Invariance groups of finite functions and orbit equivalence of permutation groups

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Joint work with

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- Reinhard Pöschel,
- Géza Makay.
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- Erik Friese,
- Keith Kearnes,
- Erkko Lehtonen,
- $P^3$ (Péter Pál Pálfy),
- Sándor Radeleczki.
Invariance groups

Definition
The invariance group of a function $f : \mathbb{k}^n \to \mathbb{m}$ is

$$S(f) = \{ \sigma \in S_n \mid f(x_1, \ldots, x_n) \equiv f(x_{1\sigma}, \ldots, x_{n\sigma}) \}.$$
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- A group $G$ is $(k, m)$-representable if there is a function $f : \mathbb{k}^n \rightarrow \mathbb{m}$ such that $S(f) = G$. 
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- A group $G$ is $(k, \infty)$-representable if $G$ is $(k, m)$-representable for some $m$. 

Special cases:

- $G$ is $(2, 2)$-representable iff $G$ is the invariance group of a Boolean function $f : 2^n \rightarrow 2$.

- $G$ is $(2, \infty)$-representable iff $G$ is the invariance group of a pseudo-Boolean function $f : 2^n \rightarrow m$. 

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Frucht 1939:
Every group is isomorphic to the automorphism group of a graph.
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Every group is isomorphic to the invariance group of some Boolean function (i.e., (2, 2)-representable).
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Proof.

\[ f : 2^n \rightarrow 2 \iff \mathcal{H} = (n, \{ E \subseteq n \mid f (\chi_E) = 1\}) \]
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Example
\[ S\left(\begin{array}{ccc}
\text{\includegraphics[width=0.2\textwidth]{graph.png}}
\end{array}\right) \cong A_3 \]
Concrete representation

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Suppose that $S(f) = A_3$ for some $f : 2^3 \rightarrow m$. 
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Suppose that $S(f) = A_3$ for some $f: 2^3 \rightarrow m$. Then $f$ must be constant on the orbits of $A_3$ acting on $2^3$:

$$
\begin{align*}
000 & \mapsto a \\
100, 010, 001 & \mapsto b \\
011, 101, 110 & \mapsto c \\
111 & \mapsto d
\end{align*}
$$

However, such a function is totally symmetric, i.e., $S(f) = S_3$. Thus $A_3$ is not $(2, \infty)$-representable.

Let $g: 3^3 \rightarrow 2^3$ such that $g(0, 1, 2) = g(1, 2, 0) = g(2, 1, 0) = 1$ and $g = 0$ everywhere else. Then $S(g) = A_3$, thus $A_3$ is $(3, 2)$-representable.
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<table>
<thead>
<tr>
<th>Orbit</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
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Ein Kleines Problem

Clote, Kranakis 1991:
If $G$ is $(2, \infty)$-representable, then $G$ is $(2, 2)$-representable.

Kisielewicz 1998:
False!

The Klein four-group $V = \{ \text{id}, (12)(34), (13)(24), (14)(23) \} \leq S_4$ is a counterexample; moreover, it is the only counterexample that one could "easily" find.

$V = S_4(2) \cap S_4(3) \Rightarrow V$ is $(2, 3)$-representable but not $(2, 2)$-representable.

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There are infinitely many groups that are $(2, \infty)$-representable but not $(2, 2)$-representable. (?)
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$V = S(\begin{array}{c}
  [1]
\end{array})$
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\text{ } & \text{ }\\
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\[ V = S\left(\begin{array}{cccc}
1 & 2 & 3 & 4 \\
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The following are equivalent for any group $G \leq S_n$:

(i) $G$ is the invariance group of a pseudo-Boolean function (i.e., $G$ is $(2, \infty)$-representable).

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The orbit closure of $G$ is the greatest element of its orbit equivalence class.
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All primitive subgroups of $S_n$ are orbit closed except for $A_n$ and $C_5$, $AGL(1,5)$, $PGL(2,5)$, $AGL(1,8)$, $AGL(1,9)$, $ASL(2,3)$, $PSL(2,8)$, $PΓL(2,8)$ and $PGL(2,9)$. 
Primitive groups

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Theorem
All primitive groups are $(3, \infty)$-representable except for the alternating groups.
For $a = (a_1, \ldots, a_n) \in k^n$ and $\sigma \in S_n$, let $a^\sigma = (a_{1\sigma}, \ldots, a_{n\sigma})$. 
A Galois connection

For $a = (a_1, \ldots, a_n) \in k^n$ and $\sigma \in S_n$, let $a^\sigma = (a_{1\sigma}, \ldots, a_{n\sigma})$.

