On the Bergman property for clones

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June 7, 2013

(joint work with Maja Pech)
Outline

**Cofinality and generating sets of clones**
- Definition
- Reduction to semigroups

**Cofinality for homogeneous structures**
- Homogeneous structures
- Dolinka’s cofinality result
- Cofinality of polymorphism clones of homogeneous structures

**The Bergman property for clones**
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Definition of cofinality for clones

Observation
If a clone $\mathbb{F}$ is non-finitely generated, then it can be approximated by a chain of proper subclones $\langle \mathbb{F}^{(1)} \rangle \leq \langle \mathbb{F}^{(2)} \rangle \leq \ldots$.

Question
In general, what is the minimal possible length of such a chain?

“Answer”
It is some regular cardinal.

Definition (Cofinality of a clone)
Let $\mathbb{F}$ be a non-finitely generated clone. By $\text{cf}(\mathbb{F})$ we denote the least cardinal $\lambda$ such that there exists a chain $(\mathbb{F}_i)_{i < \lambda}$ such that

1. $\forall i < \lambda : \mathbb{F}_i < \mathbb{F}$,
2. $\cup_{i < \lambda} \mathbb{F}_i = \mathbb{F}$. 
Observations

- Countable clones are either finitely generated or have cofinality $\aleph_0$.
- Therefore the concept of cofinality becomes interesting only for clones on infinite sets.
- Examples for very large clones are the polymorphism clones of certain homogeneous structures.

**Lemma**

If $F \leq O_A$ has uncountable cofinality, then

$$\exists n \in \mathbb{N}_+ : F = \langle F^{(n)} \rangle_{O_A}.$$
Motivating questions

1. Does the polymorphism clone of the Rado-graph have uncountable cofinality?
2. Does the clone $O_A$ of all functions on an infinite set $A$ have uncountable cofinality?
3. What about other homogeneous structures?
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Relative rank of clones

We adapt Ruškuc’ notion of relative rank for semigroups to clones:

Let $F$ be a clone, and let $M \subseteq F$.

**Definition**
A subset $N \subseteq F$ is called **generating set of $F$ modulo $M$** if

$$\langle M \cup N \rangle_{O_A} = F.$$

The **relative rank of $F$ modulo $M$** is the smallest cardinal of a generating set of $F$ modulo $M$.

It is denoted by

$$\text{rank}(F : M)$$
Cofinality and relative rank

Proposition

Let $F \leq O_A$, $S \subseteq F^{(1)}$ be a transformation semigroup. If $\text{cf}(S) > \aleph_0$ and if $\text{rank}(F : S)$ is finite, then

$$\text{cf}(F) > \aleph_0,$$

too.
Some concrete cofinality results

Let $R$ denote the Rado-graph.

Observation from Maja’s talk
The relative rank of $\text{Pol}(R)$ modulo $\text{End}(R)$ is equal to 1.

Theorem (Dolinka 2012)
\[
\text{cf}(\text{End } R) > \mathfrak{c}_0.
\]

Corollary
\[
\text{cf}(\text{Pol } R) > \mathfrak{c}_0.
\]

Theorem (Malcev, Mitchel, Ruškuc 2009)
For every infinite set $A$ holds $\text{cf}(O_A^{(1)}) > \mathfrak{c}_0$.

From the proof of Sierpiński’s Theorem we have:
The relative rank of $O_A$ modulo $O_A^{(1)}$ is equal to 1.

Corollary
For every infinite set $A$ holds $\text{cf}(O_A) > \mathfrak{c}_0$. 
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Ages

Definition
A class of finitely generated countable structures is called an age if it is obtainable as the class of all finitely generated structures that embed into a given fixed countable structure.

Hereditary property (HP)
\( \mathcal{K} \) has the (HP) if \( \forall A \in \mathcal{K} \) if \( B \hookrightarrow A \), then also \( B \in \mathcal{K} \).

