A generalization of the Kaloujnine-Krasner Theorem

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Motivating theorem from group theory:

Kaloujnine–Krasner Theorem

For any groups $N$ and $H$ every extension of $N$ by $H$ is embeddable in the wreath product of $N$ by $H$. 
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**Kaloujnine–Krasner Theorem**

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Is there any direct generalization for semigroups? If not, whether we can find „similar” theorem, which is still a generalization?
completely simple semigroups $\equiv$

Rees-matrix semigroups with normalized sandwich matrices

$S = M(G; I, \Lambda; P)$, $P$ normalized

$\rho \subseteq S \times S$ congruence

$\rho$ is a group congruence of $S$ iff $\exists N \triangleleft G$ s.t. every entry of $P$ are from $N$
completely simple semigroups \equiv

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\[ \rho \text{ is a group congruence of } S \iff \exists N \triangleleft G \text{ s.t. every entry of } P \text{ are from } N \]

moreover \( S/\rho \cong G/N \) and \( \text{Ker } \rho = \mathcal{M}(N; I, \Lambda; P) \)
completely simple semigroups $\equiv$

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moreover $S/\rho \cong G/N$ and $\text{Ker } \rho = \mathcal{M}(N; I, \Lambda; P)$

We say that $S = \mathcal{M}(G; I, \Lambda; P)$ is an extension of $K = \mathcal{M}(N; I, \Lambda; P)$ by $G/N$. 
$S$ semigroup, $H$ group, $H$ acts on $S$

multiplication on $S \times H$:

$$(s, A)(t, B) = (s \cdot A^t, AB)$$

this is $S \rtimes H$ — *semidirect product* of $S$ by $H$, with respect to the given action of $H$ on $S$
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Special construction: semidirect product $S^H \rtimes H$ with respect to the action $H$ on $S^H$ defined by, for $f \in S^H$, $A \in H$:

$^Af : H \to S, \quad B(^Af) = (BA)f$

this is the *wreath product* of $S$ by $H$, denoted by $S \wr H$
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this is the wreath product of $S$ by $H$, denoted by $S \wr H$

**Important**: $S$ and $H$ completely determines $S \wr H$. 
Kaloujnine–Krasner Theorem

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Let $G$ be an extension of $N$ by $H$.

$r_A \ (A \in H)$ — transversal of the cosets modulo $N$ in $G$

$f_g \in N^H \ (g \in G)$:

An embedding:

$$\varphi : G \rightarrow N \wr H, \ g \mapsto (f_g, gN)$$

$$f_g : H \rightarrow N, \ A \mapsto r_A g r_A^{-1} gN$$
Let $S = \mathcal{M}(G; I, \Lambda; P)$ be an extension of the semigroup $K = \mathcal{M}(N; I, \Lambda; P)$ by the group $H$.

Does there exist an embedding $S \to K \wr H$?

If $G$ is Abelian, mimic the proof of the Kaloujnine–Krasner Theorem:

$\phi : S \to K \wr H, (i, g, \lambda) \mapsto (f_i \lambda g, gN)$,

where $f_i \lambda g : H \to K, A \mapsto (i, r_Agr^{-1}A \cdot gN, \lambda)$.

If $G = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ then an embedding exists, but it is not “natural.”
Let $S = \mathcal{M}(G; I, \Lambda; P)$ be an extension of the semigroup $K = \mathcal{M}(N; I, \Lambda; P)$ by the group $H$. Does there exist an embedding $S \rightarrow K \wr H$?

