From lifting objects to lifting diagrams: Recent progress on larders and CLL

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We are addressing the following general problem. We are given categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ together with functors $\Phi\colon \mathcal{A} \to \mathcal{S}$ and $\Psi\colon \mathcal{B} \to \mathcal{S}$, as illustrated on the left hand side of the following figure.

FIGURE 1. A few categories and functors

We assume that for "many" objects A of $\mathcal{A}$, there exists an object B of $\mathcal{B}$ such that $\Phi(A) \cong \Psi(B)$. We ask in what extent the assignment $A \mapsto B$ can be made *functorial*. The ideal situation would be the existence of a functor $\Gamma\colon \mathcal{A} \to \mathcal{B}$ such that $\Gamma \circ \Phi$ is equivalent to Ψ, cf. the previous figure.

Condensate Lifting Lemma CLL [P. Gillibert and FW, 2008] *If $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, Φ, Ψ, together with full subcategories $\mathcal{A}^\dagger$ of $\mathcal{A}$ and $\mathcal{B}^\dagger$ of $\mathcal{B}$, and a subcategory $\mathcal{S}^\Rightarrow$ of $\mathcal{S}$, form a so-called "larder", then "something like this can be done".*

The objects in $\mathcal{A}^\dagger$, resp. $\mathcal{B}^\dagger$, can be thought of as "small objects" while the arrows in $\mathcal{S}^\dagger$ can be thought of as "quotient maps", for example from the congruence lattice of an algebra to the congruence lattice of a quotient. Nevertheless CLL is a result of category theory.

As consequences of either CLL or the methods leading to it, we shall discuss the following results.

Theorem 1. [P. Gillibert 2008] *For varieties $\mathcal{A}$ and $\mathcal{B}$ of algebras with $\mathcal{A}$ locally finite and $\mathcal{B}$ finitely generated congruence-distributive, the "critical point"* crit$(\mathcal{A};\mathcal{B})$, *defined as the least cardinality of a semilattice in the compact congruence class of* $\mathcal{A}$ *but not of* $\mathcal{B}$, *is smaller than* $\aleph_\omega$.

This result is proved in [2]. By using von Neumann regular rings and the dimension monoid construction, these results are extended to some explicit calculations of critical points in [3]. Theorem 1 is extended further to *quasivarieties* in [4].

Theorem 2. [P. Gillibert and FW, 2008] *The Grätzer-Schmidt Theorem* (*i.e., every algebraic lattice is isomorphic to the congruence lattice of some algebra*) *can be extended to any diagram, indexed by any finite poset* (*or, in presence of a proper class of Erdős cardinals, any poset*) *of algebraic lattices and complete* $(\vee, 0)$*-homomorphisms.*

Theorem 3. [FW 2008] *There exists a non-coordinatizable sectionally complemented modular lattice, of cardinality* $\aleph_1$, *with a large von Neumann* 4*-frame.*

Theorem 4. [P. Gillibert and FW 2009] *Let* $\mathcal{V}$ *be a nondistributive variety of lattices. Then the free lattice* (*resp., free bounded lattice*) *in* $\mathcal{V}$ *on* $\aleph_1$ *generators has no congruence-permutable, congruence-preserving extension.*

References

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[2] P. Gillibert, *Critical points of pairs of varieties of algebras*, Internat. J. Algebra Comput. **19**, no. 1 (2009), 1–40.

[3] P. Gillibert, *Critical points between varieties generated by subspace lattices of vector spaces*, preprint 2008, available online at `http://hal.archives-ouvertes.fr/hal-00324397`.

[4] P. Gillibert and F. Wehrung, *A categorical approach of critical points between quasivarieties*, manuscript.