On Zariski topologies of Abelian groups with operations

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Topologizing algebras
Topologizing groups

In 1944 Markov posed the following question:

Does every infinite group admit a non-discrete Hausdorff topology in which its multiplication and inversion are continuous?

He (implicitly) defined a $T_1$ topology on a group, called now its Zariski topology, and proved: For countable groups, the answer is positive iff the Zariski topology is non-discrete.

It was proved that the answer is affirmative for Abelian groups (Kertész and Szele, 1953) and negative in general (for uncountable groups: Shelah, 1976 (under CH), Hesse, 1979 (without CH); for countable groups: Olshanski based on Adian’s construction, 1980).

Remark. Any infinite group admits a non-discrete Hausdorff topology in which all left and right shifts and inversion are continuous (Zelenyuk, 2006).
Topologizing rings

The same question can be posed for rings (and other algebras):

*Does every infinite ring admit some non-discrete Hausdorff topology in which its operations are continuous?*

Similarly to the case of groups, Markov proved: *For countable rings, the answer is positive iff the Zariski topology is non-discrete.*

In 1970s Arnautov obtained the negative answer for uncountable rings. On the other hand, he shown: *The Zariski topology of every infinite ring is non-discrete,* thus giving the affirmative answer for countable rings.

In 1997 Protasov gave a short proof of Arnautov’s result by using Hindman’s Finite Sums Theorem, a famous statement in Ramsey-theoretic algebra obtained via ultrafilter extensions of semigroups.
Following close ideas, we prove non-discreteness of Zariski topologies for a wider class of universal algebras, called here polyrings, which includes various classical algebras besides rings.

Actually, we state a much stronger fact: If $K$ is a polyring, then $K^n$ considered as a subspace of $K^{n+1}$ with its Zariski topology is closed nowhere dense in it. Our proof uses a multidimensional generalization of Hindman’s theorem (Bergelson–Hindman, 1996).
Zariski topologies of polynomials
**Polyrings**

**Definition.** $(K, 0, +, \Omega)$ is a polyring iff $(K, 0, +)$ is an Abelian group and any operation $F \in \Omega$ (of arbitrary arity) is distributive w.r.t. the addition, i.e. the shifts

$$x \mapsto F(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{n-1})$$

are endomorphisms of $(K, 0, +)$, for all $i < n$ and $a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1} \in K$.

**Examples.** Various classical algebras: Abelian groups with operators, modules, rings, differential rings, linear algebras, etc.

**Fact.** For any Abelian group $(K, 0, +)$ there is the largest polyring $(K, 0, +, \Omega)$. 
Zariski topologies

Let $K$ be a polyring and $n < \omega$. If $F \in K[x_1, \ldots, x_n]$ is a term of $n$ variables, let

$$S_F = \{(a_1, \ldots, a_n) \in K^n : F(a_1, \ldots, a_n) = 0\}$$

denote the set of solutions of the equation $F(x_1, \ldots, x_n) = 0$ in $K$.

**Definition.** A set $S \subseteq K^n$ is closed in the Zariski topology on $K^n$ iff $S$ is an intersection of finite unions of sets $S_F$.

**Facts.** 1. The Zariski topology on $K$ is a $T_1$ topology in which all shifts are continuous.  
2. The Zariski topology on $K^{n+1}$ includes the product of the Zariski topologies on $K^n$ and $K$, and can be stronger.  
3. $K^n$ is homeomorphic to $K^n \times \{0\} \subseteq K^{n+1}$ (and will be identified with it below).
The main result

**Theorem.** Let $K$ be an infinite polyring. For any term $F \in K[x_1, \ldots, x_n]$ the mapping of $K^n$ into $K$ defined by $F$ is closed nowhere dense in $K^{n+1}$. In particular, so is $K^n$.

Roughly speaking, this shows that such spaces, although can be not Hausdorff, allow a reasonable notion of topological dimension.

**Corollary.** If $K$ is an infinite polyring, $0 < n < \omega$, then $K^n$ is non-discrete.

**Remark.** If $\Omega \subseteq \Omega'$ then the Zariski topology of $(K, 0, +, \Omega')$ is stronger than one of $(K, 0, +, \Omega)$. Since there is the largest polyring with a given $(K, 0, +)$, Theorem gives the best possible result in this direction.
Questions
We mention only a few questions. All algebras below are considered with their Zariski topologies.

1. Is every non-discrete group $K$ nowhere dense in $K^2$?

2. If $E$ is an endomorphism of a non-discrete group $K$, is $(K, \cdot, E)$ also non-discrete?

By our result, both answers are affirmative for Abelian groups.

3. Is every (non-commutative) field connected?

This fails for some rings.

4. Given a polyring $K$, classify closed subsets of $K^n$ up to: (i) homeomorphisms; (ii) local homeomorphisms.

This may be unclear even for fields.