1} v), \\ &\qquad \frac{d(z^*, T^{n_k+1} v) + d(Tz^*, T^{n_k} v)}{2s}\} \\ &= \max\{d(z^*, T^{n_k} v), d(T^{n_k} v, T^{n_k+1} v), \\ &\qquad \frac{d(z^*, T^{n_k+1} v) + d(z^*, T^{n_k} v)}{2s}\} \\ &\leq \max\{d(z^*, T^{n_k} v), d(T^{n_k} v, T^{n_k+1} v), \\ &\qquad \frac{s[d(z^*, T^{n_k}) + d(T^{n_k}, T^{n_k+1} v)] + d(z^*, T^{n_k} v)}{2s}\}. \end{aligned} \tag{2.13}$$

On taking limits as $k \to \infty$ we have $\lim\limits_{n\to\infty} M_T(z^*, T^{n_k}v) = \ell$.

We now show that $\ell = 0$. Suppose $\ell > 0$.

Since T is triangular α-orbital admissible with respect to η, we have $\alpha(v, T^n v) \geq \eta(v, T^n v)$ and hence $\alpha(z^*, T^n v) \geq \eta(z^*, T^n v)$ and $\alpha(y^*, T^n v) \geq \eta(y^*, T^n v)$ for all $n \in \mathbb{N}$.

Now, from (1.3) we have $\zeta(s^4 d(z^*, T^{n_k+1}v), M_T(z^*, T^{n_k}v)) \geq 0$.

Hence, we have $s^4 d(z^*, T^{n_k+1}v) \leq M_T(z^*, T^{n_k}v)$ which implies that

$$sd(z^*, T^{n_k+1}v) \leq s^3 d(z^*, T^{n_k+1}v) \leq M_T(z^*, T^{n_k}v).$$

Now, we have

$$d(z^*, T^{n_k}v) \leq sd(z^*, T^{n_k+1}v) + sd(T^{n_k+1}v, T^{n_k}v) \leq M_T(z^*, T^{n_k}v) + sd(z^*, T^{n_k}v).$$

On taking limits as $k \to \infty$ we have

$$\lim_{n\to\infty} sd(z^*, T^{n_k+1}v) = \ell.$$

Now, by choosing $t_k = sd(z^*, T^{n_k+1}v)$ and $s_k = M_T(z^*, T^{n_k}v)$, from property (ζ_3), it follows that

$$0 \leq \limsup_{k\to\infty} \zeta(s^4 d(z^*, T^{n_k+1}v), M_T(z^*, T^{n_k}v)) < 0,$$

a contradiction. Hence $\ell = 0$. Hence $T^{n_k}v \to z^*$ as $n \to \infty$. Therefore $z^* = x^*$.

Similarly we can prove that $y^* = x^*$.

Thus it follows that $z^* = y^*$, a contradiction. Hence T has a unique fixed point. □

3. A fixed point result in partially ordered b-metric spaces

Definition 3.1. Let $(X, \preceq)$ be a partially ordered set. If there exists a b-metric d on X with coefficient $s \geq 1$, such that (X, d) is complete, then we say that $(X, \preceq, d)$ is a partially ordered complete b-metric space with coefficient $s \geq 1$.

Theorem 3.2. Let $(X, \preceq, d)$ be a partially ordered complete b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a selfmap of X. Assume that the following conditions are satisfied:

(i) there exists a simulation mapping ζ such that

$$\zeta(s^4 d(Tx, Ty), M_T(x, y)) \geq 0, \text{ for any } x, y \in X \text{with } x \preceq y,$$

where $M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$,

(ii) T is a nondecreasing,

(iii) there exists an $x_1 \in X$ such that $x_1 \preceq Tx_1$,

(iv) either T is continuous or if $\{x_n\}$ is a decreasing sequence with $x_n \to x^*$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x^*$ for all $k \in \mathbb{N}$.

Then $\{T^n x_1\}$ converges to an element x^* of X and x^* is a fixed point of T.

Further, if for all $x \neq y \in X$, there exists $v \in X$ such that $x \preceq v, y \preceq v$ and $v \preceq Tv$, then T has a unique fixed point.

