Solution of the word problem in certain inverse monoids

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Let $F(X)$ be the free group on X and let R be a cyclically reduced word in $F(X)$. Let G be the one-relator group presented by the group presentation gp $\langle X|R\rangle$ and let M be the inverse monoid presented by the one-relator inverse monoid presentation

(1) $\mathrm{Inv}\,\big(X\big|R = 1\big)$.

While the Word Problem is known to be solvable in one-relator groups, relatively little is known on the solvability of the Word Problem in one-relator inverse monoids. See [IMM] for examples.

In this work we solve the Word Problem for a class of one-relator inverse monoids.

Recall that a word $W \in F(X)$ is called *unbordered* if no proper prefix of W is its suffix.

Main Theorem. *Let notation be as above. Assume $R = U^n$, $n \geq 4$, where U is unbordered. Let y be the first letter of U and let t be the last letter if U. If y^{-1} and t^{-1} do not occur in U then the one-relator inverse monoid $Inv\,\big(X\big|U^n = 1\big)$ has solvable Word Problem.*

The proof of the above theorem relies on Theorem 3.1 in [IMM], which reduces the solvability of the Word Problem for the monoid M presented by (1) to the Word Problem in G and the Membership Problem for the submonoid P_R of the maximal group image G of M, generated by the proper prefixes (initial subwords) of R. In fact we solve the latter problem for R as above.

Denote by P the set of proper prefixes of R in $F(X)$. The main ingredient of the proof is the following proposition.

Proposition. *Let $g \in G$ and let ℓ be the length of a shortest representative of g in $F(X)$. If $g \in P_R$ then P^* contains a word of length at most $c\ell$, where c is a computable constant depending on $|R|$, which represents g.*

The proof of this Proposition uses van Kampen diagrams and word combinatorics.

Using the above Proposition, the solution of the Membership Problem for P_R is as follows: let U be any word in $F(X)$ which represents g. Since the Word Problem in G is solvable, due to W. Magnus, we may compute ℓ and hence we may also construct the following finite set

$$S = \{W \in F(X) |\ |W| \leq c\ell,\ W \text{ represents } g\}.$$

But since P^* has a solvable Membership Problem in $F(X)$, due to M. Benois (see also [IMM], Lemma 5.4), we may construct $S \cap P^*$. Now, due to the Proposition, $S \cap P^* \neq \emptyset$ if an only if $g \in P_R$.

This completes the solution of the Membership Problem of P_R in G.

Remarks.

(1) The Main Theorem is the starting point of a project which aims to solve some of the basic decision problems in certain one-relator inverse monoids given by (1), using the fact that for some words, like R , the solutions of these problems in G are known in a very detailed manner.

(2) With much more work on the side of word combinatorics we can deal with the cases $n = 3$ and $n = 2$.

References

[IMM] S. V. Ivanov, S. W. Margolis, J. C. Meakin: On one-relator inverse monoids and one relator groups, *J. Pure Appl. Algebra* **159** (2001), 83–111.