If $f : k^n \rightarrow k$ and $\sigma \in S_n$, then we write

$$\sigma \vdash f : \iff f (a^\sigma) = f (a) \text{ for all } a \in k^n.$$

$n$: number of variables, $k$: size of domain
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\]

Let \( O_k^{(n)} = \{ f \mid f : k^n \rightarrow k \} \), and for \( F \subseteq O_k^{(n)} \) and \( G \subseteq S_n \) define

\[
F^\vdash := \{ \sigma \in S_n \mid \forall f \in F : \sigma \vdash f \}, \quad \overline{F}^{(k)} := (F^\vdash)^\vdash,
\]

\[
G^\vdash := \{ f \in O_k^{(n)} \mid \forall \sigma \in G : \sigma \vdash f \}, \quad \overline{G}^{(k)} := (G^\vdash)^\vdash.
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\]

For \( G \leq S_n \), we call \( G^{(k)} \) the Galois closure of \( G \) over \( k \).

\( n \): number of variables, \( k \): size of domain
Galois closed groups as invariance groups

Fact
The following are equivalent for any group $G \leq S_n$:

(i) $G$ is Galois closed over $k$.

(ii) $G$ is $(k, \infty)$-representable.

(iii) $G$ is the invariance group of a function $f : k^n \to \infty$.

(iv) $G$ is the intersection of invariance groups of functions $k^n \to 2$.

(v) $G$ is the intersection of invariance groups of functions $k^n \to k$.

(vi) $G$ is orbit closed with respect to the action of $S_n$ on $k^n$.

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Orbits and closures

For \( a = (a_1, \ldots, a_n) \in \mathbb{k}^n \) and \( G \leq S_n \), define

\[
a^G = \{ a^\sigma \mid \sigma \in G \}, \quad \text{Orb}^{(k)} (G) = \{ a^G \mid a \in \mathbb{k}^n \}.
\]

The case \( k = 2 \) corresponds to orbit equivalence and orbit closure.

Proposition

For all \( G \leq S_n \) we have

\[
G^{(2)} \geq G^{(3)} \geq \cdots \geq G^{(n)} = \cdots = G.
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For all \( G, H \leq S_n \) we have

\[ \overline{G}^{(k)} = \overline{H}^{(k)} \iff G^\perp = H^\perp \]

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For every \( G \leq S_n \) and \( k \geq 2 \), we have

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Proof.

$$\overline{G^{(k)}} = \{ \sigma \in S_n \mid \forall a \in k^n : a^\sigma \in a^G \}$$

$$= \{ \sigma \in S_n \mid \forall a \in k^n \exists \pi \in G : a^\sigma = a^\pi \}$$

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\overline{G}^{(k)} = \{ \sigma \in S_n | \forall a \in k^n : a^\sigma \in a^G \}
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$n$: number of variables, $k$: size of domain
A formula for the closure

Proposition

For every $G \leq S_n$ and $k \geq 2$, we have

$$G^{(k)} = \bigcap_{a \in k^n} \text{Stab}(a) \cdot G.$$ 

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The case $k = n - 1$

Theorem

If $k = n - 1 \geq 2$, then all subgroups of $S_n$ except $A_n$ are Galois closed over $k$. 

Definition (Clote, Kranakis 1991)

A group $G \leq S_n$ is weakly representable, if $G$ is $(k, \infty)$-representable for some $k < n$. 

Corollary

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Proof.

$G \leq S_n$ is weakly representable $\iff \exists k < n: G(k) = G \iff G(n - 1) = G \iff G \neq A_n$.
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- $\overline{C_4}(k) = D_4$ (for $n = 4$);
- all other subgroups of $S_n$ are closed.

$n$: number of variables, $k$: size of domain
The case $k = n - d$

**Theorem**

Let $n > \max(2^d, d^2 + d)$ and $G \leq S_n$. Then $G$ is not Galois closed over $k$ if and only if

1. $G \leq_{sd} A_L \times \Delta$ or
2. $G <_{sd} S_L \times \Delta$,

where $n = L \cup D$ with $|L| > d$, $|D| < d$ and $\Delta \leq S_D$.

The closure of these groups is $\overline{G}^{(k)} = S_L \times \Delta$.

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Using the simplicity of alternating groups, one can show that these subdirect products are of the following form:

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**Remark**

Using the simplicity of alternating groups, one can show that these subdirect products are of the following form:

1. \( G = A_L \times \Delta \);
2. \( G = (A_L \times \Delta_0) \cup ((S_L \setminus A_L) \times (\Delta \setminus \Delta_0)) \),
   where \( \Delta_0 \leq \Delta \) is a subgroup of index 2.

\( n \): number of variables, \( k \): size of domain
Interesting subgroups of $S_4$, $S_5$ and $S_6$
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<table>
<thead>
<tr>
<th>$G \leq S_n$</th>
<th>$\overline{G}^{(2)}$</th>
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<tr>
<td>$C_4$</td>
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<tr>
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$G \leq S_n$ denotes the group $G$ with $n$ elements that is a subgroup of the symmetric group $S_n$. The notation $\overline{G}^{(k)}$ refers to the $k$th derived group of $G$. The table lists some of the interesting subgroups of $S_4$, $S_5$, and $S_6$ and their derived group structures.
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<td>$Rot (\square)$</td>
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References


