Algebraic approach to coloring by oriented trees

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19% of definitions in this talk

- a relational structure: $\mathbb{A} = \langle A; R_1, \ldots, R_n \rangle$, where $R_i \subseteq A^{k_i}$
- a directed graph: $\mathbb{G} = \langle G; \rightarrow \rangle$, where $\rightarrow$ is binary
- an algebra: $A = \langle A; \mathcal{F} \rangle$, $\mathcal{F}$ is a clone of operations on $A$
- a subuniverse $(C \leq A)$: a subset closed under all operations
- an idempotent algebra: every $f \in \mathcal{F}$ satisfies $f(x, x, \ldots, x) \approx x$ (equivalently, $\{a\} \leq A$ for every $a \in A$)

*All domains in this talk are finite!*
Fixed template CSPs

- fix a finite relational structure $\mathbb{A}$
- the Constraint satisfaction problem over $\mathbb{A} = \text{membership problem for the set}$
  \[\text{CSP}(\mathbb{A}) = \{X | X \to \mathbb{A}\}\]
- goal: characterize relational structures wrt. complexity of the CSP and related algorithmic properties

Conjecture (The CSP dichotomy conjecture – Feder, Vardi ’93)

*For every $\mathbb{A}$, CSP($\mathbb{A}$) is in P or NP-complete.*
Algebra of polymorphisms

- **polymorphisms** of $A = \text{operations preserving all relations}

$$f(a_1 \ a_2 \ \ldots \ a_n) = a$$

$$\downarrow \ \downarrow \ \ \ \downarrow$$

$$f(b_1 \ b_2 \ \ldots \ b_n) = b$$

- $\langle A; \text{Pol}(A) \rangle = \text{the algebra of polymorphisms of } A$

- a **primitive positive (pp-)** formula: $\exists$, $\land$, $=$

- relations pp-definable from $A = \text{SP}_{\text{fin}}(A)$

*Relational structures are algebras, too! (See Ross Willard’s talk.*)
The algebraic approach to CSP & Maltsev conditions

- Bulatov, Jeavons, Krokhin '00-'05: complexity of CSP($\mathbb{A}$) is controlled by the equational theory of HSP($\mathbb{A}$)
- A strong Maltsev condition = finite set of equations in some operation symbols
- A weak near-unanimity (WNU) = $n$-ary operation ($n \geq 2$) satisfying
  
  \[ f(x, \ldots, x, y) \approx f(x, \ldots, x, y, x) \approx \cdots \approx f(y, x, \ldots, x) \]

- A near-unanimity (NU) = a WNU such that $f(x, \ldots, x, y) \approx x$
- A majority = ternary NU (eg. $(x \land y) \lor (y \land z) \lor (x \land z)$)
- A semilattice operation
Cores & constants

- a core structure = every endomorphism is an automorphism
- every structure has a unique (up to isomorphism) core

The algebraic approach works only for cores
- but $\text{CSP}(A) = \text{CSP}(\text{core of } A)$
- also, we can add all singleton unary relations (i.e., we can prescribe values to variables) $\Rightarrow$ idempotent algebras
Two important classes of algebras

- Taylor algebra = satisfies any nontrivial strong Maltsev condition
- Maróti, McKenzie ’06: Taylor iff has some WNU
- Bulatov, Jeavons, Krokhin ’00-05:
  - If a core $A$ is not Taylor, then \( \text{CSP}(A) \) is \( \text{NP} \)-complete.

**Algebraic dichotomy conjecture**

- If a core $A$ is Taylor, then \( \text{CSP}(A) \) is in \( \text{P} \).
- $A$ has **bounded width (BW) = \( \text{CSP}(A) \) solvable by local consistency checking (in \( \text{P} \)), “Can’t encode linear equations.”
- $SD(\wedge)$ algebra = \( \text{HSP}(A) \) has \( \wedge \)-semidistributive congruence lattices
- Maróti, McKenzie ’06: $SD(\wedge)$ iff has WNUs of almost all arities
- Barto, Kozik ’08: **Bounded width theorem**
  - A core $A$ has BW iff it is $SD(\wedge)$.
Absorption & always absorbing algebras

- an absorbing subuniverse \((C \subseteq A)\) = there exists an idempotent \(t \in \mathcal{F}\) such that

\[
\begin{align*}
  t(A, C, \ldots, C, C) & \subseteq C, \\
  t(C, A, \ldots, C, C) & \subseteq C, \\
  \vdots \\
  t(C, C, \ldots, C, A) & \subseteq C. \\
\end{align*}
\]