Joint embedding property (JEP)
\( \mathcal{K} \) has the (JEP) if

\[
\forall A, B \in \mathcal{K} \exists C \in \mathcal{K} : A \hookrightarrow C, B \hookrightarrow C.
\]

Theorem (Fraïssé)
\( \mathcal{K} \) is an age if and only if it contains up to isomorphism only countably many structures, it has the (HP) and the (JEP).
Fraïssé-classes

Amalgamation property (AP)

$\mathcal{K}$ has the (AP) if for all $A, B_1, B_2 \in \mathcal{K}$ and for all $f_1 : A \leftrightarrow B_1$, $f_2 : A \leftrightarrow B_2$, there exist $C \in \mathcal{K}$, $g_1 : B_1 \leftrightarrow C$, $g_2 : B_2 \leftrightarrow C$, such that the following diagram commutes:

\[
\begin{array}{c}
A & \xrightarrow{f_1} & B_1 \\
\uparrow & & \downarrow \\
B_2 & \xleftarrow{f_2} & \end{array}
\]

Definition

An age $\mathcal{K}$ is called a Fraïssé-class if it has the amalgamation property (AP).

Theorem (Fraïssé 1953)

1. $\mathcal{K}$ is a Fraïssé-class $\iff$ $\mathcal{K}$ is the age of a countable homogeneous structure,
2. any two countable homogeneous structures of the same age are isomorphic.
Fraïssé-classes

Amalgamation property (AP)

$\mathcal{K}$ has the (AP) if for all $A, B_1, B_2 \in \mathcal{K}$ and for all $f_1 : A \hookrightarrow B_1$, $f_2 : A \hookrightarrow B_2$, there exist $C \in \mathcal{K}$, $g_1 : B_1 \hookrightarrow C$, $g_2 : B_2 \hookrightarrow C$, such that the following diagram commutes:

\[
\begin{array}{ccc}
A & \xleftarrow{f_1} & B_1 \\
\downarrow{f_2} & & \uparrow{g_1} \\
B_2 & \xleftarrow{g_2} & C \\
\end{array}
\]

Definition

An age $\mathcal{K}$ is called **Fraïssé-class** if it has the amalgamation property (AP).

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Fraïssé-classes

Amalgamation property (AP)

\( \mathcal{K} \) has the (AP) if for all \( A, B_1, B_2 \in \mathcal{K} \) and for all \( f_1 : A \leftrightarrow B_1, f_2 : A \leftrightarrow B_2 \), there exist \( C \in \mathcal{K}, g_1 : B_1 \leftrightarrow C, g_2 : B_2 \leftrightarrow C \), such that the following diagram commutes:

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\begin{array}{ccc}
A & \xrightarrow{f_1} & B_1 \\
\uparrow & & \uparrow \\
B_2 & \xleftarrow{g_2} & C \\
\downarrow & & \downarrow \\
& f_2 & \ \\
\end{array}
\]

Definition

An age \( \mathcal{K} \) is called **Fraïssé-class** if it has the amalgamation property (AP).

Theorem (Fraïssé 1953)

1. \( \mathcal{K} \) is a Fraïssé-class \( \iff \mathcal{K} \) is the age of a countable homogeneous structure,
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Homo amalgamation property (HAP)

\( \mathcal{K} \) has the (HAP) if for all \( A, B_1, B_2 \in \mathcal{K} \), for all homomorphisms \( f_1: A \to B_1 \), \( f_2: A \to B_2 \) there exist \( C \in \mathcal{K} \), \( g_1: B_1 \to C \), and \( g_2: B_2 \to C \), such that the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{f_1} & B_1 \\
\downarrow{f_2} & & \downarrow \\
B_2
\end{array}
\]

**Theorem (Dolinka 2011)**

A countable homogeneous structure \( A \) is homomorphism homogeneous if and only if \( \text{Age}(A) \) has the (HAP).
Homo amalgamation property (HAP)

