Lattices of regular closed sets in closure spaces: semidistributivity and Dedekind-MacNeille completions

Friedrich Wehrung

LMNO (Caen, France)
E-mail: friedrich.wehrung01@unicaen.fr
URL: http://www.math.unicaen.fr/~wehrung

NSAC 2013, Novi Sad, June 2013
Joint work with Luigi Santocanale
What is the permutohedron?

- The **permutohedron on $n$ letters**, denoted by $P(n)$, can be defined as the set of all permutations of $n$ letters, with the ordering

What is the permutohedron?

The permutohedron on \( n \) letters, denoted by \( P(n) \), can be defined as the set of all permutations of \( n \) letters, with the ordering

\[
\alpha \leq \beta \iff \text{Inv}(\alpha) \subseteq \text{Inv}(\beta),
\]

where we set \([n] = \{1, 2, \ldots, n\}\) and \(I_n = \{(i, j) \in [n] \times [n] | i < j\}\). Alternate definition: \(P(n) = \{\text{Inv}(\sigma) | \sigma \in S_n\}\), ordered by \(\subseteq\).
What is the permutohedron?

The permutohedron on \( n \) letters, denoted by \( P(n) \), can be defined as the set of all permutations of \( n \) letters, with the ordering

\[
\alpha \leq \beta \iff \text{Inv}(\alpha) \subseteq \text{Inv}(\beta),
\]

where we set

\[
[n] = \{1, 2, \ldots, n\},
\]

\[
\mathcal{J}_n = \{(i, j) \in [n] \times [n] \mid i < j\},
\]

\[
\text{Inv}(\alpha) = \{(i, j) \in \mathcal{J}_n \mid \alpha^{-1}(i) > \alpha^{-1}(j)\}.
\]
What is the permutohedron?

- The permutohedron on $n$ letters, denoted by $P(n)$, can be defined as the set of all permutations of $n$ letters, with the ordering

$$\alpha \leq \beta \iff \text{Inv}(\alpha) \subseteq \text{Inv}(\beta),$$

where we set

$$[n] = \{1, 2, \ldots, n\},$$

$$J_n = \{(i, j) \in [n] \times [n] \mid i < j\},$$

$$\text{Inv}(\alpha) = \{(i, j) \in J_n \mid \alpha^{-1}(i) > \alpha^{-1}(j)\}.$$

- Alternate definition: $P(n) = \{\text{Inv}(\sigma) \mid \sigma \in \mathfrak{S}_n\}$, ordered by $\subseteq$. 
What are the $\text{Inv}(\sigma)$?

- Both $\text{Inv}(\sigma)$ and $J_n \setminus \text{Inv}(\sigma)$ are transitive relations on $[n]$. 
What are the $\text{Inv}(\sigma)$?

- Both $\text{Inv}(\sigma)$ and $\mathcal{J}_n \setminus \text{Inv}(\sigma)$ are transitive relations on $[n]$. 
  \textbf{(Proof:} let $(i, j) \in \mathcal{J}_n$. Then $(i, j) \in \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) > \sigma^{-1}(j)$; $(i, j) \notin \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) < \sigma^{-1}(j)$.)
What are the $\text{Inv}(\sigma)$?

- Both $\text{Inv}(\sigma)$ and $\mathcal{J}_n \setminus \text{Inv}(\sigma)$ are transitive relations on $[n]$.  
  \text{(Proof: let } (i, j) \in \mathcal{J}_n. \text{ Then } (i, j) \in \text{Inv}(\sigma) \text{ iff } \sigma^{-1}(i) > \sigma^{-1}(j); (i, j) \notin \text{Inv}(\sigma) \text{ iff } \sigma^{-1}(i) < \sigma^{-1}(j).)\)

- Conversely, every subset $x \subseteq \mathcal{J}_n$, such that both $x$ and $\mathcal{J}_n \setminus x$ are transitive, is $\text{Inv}(\sigma)$ for a unique $\sigma \in \mathcal{S}_n$  
  \text{(Dushnik and Miller 1941, Guilbaud and Rosenstiehl 1963).}
What are the Inv($\sigma$)?

- Both Inv($\sigma$) and $J_n \setminus \text{Inv}(\sigma)$ are transitive relations on $[n]$.  
  \textit{(Proof:} Let $(i, j) \in J_n$. Then $(i, j) \in \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) > \sigma^{-1}(j)$; $(i, j) \notin \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) < \sigma^{-1}(j)$.\textit{)}
- Conversely, every subset $x \subseteq J_n$, such that both $x$ and $J_n \setminus x$ are transitive, is Inv($\sigma$) for a unique $\sigma \in S_n$ (Dushnik and Miller 1941, Guilbaud and Rosenstiehl 1963).
- Say that $x \subseteq J_n$ is \textbf{closed} if it is transitive, \textbf{open} if $J_n \setminus x$ is closed, and \textbf{clopen} if it is both closed and open.
What are the $\text{Inv}(\sigma)$?

