On morphisms of lattice-valued formal contexts

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Outline

1. Introduction
2. Preliminaries on powerset operators
3. Categories of lattice-valued formal contexts
4. Properties of the categories of lattice-valued formal contexts
5. Conclusion
**Formal Concept Analysis (FCA)** has taken its origin as an attempt to restructure mathematics, e.g., lattice theory.

Since then, FCA has been developed as a subfield of applied mathematics, based in mathematization of concept hierarchies.

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One of the main building blocks of FCA provide *formal contexts*.

**Definition 1**

A *formal context* is a triple \((G, M, I)\), which comprises a set of objects \(G\), a set of attributes \(M\), and a binary incidence relation \(I\) between \(G\) and \(M\), where \(g I m\) means “object \(g\) has attribute \(m\)”.
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A *formal context* is a triple \((G, M, I)\), which comprises a set of objects \(G\), a set of attributes \(M\), and a binary incidence relation \(I\) between \(G\) and \(M\), where \(g \mid m\) means “object \(g\) has attribute \(m\)”.
There exist at least three (different) ways of defining a morphism between two formal contexts \((G_1, M_1, I_1)\) and \((G_2, M_2, I_2)\).

1. The theory of FCA employs pairs of maps \(G_1 \xrightarrow{\alpha} G_2, M_1 \xrightarrow{\beta} M_2\) such that \(g I_1 m \iff \alpha(g) I_2 \beta(m)\) for every \(g \in G_1, m \in M_1\).

2. The theory of *Chu spaces* uses pairs of maps \(G_1 \xrightarrow{\alpha} G_2, M_2 \xrightarrow{\beta} M_1\) such that \(g I_1 \beta(m) \iff \alpha(g) I_2 m\) for every \(g \in G_1, m \in M_2\).
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2. The theory of *Chu spaces* uses pairs of maps $G_1 \xrightarrow{\alpha} G_2$, $M_2 \xrightarrow{\beta} M_1$ such that $g I_1 \beta(m)$ iff $\alpha(g) I_2 m$ for every $g \in G_1$, $m \in M_2$. 
The theory of *Galois connections* relies on the pairs of maps $\mathcal{P}(G_1) \xrightarrow{\alpha} \mathcal{P}(G_2)$, $\mathcal{P}(M_2) \xrightarrow{\beta} \mathcal{P}(M_1)$, where $\mathcal{P}(X)$ stands for the powerset of $X$, such that the diagrams

\[
\begin{array}{ccc}
\mathcal{P}(G_1) & \xrightarrow{\alpha} & \mathcal{P}(G_2) \\
H_1 \downarrow & & \downarrow H_2 \\
\mathcal{P}(M_1) & \xleftarrow{\beta} & \mathcal{P}(M_2)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathcal{P}(M_1) & \xleftarrow{\beta} & \mathcal{P}(M_2) \\
K_1 \downarrow & & \downarrow K_2 \\
\mathcal{P}(G_1) & \xrightarrow{\alpha} & \mathcal{P}(G_2)
\end{array}
\]

commute, where $H_j(S) = \{ m \in M_j \mid s \ I_j m \text{ for every } s \in S \}$ and $K_j(T) = \{ g \in G_j \mid g \ I_j t \text{ for every } t \in T \}$ (*Birkhoff operators*).
J. T. Denniston, A. Melton, and S. E. Rodabaugh compared the approaches of items (2) and (3) by considering their respective categories of *lattice-valued formal contexts* (in the sense of R. Bělohlávek) over a fixed commutative quantale $Q$, and constructing an embedding of each category into its counterparts.

They finally arrived at the conclusion that the two viewpoints on formal context morphisms were not categorically isomorphic.
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This talk compares all three of the above-mentioned approaches to morphisms in the framework of lattice-valued formal contexts over a category of not necessarily commutative quantales.

We construct a number of embeddings between their respective categories of formal contexts, showing that the approach of item (3) falls out of the FCA setting in the lattice-valued case.
This talk compares all three of the above-mentioned approaches to morphisms in the framework of lattice-valued formal contexts over a category of not necessarily commutative quantales.

