Strongly regular types

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Definition. (M, cl) is a pregeometry if $\mathrm{cl} : \mathcal{P}(M) \to \mathcal{P}(M)$ satisfies the following axioms:

- $X \subseteq Y$ implies $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$;
- $X \subseteq \mathrm{cl}(X) = \mathrm{cl}(\mathrm{cl}(X))$;
- $\mathrm{cl}(X) = \bigcup\{\mathrm{cl}(X_0) \mid X_0 \subseteq X \text{ finite}\}$;
- $a \in \mathrm{cl}(X \cup \{b\}) \setminus \mathrm{cl}(X)$ implies $b \in \mathrm{cl}(X \cup \{a\})$.

In model theory a pregeometry usually appears in the case of regular types in stable theories. I will discuss the general case when $(M, ...)$ is a (infinite) first-order structure and the pregeometry derives from the structure: examples are vector spaces, fields and differential fields. The basic idea is that $a \in \mathrm{cl}(B)$ should be witnessed by a first-order formula whose solution set is considered as 'small' in M. Then the set of 'big' formulas generates a filter or, equivalently, a type over M. The type particularly well behaves when M is saturated and it is complete (i.e. ultrafilter) and the closure is described in terms of semi-isolation.

Now we wonder whether a type over $\emptyset$ determines a pregeometry via semi-isolation. The following definition covers both the stable and the saturated case:

Definition. Let T be a countable, complete first-order theory. A non-isolated type $p \in S_1(\emptyset)$ is strongly regular via $\phi(x) \in p$ if and only if for all $M \models T$ and $a_1 a_2 ... a_n \in \phi(M) \setminus p(M)$, $p(x) \cup \mathrm{tp}(a_1 a_2 ... a_n)$ determines a complete $(n+1)$-type.

The main result is that if $p \in S_1(\emptyset)$ is strongly regular via $x = x$ and M is saturated then $(p(M), \mathrm{Sem})$ is a pregeometry unless there is a very specific partial order definable (with parameters) in the structure.