- Both $\text{Inv}(\sigma)$ and $\mathcal{J}_n \setminus \text{Inv}(\sigma)$ are transitive relations on $[n]$.  
  \textit{(Proof: let $(i, j) \in \mathcal{J}_n$. Then $(i, j) \in \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) > \sigma^{-1}(j)$; $(i, j) \notin \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) < \sigma^{-1}(j)$.)}

- Conversely, every subset $x \subseteq \mathcal{J}_n$, such that both $x$ and $\mathcal{J}_n \setminus x$ are transitive, is $\text{Inv}(\sigma)$ for a unique $\sigma \in \mathcal{S}_n$ (Dushnik and Miller 1941, Guilbaud and Rosenstiehl 1963).

- Say that $x \subseteq \mathcal{J}_n$ is \textbf{closed} if it is transitive, \textbf{open} if $\mathcal{J}_n \setminus x$ is closed, and \textbf{clopen} if it is both closed and open.

- Hence $P(n) = \{x \subseteq \mathcal{J}_n \mid x \text{ is clopen}\}$, ordered by $\subseteq$. 
What are the $\text{Inv}(\sigma)$?

Both $\text{Inv}(\sigma)$ and $\mathcal{J}_n \setminus \text{Inv}(\sigma)$ are transitive relations on $[n]$. 
*(Proof: let $(i, j) \in \mathcal{J}_n$. Then $(i, j) \in \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) > \sigma^{-1}(j)$; $(i, j) \notin \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) < \sigma^{-1}(j)$.)

Conversely, every subset $x \subseteq \mathcal{J}_n$, such that both $x$ and $\mathcal{J}_n \setminus x$ are transitive, is $\text{Inv}(\sigma)$ for a unique $\sigma \in S_n$ (Dushnik and Miller 1941, Guilbaud and Rosenstiehl 1963).

Say that $x \subseteq \mathcal{J}_n$ is closed if it is transitive, open if $\mathcal{J}_n \setminus x$ is closed, and clopen if it is both closed and open.

Hence $\mathcal{P}(n) = \{x \subseteq \mathcal{J}_n \mid x \text{ is clopen}\}$, ordered by $\subseteq$.

Observe that each $x \in \mathcal{P}(n)$ is a strict ordering. It can be proved (Dushnik and Miller 1941) that those are exactly the finite strict orderings of order-dimension 2.
The permutohedra $P(2)$, $P(3)$, and $P(4)$. 

Lattices of regular closed sets

The precursor

Regular closed sets

Transitive binary relations

Convexity and hyperplane arrangements

Graphs

Join-semilattices
The permutohedra $P(5)$ and $P(6)$
The permutohedron $P(7)$
Permutohedra are ortholattices

Theorem (Guilbaud and Rosenstiehl 1963)

The permutohedron $P(n)$ is a lattice, for every positive integer $n$.

The assignment $x \mapsto x^c = I_n \setminus x$ defines an orthocomplementation on $P(n)$:

$x \leq y \implies y^c \leq x^c$;

$(x^c)^c = x$;

$x \wedge x^c = 0$ (equivalently, $x \vee x^c = 1$).

Hence $P(n)$ is an ortholattice.
Permutohedra are ortholattices

Theorem (Guilbaud and Rosenstiehl 1963)

The permutohedron $P(n)$ is a lattice, for every positive integer $n$. 
Permutohedra are ortholattices

Theorem (Guilbaud and Rosenstiehl 1963)

The permutohedron $P(n)$ is a lattice, for every positive integer $n$.

The assignment $x \mapsto x^c = J_n \setminus x$ defines an orthocomplementation on $P(n)$:
Permutohedra are ortholattices

Theorem (Guilbaud and Rosenstiehl 1963)

The permutohedron $P(n)$ is a lattice, for every positive integer $n$.

The assignment $x \mapsto x^c = J_n \setminus x$ defines an orthocomplementation on $P(n)$:

\[
x \leq y \Rightarrow y^c \leq x^c ; \\
(x^c)^c = x ; \\
x \wedge x^c = 0 \quad (\text{equivalently, } x \vee x^c = 1).
\]
Permutohedra are ortholattices

Theorem (Guilbaud and Rosenstiehl 1963)

The permutohedron $P(n)$ is a lattice, for every positive integer $n$.