We construct a number of embeddings between their respective categories of formal contexts, showing that the approach of item (3) falls out of the FCA setting in the lattice-valued case.
Definition 2

\textbf{CSLat}(\bigvee)\textit{ is the variety of V-semilattices, i.e., partially ordered sets (posets), which have arbitrary joins.}

Every \( \bigvee \)-semilattice homomorphism \( A_1 \xrightarrow{\varphi} A_2 \) has the upper adjoint map \( A_2 \xleftarrow{\varphi^+} A_1 \) given by \( \varphi^+(a_2) = \bigvee\{a_1 \in A_1 \mid \varphi(a_1) \leq a_2\} \).
Quantales

\( \mathbf{\lor}\)-semilattices

**Definition 2**

\( \text{CSLat}(\mathbf{\lor}) \) is the variety of \( \mathbf{\lor}\)-semilattices, i.e., partially ordered sets (posets), which have arbitrary joins.

Every \( \mathbf{\lor}\)-semilattice homomorphism \( A_1 \xrightarrow{\varphi} A_2 \) has the upper adjoint map \( A_2 \xrightarrow{\varphi^\perp} A_1 \) given by \( \varphi^\perp(a_2) = \bigvee \{ a_1 \in A_1 \mid \varphi(a_1) \leq a_2 \} \).
Quantales

Definition 3

1. **Quant** is the variety of *quantales*, i.e., triples \((Q, \lor, \otimes)\), where
   - \((Q, \lor)\) is a \(\lor\)-semilattice;
   - \((Q, \otimes)\) is a semigroup;
   - \(\otimes\) distributes across \(\lor\) from both sides.

2. **UQuant** is the variety of *unital quantales*, i.e., quantales \(Q\), which have an element \(1_Q\) such that \((Q, \otimes, 1_Q)\) is a monoid.

A quantale \(Q\) has two residuations, which are given by \(q_1 \rightarrow_l q_2 = \lor\{q \in Q \mid q \otimes q_1 \leq q_2\}\) and \(q_1 \rightarrow_r q_2 = \lor\{q \in Q \mid q_1 \otimes q \leq q_2\}\).
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Crisp forward powerset operator

Definition 4

Given a map $X_1 \xrightarrow{f} X_2$, the forward powerset operator w.r.t. $f$ is the map $\mathcal{P}(X_1) \xrightarrow{f^{-\rightarrow}} \mathcal{P}(X_2)$, which is defined by $f^{-\rightarrow}(S) = \{f(s) \mid s \in S\}$. 
Theorem 5

1. Given a variety \( L \), which extends \( \text{CSLat}(\lor) \), every subcategory \( S \) of \( L \) provides a functor \( \text{Set} \times S \xrightarrow{(-)\rightarrow} \text{CSLat}(\lor) \), which is defined by \( ((X_1, L_1) \xrightarrow{(f,\varphi)} (X_2, L_2))\rightarrow = L_{X_1}^X \xrightarrow{(f,\varphi)\rightarrow} L_{X_2}^X \), where \( ((f,\varphi)\rightarrow(\alpha))(x_2) = \varphi(\lor f(x_1) = x_2 \alpha(x_1)) \).

2. Let \( L \) be a variety, which extends \( \text{CSLat}(\lor) \), and let \( S \) be a subcategory of \( L^{\text{op}} \) such that for every \( S \)-morphism \( L_1 \xrightarrow{\varphi} L_2 \), the map \( L_1 \xrightarrow{\varphi^{\text{op}\leftarrow}} L_2 \) is \( \lor \)-preserving. Then there exists a functor \( \text{Set} \times S \xrightarrow{(\rightarrow)\leftarrow} \text{CSLat}(\lor) \) defined by \( ((X_1, L_1) \xrightarrow{(f,\varphi)} (X_2, L_2))\leftarrow = L_{X_1}^X \xrightarrow{(f,\varphi)\leftarrow} L_{X_2}^X \), where \( ((f,\varphi)\leftarrow(\alpha))(x_2) = \varphi^{\text{op}\leftarrow}(\lor f(x_1) = x_2 \alpha(x_1)) \).
Given a variety $\mathbf{L}$, which extends $\text{CSLat}(\vee)$, every subcategory $\mathbf{S}$ of $\mathbf{L}^{\text{op}}$ provides a functor $\text{Set}^{\text{op}} \times \mathbf{S} \xrightarrow{(\_\to^o)} (\text{CSLat}(\vee))^{\text{op}}$

with $((X_1, L_1) \xrightarrow{(f, \varphi)} (X_2, L_2)) \to^o = L_1^{X_1} \xrightarrow{((f, \varphi) \to^o)^{\text{op}}} L_2^{X_2}$, where $((f, \varphi) \to^o(\alpha))(x_1) = \varphi^{\text{op}}(\vee_{f^{\text{op}}(x_2)=x_1} \alpha(x_2))$.