Proof. We define functions $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 3 & \text{if } x \preceq y \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } \eta(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ 4 & \text{otherwise.} \end{cases}$$

Now, for any $x, y \in X$, $\alpha(x,y) \geq \eta(x,y)$ if and only if $x \preceq y$. By (i), we have $\zeta(s^4 d(Tx,Ty), M_T(x,y)) \geq 0$. Suppose that $\alpha(x,Tx) \geq \eta(x,Tx)$, then we have $x \preceq Tx$. Since T is nondecreasing, we have $Tx \preceq TTx$ which implies that $\alpha(Tx,TTx) \geq \eta(Tx,TTx)$, hence T is α-orbital admissible with respect to η.

Further, suppose that $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$, so that we have $x \preceq y$ and $y \preceq Ty$. It follows that $x \preceq Ty$ and hence $\alpha(x,Ty) \geq \eta(x,Ty)$. Thus T is triangular α-orbital admissible with respect to η. Hence T satisfies all the hypotheses of Theorem 2.1 (Theorem 2.2) and T has a fixed point.

Moreover, if for all $x \neq y \in X$, there exists a $v \in X$ such that $x \preceq v, y \preceq v$ and $v \preceq Tv$, then we have $\alpha(x,v) \geq \eta(x,v), \alpha(y,v) \geq \eta(y,v)$ and $\alpha(v,Tv) \geq \eta(v,Tv)$. Hence by Theorem 2.3, T has a unique fixed point. □

4. Corollaries and examples

Corollary 4.1. Let (X,d) be a complete metric space. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0,\infty)$ be mappings.

Suppose that the following conditions are satisfied:

(i) there exists a simulation mapping ζ such that for any $x, y \in X$, $\alpha(x,y) \geq \eta(x,y)$ implies $\zeta(d(Tx,Ty), M(x,y)) \geq 0$, where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$,

(ii) T is a triangular α-orbital admissible mapping with respect to η,

(iii) there exists an $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$, and

(vi) T is an α-η-continuous mapping, or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x^*) \geq \eta(x_{n_k},x^*)$ for all $k \in \mathbb{N}$.

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Moreover, if for all $x \neq y \in X$, there exists $v \in X$ such that $\alpha(x,v) \geq \eta(x,v), \alpha(y,v) \geq \eta(y,v)$ and $\alpha(v,Tv) \geq \eta(v,Tv)$, then T has a unique fixed point.

Proof. Follows from Theorem 2.3 by taking $s = 1$. □

Remark 4.2. Theorem 1.20 follows as a corollary to Corollary 4.1 by choosing $\alpha(x,y) = \eta(x,y) = 1$ for all $x, y \in X$, which in turn Theorem 1.20 follows as a corollary to Theorem 2.3.

Corollary 4.3. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0,\infty)$ be mappings. Assume that there exist two continuous function $\psi, \varphi : [0,\infty) \to [0,\infty)$ with $\psi(t) < t \leq \varphi(t)$ for all $t > 0$ and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ such that for any $x, y \in X$ with $\alpha(x,y) \geq \eta(x,y)$ implies

$$\varphi(s^4 d(Tx,Ty)) \leq \psi(M_T(x,y)), \tag{4.1}$$

where $M_T(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$.

Suppose that the following conditions are satisfied:

(i) T is a triangular α-orbital admissible mapping;

(ii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$; and

(iii) either T is an α-η-continuous mapping, or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq \eta(x_{n_k}, x^*)$ for all $k \in \mathbb{N}$.

Then $\{T^n x_1\}$ converges to an element x^* of X and x^* is a fixed point of T.

Proof. The conclusion of this corollary follows from Theorem 2.1(Theorem 2.2) by taking $\zeta(t,s) = \psi(s) - \varphi(t)$ for all $t, s \geq 0$. □

Corollary 4.4. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0,\infty)$ be mappings.

Suppose that the following conditions are satisfied:

(i) there exists a simulation mapping ζ such that for any $x, y \in X$ with $\alpha(x,y) \geq 1$ implies $\zeta(s^4 d(Tx,Ty), M_T(x,y)) \geq 0$, where $M_T(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$.

