Luzin and Sierpiński sets

Marcin Michalski and Szymon Żeberski
Wrocław University of Technology

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Let $\mathcal{I}$ be a $\sigma$-ideal of subsets of $\mathbb{R}$ ($\mathbb{R}^2$) and $\mathcal{B}$ a family of Borel sets. We say that $\mathcal{I}$:

- is translation invariant, if for each $x \in \mathbb{R}$ and $I \in \mathcal{I}$ we have $x + I \in \mathcal{I}$,
- is scale invariant, if for each $x \in \mathbb{R}$ and $I \in \mathcal{I}$ we have $xI \in \mathcal{I}$,
- has Borel base if $(\forall I \in \mathcal{I})(\exists B \in \mathcal{B} \cap \mathcal{I})(I \subseteq B)$,
- has Steinhaus property if $Int(A - B) \neq \emptyset$ for each $A, B \in \mathcal{B} \setminus \mathcal{I}$.
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**Example**

Meager sets $\mathcal{M}$ and null sets $\mathcal{N}$ have these properties.
Definition

$A$ is

- $\mathcal{I}$-nonmeasurable if $A \notin \sigma(B \cup \mathcal{I})$,
- completely $\mathcal{I}$-nonmeasurable if $A \cap B$ is $\mathcal{I}$-nonmeasurable for every $B \in B \setminus \mathcal{I}$,
- $\mathcal{I}$-Luzin set if $|A| = \mathfrak{c}$ and for every $I \in \mathcal{I}$ a set $A \cap I$ is countable,
- strong $\mathcal{I}$-Luzin set if $A$ is an $\mathcal{I}$-Luzin and its intersection with every Borel $\mathcal{I}$-positive set is uncountable.
Definition

A is:

- a Luzin set if $|L| = c$ and every intersection of $L$ and a meager set is countable,
- a strong Luzin set if $A$ is a Luzin set and every intersection of $A$ and a $\mathcal{M}$-positive Borel set is uncountable,
- a Sierpiński set if $|S| = c$ and every intersection of $S$ and a null set is countable,
- a strong Sierpiński set if $A$ is a Sierpiński set and every intersection of $A$ and a $\mathcal{N}$-positive Borel set is uncountable,
- a Bernstein set if for each perfect set $P$ we have $A \cap P \neq \emptyset$ and $A^c \cap P \neq \emptyset$. 
Fact
Let $B$ be a Borel $\mathcal{I}$-positive set and let $D$ be a countable dense set. Then $B + D$ is an $\mathcal{I}$-residual set.

Corollary
Let $L$ be a $\mathcal{I}$-Luzin set. Then $L + \mathbb{Q}$ is a strong $\mathcal{I}$-Luzin set.

Fact (CH)
There exists a partition of $\mathbb{R}$ into $c$ many strong $\mathcal{I}$-Luzin sets.
Theorem (CH)

There exists a set $A \subseteq \mathbb{R}^2$ such that each horizontal slice $A^y$ is a strong $\mathcal{I}$-Luzin set and each vertical slice $A_x$ is a cocountable set. Such a set is $\mathcal{M}$ and $\mathcal{N}$-nonmeasurable. Moreover, in the case $\mathcal{I} = \mathcal{M}$, $A$ is completely $\mathcal{M}$-nonmeasurable, and in the case $\mathcal{I} = \mathcal{N}$, $A$ is completely $\mathcal{N}$-nonmeasurable.
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Theorem (CH)

There exists a set $A \subseteq \mathbb{R}^2$ such that each vertical slice $A_x$ is cocountable and $A$ is completely $\mathcal{M}$, $\mathcal{N}$-nonmeasurable.
Theorem (CH)

There exists a set $A \subseteq \mathbb{R}^2$ such that each horizontal slice $A^y$ is a strong Luzin set and each vertical slice $A_x$ is strong Sierpiński set. Moreover, $A$ is completely $\mathcal{M}$- and $\mathcal{N}$-nonmeasurable.
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Proof
Let $\{L_\alpha : \alpha < c\}$ and $\{S_\alpha : \alpha < c\}$ be a partition of $\mathbb{R}$ into strong Luzin sets and strong Sierpiński sets respectively. Let us set:

$$A = \bigcup_{\alpha < c} (L_\alpha \times S_\alpha).$$
Theorem

- Assume that a Luzin set exists. Then there exists a set $A \subseteq \mathbb{R}^2$ such that for each straight line $l$ a set $A \cap l$ is a strong Luzin set.
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- (CH) There exists a set $A \subseteq \mathbb{R}^2$ such that for each straight line $l$ a set $A \cap l$ is a strong Luzin set and $A$ is a Hamel basis.
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- (CH) There exists a set $A \subseteq \mathbb{R}^2$ such that for each homeomorphism $h : \mathbb{R} \to \mathbb{R}^2$ on its image a set $h(\mathbb{R}) \cap A$ is a strong Luzin set and $A$ is a Hamel basis.
Theorem (CH)

