# Non-isomorphic $\mathcal{R}(\mathcal{L})$-cross-sections of the partial wreath product of finite symmetric inverse semigroups

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For $n \in \mathbb{N}$ denote by $\mathcal{N}_n$ the set $\{1, 2, \ldots, n\}$ and denote by $\mathcal{IS}_n$ the set of all partial bijections on $\mathcal{N}_n$. The set $\mathcal{IS}_n$ forms a semigroup under usual composition of maps. This semigroup is called the finite symmetric inverse semigroup.

Let $S$ be a semigroup, $(P, X)$ be a semigroup of partial transformations of the set $X$. Define the set $S^{PX}$ as a set of partial functions from $X$ into the semigroup $S$:

$$S^{PX} = \{f : A \to S |\, \mathrm{dom}(f) = A, A \subseteq X\}.$$

Given $f, g \in S^{PX}$, the product $fg$ is defined in the following way:

$\mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g), (fg)(x) = f(x)g(x)$ for all $x \in \mathrm{dom}(fg)$.

For $a \in P, f \in S^{PX}$, define $f^a$ as:

$$(f^a)(x) = f(xa),\ \mathrm{dom}(f^a) = \{x \in \mathrm{dom}(a); xa \in \mathrm{dom}(f)\}.$$

The *partial wreath product* of a semigroup $S$ with the semigroup $(P, X)$ of partial transformations of the set $X$ is the set

$$\{(f, a) \in S^{PX} \times (P, X) \mid \mathrm{dom}(f) = \mathrm{dom}(a)\}$$

with composition defined by $(f, a) \cdot (g, b) = (fg^a, ab)$. We are going to denote the partial wreath product of semigroups $S$ and $(P, X)$ by $S \wr_p P$.

The partial wreath product of inverse semigroups is also an inverse semigroup.

Further, let $S$ be an inverse semigroup with identity. Recall that Green's $\mathcal{R}(\mathcal{L})$-relation on a semigroup $S$ is defined as $a\,\mathcal{R}\,b \Leftrightarrow aS = bS$ (resp. $a\,\mathcal{L}\,b \Leftrightarrow Sa = Sb$).

A subsemigroup $T$ of a semigroup $S$ is called the $\mathcal{R}(\mathcal{L})$-cross-section provided that $T$ contains exactly one element from every $\mathcal{R}(\mathcal{L})$-class.

**Theorem.** *Let $R'$, $R''$ be $\mathcal{R}$-cross-sections of the semigroup $\mathcal{IS}_m \wr_p \mathcal{IS}_n$ and let $\varphi\colon R' \to R''$ be an isomorphism. Then there exists an element $\Theta = (\vartheta, \theta) \in \mathcal{S}_m \wr \mathcal{S}_n$ that*

$$\varphi\big((f,a)\big) = \Theta^{-1}(f,a)\Theta.$$

*In other words, if $(f,a) \in R'$ and $(g,b) = \varphi\big((f,a)\big)$, then*

$$\operatorname{dom} b = \theta\big(\operatorname{dom}(a)\big)$$

*and for any $x \in \operatorname{dom}(a)$ we have*

$$\theta\big(a(x)\big) = b\big(\theta(x)\big), \quad g\big(\theta(x)\big) = \vartheta^{-1}(x)f(x)\vartheta\big(a(x)\big).$$

Dualizing this theorem we obtain a description of $\mathcal{L}$-cross-sections of semigroup $\mathcal{IS}_m \wr_p \mathcal{IS}_n$.

Because of the recursive definition of the partial wreath product, one can generalize this theorem for $\mathcal{R}(\mathcal{L})$-cross-sections of the semigroup

$$\mathcal{IS}_{n_k} \wr_p \mathcal{IS}_{n_{k-1}} \ldots \wr_p \mathcal{IS}_{n_1}.$$

We also compute the number of all non-isomorphic $\mathcal{R}(\mathcal{L})$-cross-sections.

## References


[1] Ganyushkin, O., Mazorchuk, V., Classical Finite Transformation Semigroups. An Introduction, London: Springer 2009

[2] Kochubinska, E.A., On cross-sections of partial wreath product of inverse semigroups. *Electron. Notes Discrete Math.* **28**, (2007), 379-386.