The assignment $x \mapsto x^c = J_n \setminus x$ defines an orthocomplementation on $P(n)$:

\[
\begin{align*}
x \leq y & \Rightarrow y^c \leq x^c ; \\
(x^c)^c & = x ; \\
x \land x^c & = 0 \quad \text{(equivalently, } x \lor x^c = 1) .
\end{align*}
\]

Hence $P(n)$ is an ortholattice.
Permutohedra are even more peculiar lattices

Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)

Permutohedra are even more peculiar lattices

Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)
Permutohedra are even more peculiar lattices

Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)

The permutohedron $P(n)$ is **semidistributive**, for every positive integer $n$. Thus it is also **pseudocomplemented**.
Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)

The permutohedron $P(n)$ is \textbf{semidistributive}, for every positive integer $n$. Thus it is also \textbf{pseudocomplemented}.

\textbf{Semidistributivity} means that
\[
x \lor z = y \lor z \Rightarrow x \lor z = (x \land y) \lor z, \text{ and, dually,}
\]
x \land z = y \land z \Rightarrow x \land z = (x \lor y) \land z.

Permutohedra are even more peculiar lattices

Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)

The permutohedron $P(n)$ is semidistributive, for every positive integer $n$. Thus it is also pseudocomplemented.

**Semidistributivity** means that

$x \lor z = y \lor z \Rightarrow x \lor z = (x \land y) \lor z$, and, dually,

$x \land z = y \land z \Rightarrow x \land z = (x \lor y) \land z$.

Theorem (Caspard 2000)
Permutohedra are even more peculiar lattices

**Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)**

The permutohedron $P(n)$ is **semidistributive**, for every positive integer $n$. Thus it is also **pseudocomplemented**.

**Semidistributivity** means that

\[
x \lor z = y \lor z \Rightarrow x \lor z = (x \land y) \lor z, \text{ and, dually, } \\
x \land z = y \land z \Rightarrow x \land z = (x \lor y) \land z.
\]

**Theorem (Caspard 2000)**

The permutohedron $P(n)$ is a **bounded homomorphic image of a free lattice**, for every positive integer $n$. 
Permutohedra are even more peculiar lattices

Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)

The permutohedron $P(n)$ is semidistributive, for every positive integer $n$. Thus it is also pseudocomplemented.

Semidistributivity means that

\[ x \lor z = y \lor z \Rightarrow x \lor z = (x \land y) \lor z, \text{ and, dually,} \]
\[ x \land z = y \land z \Rightarrow x \land z = (x \lor y) \land z. \]

Theorem (Caspard 2000)

The permutohedron $P(n)$ is a bounded homomorphic image of a free lattice, for every positive integer $n$.

This means that there are a finitely generated free lattice $F$ and a surjective lattice homomorphism $f : F \to P(n)$ such that each $f^{-1}\{x\}$ has both a least and a largest element.
Regular closed sets

- **Closure space**: pair \((\Omega, \varphi)\), where \(\varphi : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)\), with 
  \[ \varphi(\emptyset) = \emptyset, \; X \subseteq Y \Rightarrow \varphi(X) \subseteq \varphi(Y), \; X \subseteq \varphi(X), \]
  \[ \varphi \circ \varphi = \varphi. \]
Regular closed sets

- **Closure space**: pair \((\Omega, \varphi)\), where \(\varphi: \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)\), with \(\varphi(\emptyset) = \emptyset\), \(X \subseteq Y \Rightarrow \varphi(X) \subseteq \varphi(Y)\), \(X \subseteq \varphi(X)\), \(\varphi \circ \varphi = \varphi\).

- **Associated interior operator**: \(\check{\varphi}(X) = \Omega \setminus \varphi(\Omega \setminus X)\).
Regular closed sets

- **Closure space**: pair $(\Omega, \varphi)$, where $\varphi: \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$, with $\varphi(\emptyset) = \emptyset$, $X \subseteq Y \Rightarrow \varphi(X) \subseteq \varphi(Y)$, $X \subseteq \varphi(X)$, $\varphi \circ \varphi = \varphi$.

- **Associated interior operator**: $\tilde{\varphi}(X) = \Omega \setminus \varphi(\Omega \setminus X)$.

- **Closed sets**: $\varphi(X) = X$. **Open sets**: $\tilde{\varphi}(X) = X$. **Clopen sets**: $\varphi(X) = \tilde{\varphi}(X) = X$. **Regular closed sets**: $X = \varphi \tilde{\varphi}(X)$. 