- an absorption-free (AF) algebra = no proper absorbing subuniverse
- an always absorbing (AA) algebra = every \(C \leq A\) has a proper absorbing subuniverse (equivalently, no AF algebra in \(\text{HSP}_{\text{fin}}(A)\))
- example: NU, semilattice
- AA algebras are \(SD(\land)\)
Not every slide needs a title

- near-unanimity
- strict width
- greedy algorithm

always absorbing
- no AF algebras

SD(∧)
- bounded width
- almost all WNUs

Taylor WNU

semilattice
- implies width 1
CSP over digraphs aka $H$-coloring problem

- **Feder, Vardi '93**: for every $A$ there exists a digraph $H$ such that $\text{CSP}(A) \equiv^P \text{CSP}(H)$

- **JB, Delić, Jackson, Niven '11**: a simple construction, almost all interesting Maltsev conditions are preserved, conjectures characterizing CSPs in $P$, $NL$, $L$ reduce to digraphs
  news! actually, $\text{CSP}(A) \equiv^L \text{CSP}(H)$ (talk to Marcel)

- why digraphs? fieldtest & inspiration for the algebraic approach, possibly interesting combinatorial facts

- **Hell, Nešetřil '90**: CSP dichotomy for undirected graphs

- **Barto, Kozik, Niven '06**: dichotomy for smooth digraphs
  in fact, core smooth Taylor digraph = disjoint union of directed cycles, thus has a majority $\Rightarrow$ is AA
Oriented trees

- oriented paths have both majority and semilattice \(\Rightarrow\) are AA
- oriented **triads** (join 3 paths in one vertex) are already hard
Special oriented trees

- oriented trees have levels; maximum level = height
- a minimal path = initial vertex has level 0, terminal vertex level \(k\), and for all other vertices \(0 < \text{level}(v) < k\)

Definition

Let \(\mathbb{T}\) be an oriented tree of height 1. A \(\mathbb{T}\)-special tree is an oriented tree obtained from \(\mathbb{T}\) by replacing all edges by minimal paths of the same height (preserving orientation).

- a special triad = \(\mathbb{T}\)-special tree where
Example of a special triad

Barto, Kozik, Maróti, Niven: Is this the smallest NP-complete oriented tree? (38 vertices)
The history of special trees

- Gutjahr, Welzl, Woeginger '92: an NP-complete oriented tree (81 vertices)
- Hell, Nešetřil, Zhu '95: invented the special triads, constructed an NP-complete one (45 vertices) and more
- Barto, Kozik, Maróti, Niven '08: CSP dichotomy for special triads, Taylor implies either majority or width 1
- Barto, JB '10: CSP dichotomy for special polyads, Taylor implies $SD(\land)$, a rather complicated proof

**Theorem (JB '13)**

_The CSP (algebraic) dichotomy holds for all special trees. Taylor special trees are $SD(\land)$. (Maybe even AA, work in progress...)_

- an easy(-ish) proof, “localization”, uses very recent algebraic tools
(I have no time for) sketch of the proof

- $\mathbb{H}$ – a $\mathbb{T}$-special tree, Taylor
  $\mathbb{T} = \langle A \cup B; E \rangle$, $E \subseteq A \times B$ – an oriented tree of height 1
- $A$, $B$ and $E$ are pp-definable from $\mathbb{H}$
- $\mathbb{H}$ is $SD(\wedge)$ iff both $A$ and $B$ are $SD(\wedge)$ (this is “special”)
- $A$ or $B$ has a singleton absorbing subuniverse (Absorption theorem!)
- WLOG $\{o\} \subseteq A$, partial ordering of $A \cup B$ by distance from $o$
  - closer elements absorb more distant ones
  - $E$-neighbourhoods of singletons are AA (this is the only technical bit; we construct nice binary polymorphisms)
  - $A$ and $B$ are AA
Conjecture

Every Taylor oriented tree is already SD(∧).

“Taylor trees cannot encode linear equations.”

Problem

Is there a homotopy-like notion for oriented trees (cf. homotopy for reflexive digraphs of Larose and Tardif)?

Problem

Characterize (finite, idempotent) AA algebras.

Thank you for your attention!