Let $\mathbf{L}$ be a variety, which extends $\text{CSLat}(\vee)$, and let $\mathbf{S}$ be a subcategory of $\mathbf{L}$ such that for every $\mathbf{S}$-morphism $L_1 \xrightarrow{\varphi} L_2$, the map $L_2 \xrightarrow{\varphi^\to} L_1$ is $\vee$-preserving. Then there exists a functor $\text{Set}^{\text{op}} \times \mathbf{S} \xrightarrow{(-) \to^\to^o} (\text{CSLat}(\vee))^{\text{op}}$ defined by

$((X_1, L_1) \xrightarrow{(f, \varphi)} (X_2, L_2)) \to^\to^o = L_1^{X_1} \xrightarrow{((f, \varphi) \to^\to^o)^{\text{op}}} L_2^{X_2}$, where $((f, \varphi) \to^\to^o(\alpha))(x_1) = \varphi^{\to^o}(\vee_{f^{\text{op}}(x_2)=x_1} \alpha(x_2))$. 
Galois connections

Definition 7

A tuple \(((X_1, \leq), f, g, (X_2, \leq))\) is an order-reversing Galois connection provided that \((X_1, \leq), (X_2, \leq)\) are posets, and \(X_1 \xleftarrow{f} X_2 \xrightarrow{g}\) are maps with \(x_1 \leq g(x_2)\) iff \(x_2 \leq f(x_1)\) for every \(x_1 \in X_1, x_2 \in X_2\).
Definition 8

Let $L$ be a variety, which extends $\text{Quant}$, and let $S$ be a subcategory of $L^{op}$. $S$-$\text{FC}^{C}$ is the category, which comprises the following data.

**Objects:** tuples $\mathcal{K} = (G, M, L, I)$ ((lattice-valued) formal contexts), where $G$ is the set of context *objects*, $M$ is the set of context *attributes*, $L$ is an $S$-object, and $G \times M \xrightarrow{I} L$ is a map, which is called the context *incidence relation*.

**Morphisms:** $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ ((lattice-valued) formal context morphisms) are triples $(G_1, M_1, L_1) \xrightarrow{f=(\alpha, \beta, \varphi)} (G_2, M_2, L_2)$ in $\text{Set} \times \text{Set}^{op} \times S$ with $l_1(g, \beta^{op}(m)) = \varphi^{op} \circ l_2(\alpha(g), m)$ for every $g \in G_1$, $m \in M_2$. 
Definition 9

Let $L$ be a variety, which extends $\textbf{Quant}$, and let $S$ be a subcategory of $L$. $S\text{-FC}_m^C$ is the category, which comprises the following data.

**Objects:** (lattice-valued) formal contexts.

**Morphisms:** $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ are triples $(G_1, M_1, L_1) \xrightarrow{f=(\alpha, \beta, \varphi)} (G_2, M_2, L_2)$ in $\textbf{Set} \times \textbf{Set}^{\text{op}} \times S$ with $\varphi \circ l_1(g, \beta^{\text{op}}(m)) = l_2(\alpha(g), m)$ for every $g \in G_1$, $m \in M_2$. 
Definition 10

Let \( L \) be a variety, which extends \( \text{Quant} \), and let \( S \) be a subcategory of \( L^{\text{op}} \). \( S\text{-FC}^{GW} \) is the category, which comprises the following data.

**Objects:** (lattice-valued) formal contexts.

**Morphisms:** \( \mathcal{K}_1 \overset{f}{\rightarrow} \mathcal{K}_2 \) are triples \( (G_1, M_1, L_1) \overset{f=(\alpha, \beta, \varphi)}{\rightarrow} (G_2, M_2, L_2) \) in \( \text{Set} \times \text{Set} \times S \) with \( l_1(g, m) = \varphi^{\text{op}} \circ l_2(\alpha(g), \beta(m)) \) for every \( g \in G_1, m \in M_1 \).
Definition 11

Let $L$ be a variety, which extends $\text{Quant}$, and let $S$ be a subcategory of $L$. $S\text{-FC}_{GW}^m$ is the category, which comprises the following data.