There exist a set $A \subseteq \mathbb{R}^2$ such that for every increasing continuous function $f$ $A \cap f$ is a strong Luzin set and for each decreasing locally absolutely continuous function $g$ $A \cap g$ is a strong Sierpiski set and $A$ is a Hamel basis.
Theorem

- Assume that a Sierpiński set exists. Then there exists a set $A \subseteq \mathbb{R}^2$ such that for each straight line $l$ a set $A \cap l$ is a strong Sierpiński set.
- (CH) There exists a set $A \subseteq \mathbb{R}^2$ such that for each straight line $l$ on the plane a set $l \cap A$ is a strong Sierpiński set and $A$ is a Hamel basis.
**Fact**

- Let $L$ be an $\mathcal{I}$-Luzin set. Then there exists a linearly independent $\mathcal{I}$-Luzin set.
- Let $L$ be an $\mathcal{I}$-Luzin set. Then there exists a linearly independent strong $\mathcal{I}$-Luzin set.

**Problem**
Does the existence of an $\mathcal{I}$-Luzin set imply the existence of an $\mathcal{I}$-Luzin set which is a Hamel base?
Fact (CH)
There is an $\mathcal{I}$-Luzin set $L$ such that $L$ is a linear subspace of $\mathbb{R}$.

Theorem
It is consistent that $2^\omega = \omega_2$ and there is a Luzin set which is a linear subspace of $\mathbb{R}$.
Fact (CH)
There is an \( \mathcal{I} \)-Luzin set \( L \) such that \( L \) is a linear subspace of \( \mathbb{R} \).

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It is consistent that \( 2^\omega = \omega_2 \) and there is a Luzin set which is a linear subspace of \( \mathbb{R} \).

Proof.
Let us work in a model \( V' \) obtained from a model \( V \) of CH by adding \( \omega_2 \) Cohen reals \( \{ c_\alpha : \alpha < \omega_2 \} \). Set
\[
L = \text{span}_\mathbb{Q}(\{ c_\alpha : \alpha < \omega_2 \}).
\]
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□

Problem

Does the existence of a Luzin set imply the existence of a Luzin set which is a linear subspace of $\mathbb{R}$?
Theorem (CH)
For each \( \mathcal{I} \)-Luzin set \( L \) there exists an \( \mathcal{I} \)-Luzin set \( X \) such that \( \{ x + L : x \in X \} \) is a partition of \( \mathbb{R} \).

Theorem (CH)
There exists an \( \mathcal{I} \)-Luzin set \( L \) such that \( L + L \) is an \( \mathcal{I} \)-Luzin set.

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Theorem (CH)
For each \( n \in \mathbb{N} \setminus \{0\} \) There exists an \( \mathcal{I} \)-Luzin set \( L \) such that \( \bigoplus^n L \) is an \( \mathcal{I} \)-Luzin set and \( \bigoplus^{n+1} L = \mathbb{R} \).

Theorem (CH)
There exists an \( \mathcal{I} \)-Luzin set \( L \) such that \( \text{span}(L) \) is an \( \mathcal{I} \)-Luzin set.
Corollary (CH)

1. There exists an $\mathcal{I}$-Luzin set $L$ such that $\bigoplus^{n+1} L$ is an $\mathcal{I}$-Luzin for each $n \in \mathbb{N}$,

2. There exists an $\mathcal{I}$-Luzin set $L$ such that $L + L = L$,

3. There exists an $\mathcal{I}$-Luzin set $L$ such that $\langle \bigoplus^{n+1} L : n \in \mathbb{N} \rangle$ is a ascending sequence of $\mathcal{I}$-Luzin sets.
Theorem (CH)

- There exists a Luzin set $L$ such that $L + L$ is a Bernstein set.
- There exists a Sierpiński set $S$ such that $S + S$ is a Bernstein set.
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Proof.

$Perf = \{ P_\alpha : \alpha < c \}$, $\mathcal{M} \cap \mathcal{B} = \{ M_\alpha : \alpha < c \}$.