Closure space: pair $(\Omega, \varphi)$, where $\varphi: \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$, with $\varphi(\emptyset) = \emptyset$, $X \subseteq Y \Rightarrow \varphi(X) \subseteq \varphi(Y)$, $X \subseteq \varphi(X)$, $\varphi \circ \varphi = \varphi$.

**Associated interior operator**: $\tilde{\varphi}(X) = \Omega \setminus \varphi(\Omega \setminus X)$.

**Closed sets**: $\varphi(X) = X$. **Open sets**: $\tilde{\varphi}(X) = X$. **Clopen sets**: $\varphi(X) = \tilde{\varphi}(X) = X$. **Regular closed sets**: $X = \varphi \tilde{\varphi}(X)$.
Closure space: pair \((\Omega, \varphi)\), where \(\varphi: \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)\), with \(\varphi(\emptyset) = \emptyset\), \(X \subseteq Y \Rightarrow \varphi(X) \subseteq \varphi(Y)\), \(X \subseteq \varphi(X)\), \(\varphi \circ \varphi = \varphi\).

Associated interior operator: \(\check{\varphi}(X) = \Omega \setminus \varphi(\Omega \setminus X)\).

Closed sets: \(\varphi(X) = X\). Open sets: \(\check{\varphi}(X) = X\). Clopen sets: \(\varphi(X) = \check{\varphi}(X) = X\). Regular closed sets: \(X = \varphi \check{\varphi}(X)\).

\(\text{Clop}(\Omega, \varphi)\) (the clopen sets) is contained in \(\text{Reg}(\Omega, \varphi)\) (the regular closed sets).
Regular closed sets

- **Closure space**: pair \((\Omega, \varphi)\), where \(\varphi: \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)\), with \(\varphi(\emptyset) = \emptyset\), \(X \subseteq Y \Rightarrow \varphi(X) \subseteq \varphi(Y)\), \(X \subseteq \varphi(X)\), \(\varphi \circ \varphi = \varphi\).

- **Associated interior operator**: \(\check{\varphi}(X) = \Omega \setminus \varphi(\Omega \setminus X)\).

- **Closed sets**: \(\varphi(X) = X\). **Open sets**: \(\check{\varphi}(X) = X\). **Clopen sets**: \(\varphi(X) = \check{\varphi}(X) = X\). **Regular closed sets**: \(X = \varphi \check{\varphi}(X)\).

- \(\text{Clop}(\Omega, \varphi)\) (the **clopen** sets) is contained in \(\text{Reg}(\Omega, \varphi)\) (the **regular closed** sets).

- \(\text{Reg}(\Omega, \varphi)\) is always an ortholattice (with \(x^\perp = \varphi(x^c)\)), but \(\text{Clop}(\Omega, \varphi)\) may not be a lattice.
Regular closed sets

- **Closure space**: pair $(\Omega, \varphi)$, where $\varphi : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$, with $\varphi(\emptyset) = \emptyset$, $X \subseteq Y \Rightarrow \varphi(X) \subseteq \varphi(Y)$, $X \subseteq \varphi(X)$, $\varphi \circ \varphi = \varphi$.

- **Associated interior operator**: $\check{\varphi}(X) = \Omega \setminus \varphi(\Omega \setminus X)$.

- **Closed sets**: $\varphi(X) = X$. **Open sets**: $\check{\varphi}(X) = X$. **Clopen sets**: $\varphi(X) = \check{\varphi}(X) = X$. **Regular closed sets**: $X = \varphi \check{\varphi}(X)$.

- $	ext{Clop}(\Omega, \varphi)$ (the clopen sets) is contained in $\text{Reg}(\Omega, \varphi)$ (the regular closed sets).

- $\text{Reg}(\Omega, \varphi)$ is always an ortholattice (with $x^\perp = \varphi(x^c)$), but $	ext{Clop}(\Omega, \varphi)$ may not be a lattice.

- Every orthoposet appears as some $	ext{Clop}(\Omega, \varphi)$ (Mayet 1982, Katrnoška 1982).
What happens for convex geometries?

**Convex geometry**: closure space \((\Omega, \varphi)\) such that (\(x\) closed, \(p, q \in \Omega \setminus x\), and \(\varphi(x \cup \{p\}) = \varphi(x \cup \{q\})\) \(\Rightarrow p = q\).
What happens for convex geometries?

**Convex geometry**: closure space \((\Omega, \varphi)\) such that \((x \text{ closed}, p, q \in \Omega \setminus x, \text{ and } \varphi(x \cup \{p\}) = \varphi(x \cup \{q\})) \Rightarrow p = q\).