**Objects:** (lattice-valued) formal contexts.

**Morphisms:** $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ are triples $(G_1, M_1, L_1) \xrightarrow{f=(\alpha, \beta, \varphi)} (G_2, M_2, L_2)$ in $\text{Set} \times \text{Set} \times S$ with $\varphi \circ l_1(g, m) = l_2(\alpha(g), \beta(m))$ for every $g \in G_1$, $m \in M_1$. 
Lattice-valued Birkhoff operators

Definition 12

Every lattice-valued formal context $\mathcal{K}$ provides the following (lattice-valued) Birkhoff operators:

1. $L^G \xrightarrow{H} L^M$ given by $(H(s))(m) = \bigwedge_{g \in G} (s(g) \rightarrow_l l(g, m))$;

2. $L^M \xrightarrow{K} L^G$ given by $(K(t))(g) = \bigwedge_{m \in M} (t(m) \rightarrow_r l(g, m))$.

Theorem 13

For every lattice-valued context $\mathcal{K}$, $(L^G, H, K, L^M)$ is an order-reversing Galois connection.
Lattice-valued formal contexts

Lattice-valued Birkhoff operators

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**Theorem 13**

*For every lattice-valued context $\mathcal{K}$, $(L^G, H, K, L^M)$ is an order-reversing Galois connection.*
Crisp Birkhoff operators

Example 14

Every crisp context $\mathcal{K}$ provides the maps

1. $\mathcal{P}(G) \xrightarrow{H} \mathcal{P}(M)$, $H(S) = \{ m \in M \mid s \vdash m \text{ for every } s \in S \}$;

2. $\mathcal{P}(M) \xrightarrow{K} \mathcal{P}(G)$, $K(T) = \{ g \in G \mid g \vdash t \text{ for every } t \in T \}$;

which are the classical Birkhoff operators of a binary relation.
Definition 15

Given a variety \( L \), which extends \( \text{Quant} \), and a subcategory \( S \) of \( L \), \( S\text{-FC}^{DMR} \) is the category, concrete over the product category \( \text{Set} \times \text{Set}^{op} \), which comprises the following data.

**Objects:** lattice-valued formal contexts \( K \) with \( L \) an object of \( S \).

**Morphisms:** \( K_1 \xrightarrow{f=(\alpha,\beta)} K_2 \) are \( \text{Set} \times \text{Set}^{op} \)-morphisms \((L^G_1, L^M_1) \xrightarrow{(\alpha,\beta)} (L^G_2, L^M_2)\), making the next diagrams commute.
Relations versus Birkhoff operators

- There is a one-to-one correspondence between relations $I \subseteq G \times M$ and order-reversing Galois connections on $(\mathcal{P}(G), \mathcal{P}(M))$.

- What about the lattice-valued case?

**Definition 16**

Given a $\bigvee$-semilattice $L$ and a set $X$, every $S \subseteq X$ and every $a \in L$ provide the map $X \xrightarrow{\chi^a_S} L$, which is defined by

$$\chi^a_S(x) = \begin{cases} a, & x \in S \\ \bot_L, & \text{otherwise.} \end{cases}$$
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Given a $\vee$-semilattice $L$ and a set $X$, every $S \subseteq X$ and every $a \in L$ provide the map $X \xrightarrow{\chi_S^a} L$, which is defined by

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There is a one-to-one correspondence between relations $I \subseteq G \times M$ and order-reversing Galois connections on $(\mathcal{P}(G), \mathcal{P}(M))$.

What about the lattice-valued case?

**Definition 16**

Given a $\vee$-semilattice $L$ and a set $X$, every $S \subseteq X$ and every $a \in L$ provide the map $X \xrightarrow{\chi_S^a} L$, which is defined by

$$\chi_S^a(x) = \begin{cases} a, & x \in S \\ \perp_L, & \text{otherwise.} \end{cases}$$
Theorem 17

Let $G$, $M$ be sets and let $L$ be a unital quantale. For every order-reversing Galois connection $(L^G, \alpha, \beta, L^M)$, equivalent are:

1. **There exists a map** $G \times M \xrightarrow{I} L$ **such that** $\alpha = H$ **and** $\beta = K$.

2. **For every** $g \in G$, $m \in M$, $a \in L$, it follows that
   - (a) $(\alpha(\chi^{1L}_{\{g\}}))(m) = (\beta(\chi^{1L}_{\{m\}}))(g)$;
   - (b) $(\alpha(a \otimes \chi^{1L}_{\{g\}}))(m) = a \rightarrow_l (\alpha(\chi^{1L}_{\{g\}}))(m)$;
   - (c) $(\beta(\chi^{1L}_{\{m\}} \otimes a))(g) = a \rightarrow_r (\beta(\chi^{1L}_{\{m\}}))(g)$.