We choose sequences $\{ l_\alpha : \alpha < c \}$, $\{ l'_\alpha : \alpha < c \}$ and $\{ p_\alpha : \alpha < c \}$ such that for each $\xi < c$:

1. $l_\xi, l'_\xi \notin \bigcup_{\alpha < \xi} M_\alpha$,
2. $(\bigcup_{\alpha \leq \xi} \{ l_\alpha, l'_\alpha \} + \bigcup_{\alpha \leq \xi} \{ l_\alpha, l'_\alpha \}) \cap \{ p_\alpha : \alpha < \xi \} = \emptyset$,
3. $l_\xi + l'_\xi \in P_\xi$,
4. $p_\xi \in P_\xi$. 
Proof...
Let us denote:

\[ M_1 = \bigcup_{\alpha < \xi} M_{\alpha}, \]
\[ M_2 = \bigcup_{\alpha < \xi} M_{\alpha} \cup \left( \{ p_{\alpha} \}_{\alpha < \xi} - \{ l_{\alpha}, l'_{\alpha} \}_{\alpha < \xi} \right) \cup \frac{1}{2} \{ p_{\alpha} \}_{\alpha < \xi}, \]
\[ P = P_{\xi}, \]

Does there exist \( l' \in M_2^{\xi} \) such that a set \( M_1^{\xi} \cap (P - l') \) has cardinality \( \mathfrak{c} \)?
Proof...

We extend our universe $V$ (via generic extension) to $V'$ such that $V' \models \text{cov}(\mathcal{M}) \geq \omega_2$.

We will work in $V'$. Let us now fix a set $A \subseteq P$ of cardinality $\omega_1$. Notice that for every $a \in A$ a set $\{l : a - l \in M_1^c\} = -M_1^c + a$ is comeager. Since $\text{cov}(\mathcal{M}) > \omega_1$

\[ \bigcap_{a \in A} \{l : a - l \in M_1^c\} \cap M_2^c \neq \emptyset. \]
Proof...

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\[ \bigcap_{a \in A} \{l : a - l \in M_1^c\} \cap M_2^c \neq \emptyset. \]

It shows that \( V' \models \exists l' \in M_2^c \ | M_1^c \cap (P - l')| \geq \omega_1 \).

So, \( V' \) models the following sentence:

\[ (\exists l')_R (\exists T)_{\text{Perf}} (\forall x)_{R} (l' \in M_2^c \land (x \in T \rightarrow x \in M_1^c \land x + l' \in P)) \]

By Shoenfield absoluteness theorem it is also true in \( V \). \( \Box \)
Theorem
There are no Luzin set $L$ and Sierpiński set $S$ such that $L + S$ is a Bernstein set.

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Follows from
Lemma
Let $A$ be a null set. We can find a perfect set $P$ such that for every $n$

\[ A + P + P + \cdots + P \in \mathcal{N}. \]
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$$A + P + P + \cdots + P \in \mathcal{N}.$$ 

Proof of lemma
We can assume that $A$ is Borel. Let $V$ be our universe. We enlarge it (via forcing) to $V'$ satisfying $V' \models add(\mathcal{N}) = \omega_3$. 
Proof of lemma...

Let us work now in $V'$. Take $X \subseteq \mathbb{R}$ of cardinality $\omega_2$. Then $A + X \in \mathcal{N}'$, so we can find a null Borel set $B$, such that $A + X \subseteq B$. Notice that $\{x : x + A \subseteq B\}$ is a coanalytic set of cardinality $\omega_2$, hence, it contains a perfect set $P_0$. 
Proof of lemma...
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Now, set $A_1 = A_0 + P_0$. We want to find a perfect set $P_1 \subseteq P_0$ such that $A_1 + P_1 \in \mathcal{N}$. Moreover, we require that the first splitting node in $P_0$ is still a splitting node in $P_1$. 
Proof of lemma...
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Proof of lemma...

We get a sequence of perfect sets \((P_n, n \in \omega)\) such that
\[ P = \bigcap_{n \in \omega} P_n \] is a perfect set. Moreover, we can find a null \(G_\delta\) \(B\) such that \(B \supseteq \bigcup_{n \in \omega} A_n\). Notice that

\[ V' \models (\exists P \in \text{Perf})(\exists B)(\forall n)(\forall x)(\forall a)(\forall b)(B \text{ is null } G_\delta \land (a \in A \land b \not\in B \land x_0, x_1, \ldots, x_n \in P \rightarrow a + x_0 + \cdots + x_n \neq b)), \]

where \(x_0, x_1, \ldots, x_n\) are naturally coded by \(x\) e.g. by the formula
\[ x_i(k) = x(kn + i). \]

Above formula is \(\Sigma^1_2\). \(\square\)
Marcin Michalski, Szymon Żeberski, “Luzin and Sierpiński sets, some nonmeasurable subsets of the plane“, arXiv.org/abs/1406.3062