**Theorem (Santocanale and W. 2012)**

For (more general spaces than) finite convex geometries, the lattice \(\text{Reg}(\Omega, \varphi)\) is always **pseudocomplemented**.
For a transitive binary relation $e \subseteq P \times P$, set $\Omega = e$, $\varphi(a) = \text{cl}(a) =$ transitive closure of $a$ ($\forall a \subseteq e$).
Transitive binary relations

- For a transitive binary relation $e \subseteq P \times P$, set $\Omega = e$, $\varphi(a) = \text{cl}(a)$ = transitive closure of $a$ ($\forall a \subseteq e$).
- For $e = J_n = \text{natural strict ordering on } [n]$, $\text{Reg}(e, \text{cl}) = \text{Clop}(e, \text{cl}) = P(n)$, the permutohedron.
Transitive binary relations

- For a transitive binary relation $e \subseteq P \times P$, set $\Omega = e$, $\varphi(a) = \text{cl}(a) = \text{transitive closure of } a$ ($\forall a \subseteq e$).
- For $e = J_n = \text{natural strict ordering on } [n]$, $\text{Reg}(e, \text{cl}) = \text{Clop}(e, \text{cl}) = P(n)$, the permutohedron.
- For $e = [n] \times [n]$, $\text{Reg}(e, \text{cl}) = \text{Clop}(e, \text{cl}) = \text{Bip}(n)$, the bipartition lattice on $[n]$ (Foata and Zeilberger 1996, Han 1996, Hetyei and Krattenthaler 2011).
Transitive binary relations

- For a transitive binary relation \( e \subseteq P \times P \), set \( \Omega = e \), \( \varphi(a) = \text{cl}(a) = \text{transitive closure of } a \) (\( \forall a \subseteq e \)).

- For \( e = J_n = \text{natural strict ordering on } [n] \), \( \text{Reg}(e, \text{cl}) = \text{Clop}(e, \text{cl}) = P(n) \), the permutohedron.

- For \( e = [n] \times [n] \), \( \text{Reg}(e, \text{cl}) = \text{Clop}(e, \text{cl}) = \text{Bip}(n) \), the bipartition lattice on \([n] \) (Foata and Zeilberger 1996, Han 1996, Hetyei and Krattenthaler 2011).

- \( \text{Bip}(n) \) contains an \( M_3 \) whenever \( n \geq 3 \).
A few things about $\text{Reg}(e, \text{cl})$

**Theorem (Santocanale and W. 2012)**

1. $\text{Reg}(e, \text{cl})$ is always the Dedekind-MacNeille completion of $\text{Clop}(e, \text{cl})$. Both are equal iff $e$ is square-free.

2. The lattice $\text{Reg}(e, \text{cl})$ is spatial (i.e., every element is a join of completely join-irreducible elements).

3. For $e$ finite, $\text{Reg}(e, \text{cl})$ is semidistributive iff it is a bounded homomorphic image of a free lattice, iff every connected component of $e$ is either antisymmetric or $E \times E$ with $|E| = 2$. 
A few things about Reg(e, cl)

Theorem (Santocanale and W. 2012)

1. Reg(e, cl) is always the Dedekind-MacNeille completion of Clop(e, cl). Both are equal iff e is square-free.
A few things about $\text{Reg}(e, \text{cl})$

**Theorem (Santocanale and W. 2012)**

1. $\text{Reg}(e, \text{cl})$ is always the Dedekind-MacNeille completion of $\text{Clop}(e, \text{cl})$. Both are equal iff $e$ is square-free.
2. The lattice $\text{Reg}(e, \text{cl})$ is spatial (i.e., every element is a join of completely join-irreducible elements).
A few things about \( \text{Reg}(e, \text{cl}) \)

Theorem (Santocanale and W. 2012)

1. \( \text{Reg}(e, \text{cl}) \) is always the Dedekind-MacNeille completion of \( \text{Clop}(e, \text{cl}) \). Both are equal iff \( e \) is square-free.

2. The lattice \( \text{Reg}(e, \text{cl}) \) is spatial (i.e., every element is a join of completely join-irreducible elements).

3. For \( e \) finite, \( \text{Reg}(e, \text{cl}) \) is semidistributive iff it is a bounded homomorphic image of a free lattice, iff every connected component of \( e \) is either antisymmetric or \( E \times E \) with \( \text{card } E = 2 \).
The lattice Bip(3)
The lattice Bip(4)
We are given a real affine space $\Delta$, and a subset $E \subseteq \Delta$. 
We are given a real affine space $\Delta$, and a subset $E \subseteq \Delta$.

Setting $\text{conv}_E(X) = \text{conv}(X) \cap E$, it is well-known that $(E, \text{conv}_E)$ is a convex geometry.
We are given a real affine space $\Delta$, and a subset $E \subseteq \Delta$.