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where $f_{i\lambda} g: H \rightarrow K$, $A \mapsto (i, r_A g r_{\lambda^{-1} A} \cdot g N, \lambda)$. If $G = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ then an embedding exists, but it is not "natural."
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If $G$ is Abelian, mimic the proof of the Kaloujnine–Krasner Theorem:

$$\varphi: S \rightarrow K \wr H, \ (i, g, \lambda) \mapsto (f_{g}^{i\lambda}, gN),$$

where

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Let \( S = \mathcal{M}(G; I, \Lambda; P) \) be an extension of the semigroup \( K = \mathcal{M}(N; I, \Lambda; P) \) by the group \( H \). Does there exist an embedding

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If \( G \) is Abelian, mimic the proof of the Kaloujnine–Krasner Theorem:

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\varphi: S \to K \wr H, \ (i, g, \lambda) \mapsto (f^i_\lambda g, gN),
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If \( G = \mathbb{Z}_3 \rtimes \mathbb{Z}_2 \) then an embedding exists, but it is not „natural.”
conjecture: embedding does not exist in general
⇒ look for a counterexample
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first we would like to express the wreath product in a semidirect product form:

\[
K \wr H = K^H \rtimes H \cong \mathcal{M}(N^H; I^H, \Lambda^H; P^H) \rtimes H,
\]

where \( P^H = (p^H_{\xi \eta}) \) and for any \( \xi \in \Lambda^H, \eta \in I^H \):

\[
p^H_{\xi \eta} : H \to N, A p^H_{\xi \eta} = p_{A \xi, A \eta} \quad (A \in H)
\]

\( \mathbb{Z}_n \rtimes \mathbb{Z}_2 \) is not good because of \( \mathbb{Z}_2 \) is too „small”
the source of the problem is in the sandwich matrix of \( K \wr H \), where the entries are strongly related to each other
it suffices to work a $2 \times 2$ sandwich matrix if $\mathbb{Z}_2$ is replaced by $\mathbb{Z}_3$

the proof uses that one entry of $G$ has order 3, and the image of this element can be expressed by means of the entries of $P^H$
so we do not have enough freedom to choose it appropriately
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The proof uses that one entry of $G$ has order 3, and the image of this element can be expressed by means of the entries of $P^H$ so we do not have enough freedom to choose it appropriately.

$$h = p^H_{\xi_1 \eta_1} (p^H_{\xi_2 \eta_1})^{-1} p^H_{\xi_2 \eta_2} (p^H_{\xi_1 \eta_2})^{-1}$$
Theorem

Let $G = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, $I = \Lambda = \{1, 2\}$, $P$ be the sandwich matrix for which $p_{11} = p_{12} = p_{21} = (0, 0)$ and $p_{22} = (1, 0)$, and $N = \{(a, 0) : a \in \mathbb{Z}_7\}$. Let $S = \mathcal{M}(G; I, \Lambda; P)$ and $K = \mathcal{M}(N; I, \Lambda; P)$. Then there exists no embedding

$$S \rightarrow K \wr H.$$
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$S = \mathcal{M}(G; I, \Lambda; P)$ and $K = \mathcal{M}(N; I, \Lambda; P)$. Then there exists no embedding

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Important: there is no embedding at all, not just a „nice” embeddings like in the Kaloujnine–Krasner Theorem
How can we obtain a positive result with a similar construction?
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Idea: wreath product $\rightarrow$ semidirect product
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We are looking for an embedding

$$ S = \mathcal{M}(G; I, \Lambda; P) \rightarrow \mathcal{M}(N'; I', \Lambda'; P') \rtimes H, $$

and we don’t want to go far from the Kaloujnine–Krasner Theorem
let $N' = N^H$, $I' = I$, $\Lambda' = H \times \Lambda$, and the entries of $P'$ are „nice” maps
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**Theorem**

For any extension $S = \mathcal{M}(G; I, \Lambda; P)$ of $K = \mathcal{M}(N; I, \Lambda; P)$ by a group $H$, there exists an embedding

$$S \to \mathcal{M}(N^H; I, H \times \Lambda; Q) \rtimes H,$$

where the restriction of this embedding to maximal subgroups of $S$ coincides with that in the proof of the Kaloujnine–Krasner Theorem, and the entries of $Q$ can be expressed by means of the ingredients there, too.