3. **For every** $g \in G$, $m \in M$, $a \in L$, it follows that
   - (a) $(\alpha(a \otimes \chi^{1L}_{\{g\}}))(m) = a \rightarrow_l (\beta(\chi^{1L}_{\{m\}}))(g)$;
   - (b) $(\beta(\chi^{1L}_{\{m\}} \otimes a))(g) = a \rightarrow_r (\alpha(\chi^{1L}_{\{g\}}))(m)$. 
Consequences

Every map $G \times M \rightarrow L$ gives rise to an order-reversing Galois connection, but the converse way needs additional requirements.

Counterexample

Let $L$ be the unit interval $\mathbb{I} = ([0, 1], \lor, \land, 1)$, and let both $G$ and $M$ be singletons. One can assume that both $\mathbb{I}^G$ and $\mathbb{I}^M$ is $\mathbb{I}$. The order-reversing involution map $\mathbb{I} \xrightarrow{\alpha} \mathbb{I}$, $\alpha(a) = 1 - a$ is a part of the order-reversing Galois connection $(\mathbb{I}, \alpha, \alpha, \mathbb{I})$. The condition of, e.g., Theorem 17(3)(a) gives $\alpha(a) = a \rightarrow \alpha(1)$ for every $a \in \mathbb{I}$. However, for $a = \frac{1}{2}$, one obtains that $\alpha(\frac{1}{2}) = \frac{1}{2} \neq 0 = \frac{1}{2} \rightarrow 0 = \frac{1}{2} \rightarrow \alpha(1)$. 
Every map $G \times M \rightarrow L$ gives rise to an order-reversing Galois connection, but the converse way needs additional requirements.

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From $\mathbf{S}$-$\mathbf{FC}^C$ to $\mathbf{S}$-$\mathbf{FC}^{DMR}$

**Definition 18**

- $\mathbf{S}$-$\mathbf{FC}_*^C$ is a subcategory of $\mathbf{S}$-$\mathbf{FC}^C$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have surjective maps $G_1 \xrightarrow{\alpha} G_2$, $M_2 \xrightarrow{\beta^{op}} M_1$, and an $\mathbf{S}$-isomorphism $L_1 \xrightarrow{\varphi} L_2$.

- Let $\mathbf{L}$ extend $\mathbf{UQuant}$. $\mathbf{S}$-$\mathbf{FC}_{**}^C$ (resp. $\mathbf{S}$-$\mathbf{FC}_{**}^C$) is a full subcategory of $\mathbf{S}$-$\mathbf{FC}_*^C$, whose objects $\mathcal{K} = (G, M, L, I)$ have non-empty $G$ (resp. $M$) and, moreover, $1_L \neq \bot_L$.

**Theorem 19**

There exists a functor $\mathbf{S}$-$\mathbf{FC}_*^C \xrightarrow{H_{CD}} \mathbf{S}$-$\mathbf{FC}^{DMR}$, which is given by $H_{CD}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = \mathcal{K}_1 \xrightarrow{((\alpha, \varphi)^{op}, ((\beta, \varphi) \xrightarrow{o}^{op})} \mathcal{K}_2$. Its restriction to $\mathbf{S}$-$\mathbf{FC}_{**}^C$ (resp. $\mathbf{S}$-$\mathbf{FC}_{**}^C$) is a (non-full) embedding.
From $S-\text{FC}^C$ to $S-\text{FC}^{DMR}$

**Definition 18**

- $S-\text{FC}^*_C$ is a subcategory of $S-\text{FC}^C$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have surjective maps $G_1 \xrightarrow{\alpha} G_2$, $M_2 \xrightarrow{\beta^{op}} M_1$, and an $S$-isomorphism $L_1 \xrightarrow{\varphi} L_2$.

- Let $L$ extend $U\text{Quant}$. $S-\text{FC}^*_C^{**}$ (resp. $S-\text{FC}^{C_\bullet}_*$) is a full subcategory of $S-\text{FC}^*_C$, whose objects $\mathcal{K} = (G, M, L, I)$ have non-empty $G$ (resp. $M$) and, moreover, $1_L \neq \bot_L$.