Setting $\text{conv}_E(X) = \text{conv}(X) \cap E$, it is well-known that $(E, \text{conv}_E)$ is a convex geometry.

A subset $X \subseteq E$ is relatively convex if $X = \text{conv}_E(X)$; bi-convex if $X$ and $E \setminus X$ are both relatively convex; strongly bi-convex if $\text{conv}(X) \cap \text{conv}(E \setminus X) = \emptyset$. 

Clop$^*$ $(E, \text{conv}_E) = \{X \subseteq E | X$ is strongly bi-convex $\}$. 

The precursor
Regular closed sets
Transitive binary relations
Convexity and hyperplane arrangements
Graphs
Join-semilattices
We are given a real affine space $\Delta$, and a subset $E \subseteq \Delta$.

Setting $\text{conv}_E(X) = \text{conv}(X) \cap E$, it is well-known that $(E, \text{conv}_E)$ is a convex geometry.

A subset $X \subseteq E$ is relatively convex if $X = \text{conv}_E(X)$; bi-convex if $X$ and $E \setminus X$ are both relatively convex; strongly bi-convex if $\text{conv}(X) \cap \text{conv}(E \setminus X) = \emptyset$.

Strongly bi-convex $\Rightarrow$ bi-convex $\Rightarrow$ relatively convex.
Relatively convex sets

- We are given a real affine space $\Delta$, and a subset $E \subseteq \Delta$.
- Setting $\text{conv}_E(X) = \text{conv}(X) \cap E$, it is well-known that $(E, \text{conv}_E)$ is a convex geometry.
- A subset $X \subseteq E$ is relatively convex if $X = \text{conv}_E(X)$; bi-convex if $X$ and $E \setminus X$ are both relatively convex; strongly bi-convex if $\text{conv}(X) \cap \text{conv}(E \setminus X) = \emptyset$.
- Strongly bi-convex $\Rightarrow$ bi-convex $\Rightarrow$ relatively convex.
- $\text{Clo}^*(E, \text{conv}_E) = \{X \subseteq E \mid X$ is strongly bi-convex$\}$. 
Theorem (Santocanale and W. 2013)

Let $E$ be a subset in a real affine space $\Delta$. Then $\text{Reg}(E, \text{conv}_E)$ is the Dedekind-MacNeille completion of $\text{Clop}^*(E, \text{conv}_E)$ (thus of $\text{Clop}(E, \text{conv}_E)$).
Poset of regions of a central hyperplane arrangement

- **Central hyperplane arrangement** in $\mathbb{R}^d$: finite set $\mathcal{H}$ of hyperplanes through 0. **Regions** (set $\mathcal{R}$): connected components of $\mathbb{R}^d \setminus \bigcup \mathcal{H}$ (necessarily open). **Base region** $B \in \mathcal{R}$. 

Poset of regions: $\text{Pos}(\mathcal{H}, B) = \{ (\mathcal{R}, \leq) \}$, where $X \leq Y$ if $\text{sep}(B, X) \subseteq \text{sep}(B, Y)$. 

Theorem (Santocanale and W. 2013) $\text{Pos}(\mathcal{H}, B) \sim = \text{Clop}^\ast (E, \text{conv } E)$, for a suitably defined finite $E \subseteq \mathbb{R}^d$. 

- Lattices of regular closed sets
- The precursor
- Regular closed sets
- Transitive binary relations
- Convexity and hyperplane arrangements
- Graphs
- Join-semilattices
Poset of regions of a central hyperplane arrangement

- Central hyperplane arrangement in $\mathbb{R}^d$: finite set $\mathcal{H}$ of hyperplanes through 0. Regions (set $\mathcal{R}$): connected components of $\mathbb{R}^d \setminus \bigcup \mathcal{H}$ (necessarily open). Base region $B \in \mathcal{R}$.

- $\text{sep}(X, Y) = \{ H \in \mathcal{H} \mid H \text{ separates } X \text{ and } Y \}$, for $X, Y \in \mathcal{R}$.
Poset of regions of a central hyperplane arrangement

- **Central hyperplane arrangement** in $\mathbb{R}^d$: finite set $\mathcal{H}$ of hyperplanes through 0. Regions (set $\mathcal{R}$): connected components of $\mathbb{R}^d \setminus \bigcup \mathcal{H}$ (necessarily open). Base region $B \in \mathcal{R}$.

- $\text{sep}(X, Y) \triangleq \{ H \in \mathcal{H} \mid H \text{ separates } X \text{ and } Y \}$, for $X, Y \in \mathcal{R}$.