$\mathcal{K}$ has the (HAP) if for all $A, B_1, B_2 \in \mathcal{K}$, for all homomorphisms $f_1 : A \to B_1$, $f_2 : A \hookrightarrow B_2$ there exist $C \in \mathcal{K}$, $g_1 : B_1 \hookrightarrow C$, and $g_2 : B_2 \to C$, such that the following diagram commutes:

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B_2 & \xleftarrow{g_2} & C
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B_2 & \xrightarrow{g_2} & C
\end{array}
\]

Theorem (Dolinka 2011)

A countable homogeneous structure $A$ is homomorphism homogeneous if and only if $\text{Age}(A)$ has the (HAP).
If $\mathcal{K}$ is an age, then $\overline{\mathcal{K}} := \{ A \mid A$ countable, $\text{Age}(A) \subseteq \mathcal{K} \}$.

**Definition (Dolinka 2011)**

A Fraïssé-class $\mathcal{K}$ of relational structures is called **strict Fraïssé-class** if every pair of morphisms in $(\mathcal{K}, \leftrightarrow)$ with the same domain has a pushout in $(\overline{\mathcal{K}}, \rightarrow)$.

**Observation**

Note that these pushouts will always be amalgams. Thus the strict amalgamation property postulates canonical amalgams.
Theorem (Dolinka 2011)

Let $U$ be a countable homogeneous structure of age $\mathcal{K}$. If

1. $\mathcal{K}$ has the strict amalgamation property,
2. $\mathcal{K}$ has the (HAP),
3. the coproduct of $\aleph_0$ copies of $U$ exists and if its age is contained in $\mathcal{K}$,
4. $|\text{End } U| > \aleph_0$.

Then $\text{cf}(\text{End } U) > \aleph_0$.

Remark

Dolinka shows more: that $\text{End } U$ has uncountable strong cofinality.
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Kubiś’s amalgamated extension property

Let $\mathcal{K}$ be a class of countable, finitely generated structures. We say that $\mathcal{K}$ has the **amalgamated extension property** if

![Diagram]

**Remark**

The strict amalgamation property implies the amalgamated extension property.
Kubiš’s amalgamated extension property

Let \( \mathcal{K} \) be a class of countable, finitely generated structures. We say that \( \mathcal{K} \) has the **amalgamated extension property** if

\[
\begin{align*}
B_1 & \xrightarrow{f_1} C \xleftarrow{g_1} B_2 \\
A & \xleftarrow{f_2} B_2
\end{align*}
\]

\[
\begin{align*}
T & \xrightarrow{h_1} T' \\
A & \xleftarrow{f_2} B_2
\end{align*}
\]

\[
\begin{align*}
B_1 & \xrightarrow{g_1} C \xleftarrow{g_2} B_2
\end{align*}
\]

**Remark**

*The strict amalgamation property implies the amalgamated extension property.*
Generating polymorphism clones of homogeneous structures

Let us recall a Theorem from Maja’s talk:

**Theorem**

*Let* $\mathbf{U}$ *be a countable homogeneous structure of age* $\mathcal{K}$ *such that*

1. $\mathcal{K}$ *is closed with respect to finite products,*
2. $\mathcal{K}$ *has the (HAP),*
3. $\mathcal{K}$ *has the amalgamated extension property.*

*Then* $\text{rank}(\text{Pol} \; \mathbf{U} : \text{End} \; \mathbf{U}) = 1$

Now we are ready to combine Dolinka’s result with the above given Theorem:
Cofinality of polymorphism clones of homogeneous structures

Theorem
Let $U$ be a countable homogeneous structure of age $\mathcal{K}$. If

1. $\mathcal{K}$ has the strict amalgamation property,
2. $\mathcal{K}$ is closed with respect to finite products,
3. $\mathcal{K}$ has the (HAP),
4. the coproduct of $\aleph_0$ copies of $U$ exists and its age is contained in $\mathcal{K}$,
5. $|\text{End } U| > \aleph_0$.