(ii) T is a triangular α-orbital admissible mapping,

(iii) there exists an $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$, and

(iv) T is an α-continuous mapping, or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Moreover, if for all $x \neq y \in X$, there exists $v \in X$ such that $\alpha(x,v) \geq 1$, $\alpha(y,v) \geq 1$ and $\alpha(v,Tv) \geq 1$, then T has a unique fixed point.

Proof. Follows from Theorem 2.1(Theorem 2.2) and Theorem 2.3 by taking $\eta(x,y) = 1$ for all $x, y \in X$. □

Example 4.5. Let $X = [0,\infty)$ and let $d : X \times X \to [0,\infty)$ be defined by $d(x,y) = |x-y|^2$. Clearly (X,d) is a b-metric space with coefficient $s = 2$. We define $T : X \to X$ by

$$Tx = \begin{cases} 1 - \frac{x}{6} & \text{if } x \in [0,1] \\ 2x - 2 & \text{if } x \in (1,\infty) \end{cases}$$

and $\alpha, \eta : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 2 + xy & \text{if } x,y \in [0,1] \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } \eta(x,y) = \begin{cases} 1 + xy & \text{if } x,y \in [0,1] \\ 4 & \text{otherwise.} \end{cases}$$

We now have $\alpha(x,y) \geq \eta(x,y) \iff x,y \in [0,1]$. Let $\alpha(x,Tx) \geq \eta(x,Tx)$, then $x, Tx \in [0,1]$ and hence $Tx, TTx \in [0,1]$, since for any $x \in [0,1]$ we have $Tx \in [0,1]$. therefore $\alpha(Tx,TTx) \geq \eta(Tx,TTx)$. Hence T is α-orbital admissible with respect to η.

Suppose that $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$, then $x, y, Ty \in [0,1]$ which implies that $\alpha(x,Ty) \geq \eta(x,Ty)$. Hence T is triangular α-orbital admissible with respect to η.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Then $\{x_n\} \subseteq [0,1]$ for all $n \in \mathbb{N}$. Then we have
$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty}(1 - \frac{x_n}{6}) = 1 - \lim_{n\to\infty} \frac{x_n}{6} = 1 - \frac{x}{6} = Tx$. Hence T is α-η-continuous.

Now, we verify the inequality (1.3) for $x, y \in [0,1]$. For $x = y$ the inequality holds trivially, hence we verify for $x \neq y$.

We define $\zeta : [0,\infty) \times [0,\infty) \to [0,\infty)$ by $\zeta(t,s) = \frac{s}{1+s} - t$.

Since $\alpha(x,y) \geq \eta(x,y)$ if and only if $x, y \in [0,1]$, we have $Tx = 1 - \frac{x}{5}$ and $Ty = 1 - \frac{y}{5}$. Hence

$$\begin{aligned}
&\zeta(s^4 d(Tx,Ty), M_T(x,y)) \\
&= \frac{M_T(x,y)}{1 + M_T(x,y)} - s^4 d(Tx,Ty) \\
&\geq \frac{d(x,y)}{1 + d(x,y)} - 16 d(Tx,Ty) \\
&= \frac{|x-y|^2}{1 + |x-y|^2} - \frac{16}{36}|x-y|^2 \\
&\geq \frac{16}{36}|x-y|^2 - \frac{16}{36}|x-y|^2 = 0.
\end{aligned}$$

Hence T satisfies all the hypothesis of Theorem 2.1 with $x = \frac{6}{7}$ and $x = 2$ are fixed points of T.

Here we observe that 'Condition (H)' of Theorem 2.3 fails to hold. For, choose $x = 5$ and $x = 6$, then there is no $v \in X$ such that $\alpha(5,v) \geq \eta(5,v)$ and $\alpha(6,v) \geq \eta(6,v)$.

Remark 4.6. In the usual metric, the inequality (1.2) fails. For, by choosing $x = 3$ and $y = 4$, we have $M_T(3,4) = 2$ and $d(T3, T4) = 2$ and hence we have $\zeta(d(T3,T4), M_T(3,4)) = \zeta(2,2) < 0$, for any simulation function ζ.