- **Poset of regions**: $\text{Pos}(\mathcal{H}, B) = (\mathcal{R}, \leq_B)$, where $X \leq_B Y$ if $\text{sep}(B, X) \subseteq \text{sep}(B, Y)$.

\[ \text{Poset of regions of a central hyperplane arrangement} \]
Poset of regions of a central hyperplane arrangement

- **Central hyperplane arrangement** in $\mathbb{R}^d$: finite set $\mathcal{H}$ of hyperplanes through 0. Regions (set $\mathcal{R}$): connected components of $\mathbb{R}^d \setminus \bigcup \mathcal{H}$ (necessarily open). Base region $B \in \mathcal{R}$.

- $\text{sep}(X, Y) = \{ H \in \mathcal{H} \mid H \text{ separates } X \text{ and } Y \}$, for $X, Y \in \mathcal{R}$.

- **Poset of regions**: $\text{Pos}(\mathcal{H}, B) = (\mathcal{R}, \leq_B)$, where $X \leq_Y Y$ if $\text{sep}(B, X) \subseteq \text{sep}(B, Y)$.

**Theorem (Santocanale and W. 2013)**

$\text{Pos}(\mathcal{H}, B) \cong \text{Clop}^*(E, \text{conv}_E)$, for a suitably defined finite $E \subseteq \mathbb{R}^d$. 
Partitions in graphs

- **Graph**: \((G, \sim)\), where \(\sim\) is an irreflexive, symmetric binary relation on \(G\).
Partitions in graphs

- **Graph**: \((G, \sim)\), where \(\sim\) is an irreflexive, symmetric binary relation on \(G\).
- \(\delta_G = \{X \subseteq G \text{ nonempty} \mid X \text{ is connected}\}\).
Partitions in graphs

- **Graph**: \((G, \sim)\), where \(\sim\) is an irreflexive, symmetric binary relation on \(G\).
- \(\delta_G = \{ X \subseteq G \text{ nonempty} \mid X \text{ is connected} \}\).
- \(X = X_1 \sqcup \cdots \sqcup X_n\) if \(X = X_1 \cup \cdots \cup X_n\) (disjoint union) and \(X\) and all the \(X_i\) are connected.
Partitions in graphs

- **Graph**: \((G, \sim)\), where \(\sim\) is an irreflexive, symmetric binary relation on \(G\).
- \(\delta_G = \{X \subseteq G \text{ nonempty} \mid X \text{ is connected}\}\).
- \(X = X_1 \sqcup \cdots \sqcup X_n\) if \(X = X_1 \cup \cdots \cup X_n\) (disjoint union) and \(X\) and all the \(X_i\) are connected.
- \(\text{cl}(x) = \text{closure of } x \text{ under } \sqcup, \forall x \subseteq \delta_G\).
Partitions in graphs

- **Graph**: \((G, \sim)\), where \(\sim\) is an irreflexive, symmetric binary relation on \(G\).
- \(\delta_G = \{X \subseteq G \text{ nonempty} \mid X \text{ is connected}\}\).
- \(X = X_1 \sqcup \cdots \sqcup X_n\) if \(X = X_1 \cup \cdots \cup X_n\) (disjoint union) and \(X\) and all the \(X_i\) are connected.
- \(\text{cl}(x) =\text{closure of } x \text{ under } \sqcup, \forall x \subseteq \delta_G\).
- \((\delta_G, \text{cl})\) is a convex geometry.
Semidistributivity and Dedekind-MacNeille

Theorem (Santocanale and W. 2013)

If $G$ is finite, then $\text{Reg}(\delta_G, \text{cl})$ is a bounded homomorphic image of a free lattice.
Semidistributivity and Dedekind-MacNeille

**Theorem (Santocanale and W. 2013)**

If $G$ is finite, then $\text{Reg}(\delta_G, \text{cl})$ is a **bounded homomorphic image of a free lattice**.

**Theorem (Santocanale and W. 2013)**

If $G$ is either a **finite block graph** or a **cycle**, then the “**extended permutohedron**” $\text{Reg}(\delta_G, \text{cl})$ on $G$ is the **Dedekind-MacNeille completion** of $\text{Clop}(\delta_G, \text{cl})$. 

Does not extend to all finite graphs (e.g., $K_3$, $3 - \text{edge}$).

For $G$ the underlying graph of a Dynkin diagram $G$, $\text{Clop}(\delta_G, \text{cl}) = \text{Reg}(\delta_G, \text{cl})$ and this lattice bears **mysterious connections** with the Coxeter lattice of type $G$ (thus with hyperplane arrangements).
Theorem (Santocanale and W. 2013)

If $G$ is finite, then $\text{Reg}(\delta_G, \text{cl})$ is a bounded homomorphic image of a free lattice.

