Möbius number systems

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INTRODUCTION. A Möbius number system (MNS) assigns real numbers to sequences of Möbius transformations. The result is a continuous projection $\Phi : \Sigma \to \overline{\mathbb{R}}$ where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the extended real line and the set $\Sigma \subset A^\omega$ is a subshift. (Compare this situation to the standard binary representation of numbers which yields continuous projection $\{0,1\}^\omega \to [0,1]$.)

Möbius number systems are quite flexible, for example we can implement continued fractions as a Möbius number system.

A fundamental complication of Möbius number systems is that, unlike in ordinary numeration systems, we must always forbid some words from the set Σ, therefore $\Sigma \subsetneq A^\omega$.

The purpose of this presentation is to provide the definitions and examples of Möbius number systems and partially answer the question which subshifts can accommodate such a system.

MAIN POINTS. In the following, we cover the most important points of information on MNS to be used as a quick reference during the talk.

Definition. A Möbius transformation (MT) is any nonconstant function $M : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ of the form

$$M(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$.

Definition. A sequence $M_1, M_2, \dots$ of (complex) upper half-plane preserving MTs *represents the number* $x \in \overline{\mathbb{R}}$ if $M_n(i) \to x$ for $n \to \infty$ (here i is the complex unit).

Let A be a finite alphabet. A finite or infinite sequence of symbols from A is called a *word*. By w_i we mean the i-th letter of the word w. For

u, v finite words (or letters), denote by uv the concatenation of u and v. Denote by A^ω the set of all one-sided infinite words over A.

The set $\Sigma \subset A^\omega$ is a *subshift* if there exists a set S of finite words such that $w \in \Sigma$ iff w does not contain any $s \in S$ as a factor (i.e. there are no indices i, j such that $w_i w_{i+1} \dots w_j = s$).

Assume that we have assigned to every letter $a \in A$ a corresponding MT F_a. We then define F_v for any finite word v by $F_v = F_{v_1} \circ F_{v_2} \circ \cdots \circ F_{v_n}$ (we compose mappings as $(F \circ G)(z) = F(G(z))$).

Definition. Assume we are given a system of MTs $\{F_a : a \in A\}$. A subshift $\Sigma \subset A^\omega$ is a *Möbius number system* if:

(1) For every $w \in \Sigma$, the sequence $\{F_{w_1 \dots w_n}\}_{n=1}^\infty$ represents some point $\Phi(w) \in \overline{\mathbb{R}}$.
(2) The function $\Phi : \Sigma \to \overline{\mathbb{R}}$ is continuous and surjective.

A substitution is a monoid homomorphism $\rho : A^* \to B^*$. We consider only non-erasing substitutions. Every such substitution can be extended in a natural way to a map $\rho : A^\omega \to B^\omega$.

Theorem. *If Σ is a MNS then $\Sigma \neq \rho(A^\omega)$ for all alphabets A and all substitutions ρ.*

References

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