Hence Theorem 1.20 is not applicable.

Example 4.7. Let $X = [0,\infty)$ and let $d : X \times X \to [0,\infty)$ be defined by $d(x,y) = |x-y|^2$. Hence (X,d) is a complete b-metric space with coefficient $s = 2$. We define $T : X \to X$ by

$$Tx = \begin{cases} \frac{2}{11}x & \text{if } x \in [0,6] \\ \frac{x}{6} - 1 & \text{if } x \in (6,\infty), \end{cases}$$

and $\alpha, \eta : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 2 & \text{if } x, y \in [0,6], \\ 3 & \text{if } x \in (6,\infty), y = 0, \\ 1 + xy & \text{otherwise,} \end{cases}$$

and

$$\eta(x,y) = \begin{cases} 0 & \text{if } x,y \in [0,6], \\ 1 & \text{if } x \in (6,\infty), y = 0, \\ 4+xy & \text{otherwise.} \end{cases}$$

We now have $\alpha(x,y) \geq \eta(x,y) \iff x,y \in [0,6]$ and $x \in (6,\infty), y = 0$.

Suppose that $\alpha(x,Tx) \geq \eta(x,Tx)$, then we have $x \in [0,6)$ and hence $\alpha(Tx,TTx) \geq \eta(Tx,TTx)$. Therefore T is α-orbital admissible with respect to η.

Suppose that $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$, then we have $x,y \in [0,6]$, or $x \in (6,\infty)$ and $y = 0$, which implies that $x,Ty \in [0,6]$, or $x \in (6,\infty)$ and $Ty = y = 0$ and hence $\alpha(x,Ty) \geq \eta(x,Ty)$. Therefore T is triangular α-orbital admissible with respect to η.

We now verify the inequality (1.3). For this purpose we define $\zeta : [0,\infty) \times [0,\infty) \to (-\infty,\infty)$ by $\zeta(t,s) = \frac{3}{4}s - t$.

Now we have the following cases.

<u>Case</u> (i) : $x,y \in [0,6)$

In this case $Tx = \frac{2}{11}x, Ty = \frac{2}{11}y$, then we have

$$\begin{aligned} &\zeta(2^4 d(Tx,Ty) = M_T(0,y)) \qquad \zeta(16d(Tx,Ty), M_T(x,y)) \\ &= \tfrac{3}{4}M_T(x,y) - 16d(Tx,Ty) \\ &\geq \frac{3}{4}d(x,y) - 16(\frac{4}{121}|x-y|^2) \\ &= \frac{3}{4}|x-y|^2 - 16(\frac{4}{121}|x-y|^2) \geq 0. \end{aligned}$$

<u>Case</u> (ii) : $x \in (6,\infty), y = 0$

In this case $Tx = \frac{x}{6} - 1, T0 = 0$, then we have

$$\begin{aligned} \zeta(16d(Tx,T0), M_T(x,0)) &= \frac{3}{4}M_T(x,0) - 16(|\frac{y}{6} - 1|^2) \\ &= \frac{3}{4}M_T(x,0) - 16(\frac{1}{36}|y-6|^2) \\ &\geq \frac{3}{4}y^2 - \frac{16}{36}|y-6|^2 \geq 0. \end{aligned}$$

Hence T satisfies the inequality (1.3). Also, since for any $x \neq y \in X$ we have $\alpha(x,0) \geq \eta(x,0), \alpha(y,0) \geq \eta(y,0)$ and $\alpha(0,T0) \geq \eta(0,T0)$, T satisfies 'Condition (H)'. Hence T satisfies all the hypotheses of Theorem 2.3, and $x = 0$ is the unique fixed point of T.

Example 4.8. Let $X = [0,\infty)$ and a b-metric be as defined in Example 1.22.

Further, let T, α, η be as in Example 1.22. Then clearly T satisfies all the hypotheses of Theorem 2.1 and $x = 0$ and $x = 1$ are two fixed points.

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