**Theorem 19**

There exists a functor $S-\text{FC}^*_C \xrightarrow{\text{H}_{CD}} S-\text{FC}^{DMR}^*$, which is given by $\text{H}_{CD}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = \mathcal{K}_1^{((\alpha, \varphi)^{op}),((\beta, \varphi)\rightarrow\circ)^{op}} \rightarrow \mathcal{K}_2$. Its restriction to $S-\text{FC}^*_C^{**}$ (resp. $S-\text{FC}^{C_\bullet}_*$) is a (non-full) embedding.
Relationships between the categories of lattice-valued formal contexts

From $\text{S-FC}_m^C$ to $\text{S-FC}_{DMR}^C$

**Definition 20**

- $\text{S-FC}_m^C$ is a subcategory of $\text{S-FC}_m^C$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have surjective maps $G_1 \xrightarrow{\alpha} G_2$, $M_2 \xrightarrow{\beta^{op}} M_1$, and an $\text{S}$-isomorphism $L_1 \xrightarrow{\varphi} L_2$.

- Let $L$ extend $\text{UQuant}$. $\text{S-FC}_{m^{**}}^C$ (resp. $\text{S-FC}_{m^{*\bullet}}^C$) is a full subcategory of $\text{S-FC}_{m^*}^C$, whose objects $\mathcal{K} = (G, M, L, I)$ have non-empty $G$ (resp. $M$) and, moreover, $1_L \neq \perp_L$.

**Theorem 21**

There exists a functor $\text{S-FC}_m^{C} \xrightarrow{H_{\text{CM}_D}} \text{S-FC}_{DMR}^C$, which is given by $H_{\text{CM}_D}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = \mathcal{K}_1 \xrightarrow{((\alpha, \varphi) \rightarrow, ((\beta, \varphi)^{\text{op}} \rightarrow)^{\text{op}})} \mathcal{K}_2$. Its restriction to $\text{S-FC}_{m^{**}}^C$ (resp. $\text{S-FC}_{m^{*\bullet}}^C$) is a (non-full) embedding.
From $S\text{-FC}^C_m$ to $S\text{-FC}^{DMR}$

**Definition 20**

- $S\text{-FC}^C_{m^*}$ is a subcategory of $S\text{-FC}^C_m$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have surjective maps $G_1 \xrightarrow{\alpha} G_2$, $M_2 \xrightarrow{\beta^{op}} M_1$, and an $S$-isomorphism $L_1 \xrightarrow{\varphi} L_2$.

- Let $L$ extend $U\text{Quant}$. $S\text{-FC}^C_{m^{**}}$ (resp. $S\text{-FC}^C_{m^{*\cdot}}$) is a full subcategory of $S\text{-FC}^C_{m^*}$, whose objects $\mathcal{K} = (G, M, L, I)$ have non-empty $G$ (resp. $M$) and, moreover, $1_L \neq \bot_L$.

**Theorem 21**

There exists a functor $S\text{-FC}^C_{m^*} \xrightarrow{H_{CmD}} S\text{-FC}^{DMR}$, which is given by $H_{CmD}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = \mathcal{K}_1 \xrightarrow{((\alpha,\varphi) \rightarrow ((\beta,\varphi)^{\leftarrow \cdot \cdot \cdot} \circ \cdot)^{op})} \mathcal{K}_2$. Its restriction to $S\text{-FC}^C_{m^{**}}$ (resp. $S\text{-FC}^C_{m^{*\cdot}}$) is a (non-full) embedding.
Formal concepts, protoconcepts, and preconcepts

Definition 22

Let $\mathcal{K}$ be a lattice-valued formal context, and let $s \in L^G$, $t \in L^M$. The pair $(s, t)$ is called a

- (lattice-valued) formal concept of $\mathcal{K}$ provided that $H(s) = t$ and $K(t) = s$;
- (lattice-valued) formal protoconcept of $\mathcal{K}$ provided that $K \circ H(s) = K(t)$ (equivalently, $H \circ K(t) = H(s)$);
- (lattice-valued) formal preconcept of $\mathcal{K}$ provided that $s \leq K(t)$ (equivalently, $t \leq H(s)$).
From $\textbf{S-FC}^{DMR}$ to $\textbf{S-FC}^C$

**Definition 23**

- Given an $\textbf{L}$-algebra $L$, $\textbf{L-FC}_i^{DMR}$ is a subcategory of $\textbf{L-FC}^{DMR}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have injective maps $L^G_1 \xrightarrow{\alpha} L^G_2$, $L^M_2 \xrightarrow{\beta^{op}} L^M_1$.

- An $\textbf{L}$-algebra $L$ is called **quasi-strictly right-sided (qsrs-algebra)** provided that $a \leq (\top_L \rightarrow_1 a) \otimes \top_L$ for every $a \in L$.