Theorem (Santocanale and W. 2013)

If $G$ is either a finite block graph or a cycle, then the “extended permutohedron” $\text{Reg}(\delta_G, \text{cl})$ on $G$ is the Dedekind-MacNeille completion of $\text{Clop}(\delta_G, \text{cl})$.

- Does not extend to all finite graphs (e.g., $\mathcal{K}_{3,3}$ — edge).
Semidistributivity and Dedekind-MacNeille

Theorem (Santocanale and W. 2013)

If $G$ is finite, then $\text{Reg}(\delta_G, \text{cl})$ is a bounded homomorphic image of a free lattice.

Theorem (Santocanale and W. 2013)

If $G$ is either a finite block graph or a cycle, then the “extended permutohedron” $\text{Reg}(\delta_G, \text{cl})$ on $G$ is the Dedekind-MacNeille completion of $\text{Clop}(\delta_G, \text{cl})$.

- Does not extend to all finite graphs (e.g., $K_{3,3}$ – edge).
- For $G$ the underlying graph of a Dynkin diagram $\mathcal{G}$, $\text{Clop}(\delta_G, \text{cl}) = \text{Reg}(\delta_G, \text{cl})$ and this lattice bears mysterious connections with the Coxeter lattice of type $\mathcal{G}$ (thus with hyperplane arrangements).
The extended permutohedron on $D_4$, and the corresponding Coxeter lattice
The extended permutohedron on $\mathcal{K}_3$
The extended permutohedron on $K_4$
Join-semilattices

- For a join-semilattice $S$, set $\text{cl}(x) =$ join-closure of $x$. 

Theorem (Santocanale and W. 2013)

The following hold, for any join-semilattice $S$.

1. $\text{Reg}(S, \text{cl})$ is always the Dedekind-MacNeille completion of $\text{Clop}(S, \text{cl})$.
2. If $S$ is finite, then $\text{Reg}(S, \text{cl})$ is a bounded homomorphic image of a free lattice.
3. However, $\text{Reg}(S, \text{cl})$ may not be spatial.
Join-semilattices

- For a join-semilattice $S$, set $\text{cl}(x) =$join-closure of $x$.
- $(S, \text{cl})$ is a convex geometry.
Join-semilattices

- For a join-semilattice $S$, set $\text{cl}(x) =$ join-closure of $x$.
- $(S, \text{cl})$ is a convex geometry.

**Theorem (Santocanale and W. 2013)**

The following hold, for any join-semilattice $S$.

- $\text{Reg}(S, \text{cl})$ is always the Dedekind-MacNeille completion of $\text{Clop}(S, \text{cl})$.
- If $S$ is finite, then $\text{Reg}(S, \text{cl})$ is a bounded homomorphic image of a free lattice.
- However, $\text{Reg}(S, \text{cl})$ may not be spatial.
Join-semilattices

- For a join-semilattice $S$, set $\text{cl}(x) =$ join-closure of $x$.
- $(S, \text{cl})$ is a convex geometry.

**Theorem (Santocanale and W. 2013)**

The following hold, for any join-semilattice $S$.
- $\text{Reg}(S, \text{cl})$ is always the *Dedekind-MacNeille completion* of $\text{Clop}(S, \text{cl})$. 
Join-semilattices

- For a join-semilattice $S$, set $\text{cl}(x) =$ join-closure of $x$.
- $(S, \text{cl})$ is a convex geometry.

**Theorem (Santocanale and W. 2013)**

The following hold, for any join-semilattice $S$.

- $\text{Reg}(S, \text{cl})$ is always the Dedekind-MacNeille completion of $\text{Clop}(S, \text{cl})$.
- If $S$ is finite, then $\text{Reg}(S, \text{cl})$ is a bounded homomorphic image of a free lattice.
For a join-semilattice $S$, set $\text{cl}(x) =$join-closure of $x$.

$(S, \text{cl})$ is a convex geometry.

**Theorem (Santocanale and W. 2013)**

The following hold, for any join-semilattice $S$.

- $\text{Reg}(S, \text{cl})$ is always the Dedekind-MacNeille completion of $\text{Clop}(S, \text{cl})$.
- If $S$ is finite, then $\text{Reg}(S, \text{cl})$ is a bounded homomorphic image of a free lattice.

However, $\text{Reg}(S, \text{cl})$ may not be spatial.
Lattices of regular closed sets

The precursor

Regular closed sets

Transitive binary relations

Convexity and hyperplane arrangements

Graphs

Join-semilattices

The extended permutohedron on $S_3$