Then

$$\text{cf}(\text{Pol } U) > \aleph_0.$$
Examples

The polymorphism clones of the following structures have uncountable cofinality:

- the Rado graph,
- the countable generic poset $\mathbb{P} = (P, \leq)$,
- the countable atomless Boolean algebra,
- the countable universal homogeneous semilattice,
- the countable universal homogeneous distributive lattice,
- the vector-space $\mathbb{F}^\omega$ for any countable field $\mathbb{F}$. 
Theorem (Bergman 2006)

Let $A$ be an infinite set. $G = \text{Sym}(A)$ be the group of all permutations of $A$. Then every connected Cayley graph of $G$ has finite diameter.

Definition

Any group with this property if said to have the Bergman property.

Remark

- Bergman showed the Bergman-property of $\text{Sym}(A)$ to give an alternative proof for the uncountable cofinality of $\text{Sym}(A)$ (original proof by Macpherson and Neumann),
- Droste and Göbel generalized Bergman’s ideas to many other groups,
- The Bergman property was defined for semigroups by Maltcev, Mitchel, and Ruškuc.
Definition (Maltcev, Mitchel, Ruškuc 2009)
A semigroup $S$ has the **Bergman-property** if for every $U \subseteq S$ holds

$$U^+ = S \Rightarrow \exists n \in \mathbb{N}_+: S = \bigcup_{i=1}^{n} U^i.$$ 

Remark
Dolinka (2011) showed the Bergman property for the endomorphism monoids of many homogeneous structures (with the HAP).
The Bergman property for clones

Definition
A clone $\mathcal{F}$ is said to have the **Bergman-property** if for every generating set $H$ of $\mathcal{F}$ and every $k \in \mathbb{N} \setminus \{0\}$ there exists some $n \in \mathbb{N}$ such that every $k$-ary function from $\mathcal{F}$ can be represented by a term of depth at most $n$ from the functions in $H$. 
The main result

Theorem
Let $U$ be a countable homogeneous structure of age $\mathcal{K}$, such that

1. $\mathcal{K}$ has the strict amalgamation property,
2. $\mathcal{K}$ is closed with respect to finite products,
3. $\mathcal{K}$ has the HAP,
4. the coproduct of countably many copies of $U$ in $(\mathcal{K}, \to)$ exists,
5. $\text{End } U$ is not finitely generated.

Then $\text{Pol } U$ has the Bergman property.
Strategy of the proof

- We define the notion of strong cofinality for clones,
- we show that a clone has uncountable strong cofinality if and only if it has uncountable cofinality and the Bergman property,
- we show that the clones in question have uncountable strong cofinality.
Definition of strong cofinality for clones

For a set $U$ of functions, by $U^{[k,2]}$ we denote the set of $k$-ary functions definable from $U$ by terms of depth at most 2.

Definition

For a clone $\mathbb{F} \leq O_A$ and a cardinal $\lambda$, a chain $(U_i)_{i<\lambda}$ of proper subsets of $\mathbb{F}$ is called **strong cofinal chain** of length $\lambda$ for $\mathbb{F}$ if

1. $\bigcup_{i<\lambda} U_i = F$,
2. there exists a $k_0 \in \mathbb{N} \setminus \{0\}$ such that for all $i < \lambda$ and $k \in \mathbb{N} \setminus \{0\}$ with $k \geq k_0$ holds $U_i^{(k)} \subsetneq F^{(k)}$,
3. for all $i < \lambda$ there exists some $j < \lambda$ such that for all $k \in \mathbb{N} \setminus \{0\}$ holds $U_i^{[k,2]} \subseteq U_j$.

The **strong cofinality** of $\mathbb{F}$ is the least cardinal $\lambda$ such that there exists a strong cofinal chain of length $\lambda$ for $\mathbb{F}$. 
Examples

The polymorphism clones of the following structures have the Bergman property:

- the Rado graph,
- the countable generic poset $\mathbb{P} = (P, \leq)$,
- the countable atomless Boolean algebra $\mathbb{B}$,
- the countable universal homogeneous semilattice,
- the countable universal homogeneous distributive lattice,
- the vector-space $\mathbb{F}^\omega$ for any countable field $\mathbb{F}$. 