**Theorem 24**

There exists a functor $\textbf{L-FC}_i^{DMR} \xrightarrow{H^i_{DC}} \textbf{S-FC}^C$, which is given by $H^i_{DC}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L^G_1, L^M_1, L, \hat{i}_1) \xrightarrow{(\alpha, \beta, 1_L)} (L^G_2, L^M_2, L, \hat{i}_2)$, where $\hat{i}_j(s, t) = \top_L$ if $(s, t)$ is a formal concept of $\mathcal{K}_j$, and $\perp_L$ otherwise. If $L$ is a qsrs-algebra, then $H^i_{DC}$ is a (non-full) embedding.
From $\mathbf{S-FC}^{DMR}$ to $\mathbf{S-FC}^C$

**Definition 23**

- Given an $\mathbf{L}$-algebra $L$, $\mathbf{L-FC}^{DMR}_i$ is a subcategory of $\mathbf{L-FC}^{DMR}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have injective maps $L^{G_1} \xrightarrow{\alpha} L^{G_2}$, $L^{M_2} \xrightarrow{\beta^{op}} L^{M_1}$.

- An $\mathbf{L}$-algebra $L$ is called **quasi-strictly right-sided (qsrs-algebra)** provided that $a \leq (\top_L \rightarrow_1 a) \otimes \top_L$ for every $a \in L$.

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There exists a functor $\mathbf{L-FC}^{DMR}_i \xrightarrow{H^i_{DC}} \mathbf{S-FC}^C$, which is given by $H^i_{DC}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L^{G_1}, L^{M_1}, L, \hat{1}_1) \xrightarrow{(\alpha, \beta, 1_L)} (L^{G_2}, L^{M_2}, L, \hat{1}_2)$, where $\hat{1}_j(s, t) = \top_L$ if $(s, t)$ is a formal concept of $\mathcal{K}_j$, and $\bot_L$ otherwise. If $L$ is a qsrs-algebra, then $H^i_{DC}$ is a (non-full) embedding.
From $S$-$\text{FC}^{DMR}$ to $S$-$\text{FC}^C$

**Definition 25**

Given an $L$-algebra $L$, $L$-$\text{FC}_{rfp}^{DMR}$ is a subcategory of $L$-$\text{FC}^{DMR}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have maps $L^{G_1} \xrightarrow{\alpha} L^{G_2}$, $L^{M_2} \xrightarrow{\beta^{\text{op}}} L^{M_1}$ such that $K_2 \circ H_2 \circ \alpha(s) = \alpha(s)$ implies $K_1 \circ H_1(s) = s$, and $H_1 \circ K_1 \circ \beta^{\text{op}}(t) = \beta^{\text{op}}(t)$ implies $H_2 \circ K_2(t) = t$, for every $s \in L^{G_1}_1$, $t \in L^{M_2}_2$.

**Theorem 26**

There exists a functor $L$-$\text{FC}_{rfp}^{DMR} \xrightarrow{H^{rfp}_{DC}} S$-$\text{FC}^C$, which is given by $H^{rfp}_{DC}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L^{G_1}, L^{M_1}, L, \hat{I}_1) \xrightarrow{(\alpha, \beta, \mathbf{1}_L)} (L^{G_2}, L^{M_2}, L, \hat{I}_2)$, where $\hat{I}_j(s, t) = \top_L$ if $(s, t)$ is a formal concept of $\mathcal{K}_j$, and $\bot_L$ otherwise. If $L$ is a qsrs-algebra, then the functor is a (non-full) embedding.
From $\textbf{S-FC}^{DMR}$ to $\textbf{S-FC}^C$

**Definition 25**

Given an $L$-algebra $L$, $L$-$\textbf{FC}_{rfp}$ is a subcategory of $L$-$\textbf{FC}^{DMR}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have maps $L^G_1 \xrightarrow{\alpha} L^G_2$, $L^M_2 \xrightarrow{\beta^{op}} L^M_1$ such that $K_2 \circ H_2 \circ \alpha(s) = \alpha(s)$ implies $K_1 \circ H_1(s) = s$, and $H_1 \circ K_1 \circ \beta^{op}(t) = \beta^{op}(t)$ implies $H_2 \circ K_2(t) = t$, for every $s \in L^G_1, t \in L^M_2$.

