Free spectra of intermediate growth

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The *spectrum* of an algebra **A** is the cardinal-valued function $\mathrm{Spec}_{\mathbf{A}} : \omega \to$ **Card** whose value at n is the number of functions $f \colon A^n \to A$ that are the interpretations of terms. (It is sometimes called the free spectrum of **A**, because $\mathrm{Spec}_{\mathbf{A}}(n)$ is the size of the n-generated free algebra in the variety generated by **A**.) This talk will be about the possible growth rates of $\mathrm{Spec}_{\mathbf{A}}$ when **A** is a finite algebra.

The number of functions $f \colon A^n \to A$ is $|A|^{|A|^n}$, so $\mathrm{Spec}_{\mathbf{A}}(n) \leq 2^{2^{cn}}$ for some constant c whenever **A** is finite. If also $\mathrm{Spec}_{\mathbf{A}}(n) \geq 2^{2^{dn}}$ for some constant d and all large n, then we call the spectrum of **A** 'large'. Algebras with large spectrum are considered unclassifiable.

We call the spectrum of **A** 'small' if it has a polynomial upper bound. Joel Berman proved that if **A** is finite and $\mathrm{Spec}_{\mathbf{A}}$ is small, then $\mathrm{Spec}_{\mathbf{A}}$ is itself a polynomial, and that if $\mathrm{Spec}_{\mathbf{A}}$ is not small, then $\mathrm{Spec}_{\mathbf{A}}(n) \geq 2^{n-|A|}$ for all positive n. Finite algebras with small spectrum were classified by Kearnes and Kiss: they are the strongly nilpotent algebras of essentially finite signature.

What remains to understand is the class of finite algebras whose spectrum is not a polynomial but has no lower bound of the form $2^{2^{dn}}$. These are spectra of intermediate growth. About 10 years ago I showed that if a finite algebra **A** belongs to a congruence modular variety, then **A** has spectrum of intermediate growth iff **A** is supernilpotent, in which case $\mathrm{Spec}_{\mathbf{A}}(n) = 2^{\Theta(n^k)}$ for some positive integer k. This result suggested that the possible spectra of intermediate growth could be described, and the algebras with such spectra could be classified. Since 2001 the conjecture has been that up to log asymptotic equivalence the only spectra of intermediate growth are $2^{\Theta(n^k)}$ and $2^{\Theta(n^k \lg(n))}$ for some positive integer k, and that all of these spectra could be realized by simple algebras of type **5**.

In this talk we will explain how to construct a 3-element algebra out of any class of hypergraphs closed under induced quotients. The construction preserves the spectrum (or 'speed') of the class up to a log linear factor. This construction allows us to import results from hypergraph theory to show that spectra of finite algebras can be much wilder than previously thought: there are continuumly many spectra of 3-element algebras up to log asymptotic equivalence, spectra can oscillate wildly, there are 3-element algebras whose spectra are log asymptotically incomparable, etc.