**Theorem 26**

There exists a functor $L$-$\textbf{FC}_{rfp}^{DMR} \xrightarrow{H_{rfp}^{DC}} \textbf{S-FC}^C$, which is given by $H_{rfp}^{DC}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L^G_1, L^M_1, L, \hat{l}_1) \xrightarrow{(\alpha, \beta, 1_L)} (L^G_2, L^M_2, L, \hat{l}_2)$, where $\hat{l}_j(s, t) = \top_L$ if $(s, t)$ is a formal concept of $\mathcal{K}_j$, and $\bot_L$ otherwise. If $L$ is a qsrs-algebra, then the functor is a (non-full) embedding.
Relationships between the categories of lattice-valued formal contexts

**From \( S-\text{FC}^{DMR} \) to \( S-\text{FC}^C \)**

**Definition 27**

Given an \( L \)-algebra \( L \), \( L-\text{FC}_{orp}^{DMR} \) is a subcategory of \( L-\text{FC}^{DMR} \), with the same objects, and whose morphisms \( \mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2 \) have order-preserving maps \( LG_1 \xrightarrow{\alpha} LG_2 \), \( LM_2 \xrightarrow{\beta^{op}} LM_1 \).

**Theorem 28**

There exists a functor \( L-\text{FC}_{orp}^{DMR} \xrightarrow{H^{orp}_{DC}} S-\text{FC}^C \), which is given by \( H^{orp}_{DC}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (LG_1, LM_1, L, \hat{I}_1) \xrightarrow{(\alpha, \beta, 1_L)} (LG_2, LM_2, L, \hat{I}_2) \), where \( \hat{I}_j(s, t) = \top_L \) if \((s, t)\) is a formal preconcept of \( \mathcal{K}_j \), and \( \bot_L \) otherwise. If \( L \) is a qsrs-algebra, then the functor is a (non-full) embedding.
Relationships between the categories of lattice-valued formal contexts

From $\textbf{S-FC}^{DMR}$ to $\textbf{S-FC}^C$

**Definition 27**

Given an $\textbf{L}$-algebra $L$, $L\textbf{-FC}_{orp}^{DMR}$ is a subcategory of $L\textbf{-FC}^{DMR}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have order-preserving maps $L G_1 \xrightarrow{\alpha} L G_2$, $L M_2 \xrightarrow{\beta^{op}} L M_1$.

**Theorem 28**

There exists a functor $L\textbf{-FC}_{orp}^{DMR} \xrightarrow{H_{DC}^{orp}} \textbf{S-FC}^C$, which is given by $H_{DC}^{orp}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L G_1, L M_1, L, \hat{I}_1) \xrightarrow{(\alpha, \beta, 1_L)} (L G_2, L M_2, L, \hat{I}_2)$, where $\hat{I}_j(s, t) = \top_L$ if $(s, t)$ is a formal preconcept of $\mathcal{K}_j$, and $\bot_L$ otherwise. If $L$ is a qsrs-algebra, then the functor is a (non-full) embedding.
Theorem 29

There exists a functor \( L\text{-FC}^{DMR} \xrightarrow{H_{DC}} S\text{-FC}^C \), which is given by

\[
H_{DC}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L^{G_1}, L^{M_1}, L, \hat{I}_1) \xrightarrow{(\alpha, \beta, 1_L)} (L^{G_2}, L^{M_2}, L, \hat{I}_2),
\]

where \( \hat{I}_j(s, t) = \top_L \) if \((s, t)\) is a formal protoconcept of \( \mathcal{K}_j \), and \( \perp_L \) otherwise. If \( L \) is a qsrs-algebra, then \( H_{DC} \) is a (non-full) embedding.
This talk considered some approaches to morphisms of lattice-valued formal contexts of Formal Context Analysis (FCA).

We constructed several categories, whose objects are lattice-valued analogues of formal contexts of FCA, and whose morphisms reflect the crisp setting of Chu spaces, the lattice-valued setting of J. T. Denniston, A. Melton, and S. E. Rodabaugh, as well as the many-valued setting of B. Ganter and R. Wille.

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The difference between the settings of relations and Galois connections in the lattice-valued case, motivates the following problem.

**Problem 30**

*Is it possible to build a lattice-valued approach to FCA, which is based in order-reversing Galois connections on lattice-valued powersets, which are not generated by lattice-valued relations on their respective sets of objects and their attributes?*
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Is it possible to build a lattice-valued approach to FCA, which is based in order-reversing Galois connections on lattice-valued powersets, which are not generated by lattice-valued relations on their respective sets of objects and their attributes?
References I


Thank you for your attention!