A remark on the general nature of the Katětov’s construction

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The Urysohn space

P. URYSOHN: *Sur un espace métrique universel.*

\[ U \] — complete separable metric space which is homogeneous and embeds all separable metric spaces.

\[ U = \overline{U_Q} \]
Katětov’s construction of the Urysohn space

**M. Katětov:** *On universal metric spaces.*
General topology and its relations to modern analysis and algebra. VI (Prague, 1986),

A Katětov function over a finite rational metric space $X$ is every function $\alpha : X \to \mathbb{Q}$ such that

$$|\alpha(x) - \alpha(y)| \leq d(x, y) \leq \alpha(x) + \alpha(y)$$

$K(X) =$ all Katětov functions over $X$, which is a rational metric space under sup metric

$$\text{colim}(X \hookrightarrow K(X) \hookrightarrow K^2(X) \hookrightarrow K^3(X) \hookrightarrow \cdots) = U_{\mathbb{Q}}$$
M. Katětov: *On universal metric spaces.*

**Observation 1.** $U_\mathbb{Q}$ is countable and homogeneous.

**Observation 2.** $K(X)$ contains all 1-point extensions of $X$.

**Observation 3.** $K$ is functorial.
Homogeneity

A

automorphism

isomorphism
**Fraïssé theory**

\textbf{age}(A) — the class of all finitely generated struct’s which embed into A

\textit{amalgamation class} — a class \( \mathcal{K} \) of fin. generated struct’s s.t.

- there are countably many pairwise noniso struct’s in \( \mathcal{K} \);
- \( \mathcal{K} \) has (HP);
- \( \mathcal{K} \) has (JEP); and
- \( \mathcal{K} \) has (AP):
  for all \( A, B, C \in \mathcal{K} \) and embeddings \( f : A \hookrightarrow B \) and \( g : A \hookrightarrow C \), there exist \( D \in \mathcal{K} \) and embeddings \( u : B \hookrightarrow D \) and \( v : C \hookrightarrow D \) such that \( u \circ f = v \circ g \).
Fraïssé theory

**Theorem.** [Fraïssé, 1953]

1. If $A$ is a countable homogeneous structure, then $\text{age}(A)$ is an amalgamation class.

2. If $\mathcal{K}$ is an amalgamation class, then there is a unique (up to isomorphism) countable homogeneous structure $A$ such that $\text{age}(A) = \mathcal{K}$.

3. If $B$ is a countable structure *younger than* $A$ (that is, $\text{age}(B) \subseteq \text{age}(A)$), then $B \hookrightarrow A$.

**Definition.** If $\mathcal{K}$ is an amalgamation class and $A$ is the countable homogeneous structure such that $\text{age}(A) = \mathcal{K}$, we say that $A$ is the *Fraïssé limit* of $\mathcal{K}$ and write $A = \text{Flim}(\mathcal{K})$. 
Some prominent Fraïssé limits

$\mathbb{Q}$ — the Fraïssé limit of the class of all linear orders

$U_\mathbb{Q}$ — the Fraïssé limit of the class of finite metric spaces with rational distances (the rational Urysohn space)

$R$ — the Fraïssé limit of the class of all finite graphs (the Rado graph)

$H_n$ — the Fraïssé limit of the class of all finite $K_n$-free graphs, $n \geq 3$ (Henson graphs)

$P$ — the Fraïssé limit of the class of all finite posets (the random poset)
Recall:

M. Katětov: *On universal metric spaces.*
General topology and its relations to modern analysis and algebra. VI (Prague, 1986),

Katětov’s construction

\[ \text{colim}(X \hookrightarrow K(X) \hookrightarrow K^2(X) \hookrightarrow K^3(X) \hookrightarrow \cdots) = U_{\mathbb{Q}} \]

**Observation 1.** $U_{\mathbb{Q}}$ is countable and homogeneous.

**Observation 2.** $K(X)$ contains all 1-point extensions of $X$.

**Observation 3.** $K$ is functorial.
Katětov functors

\( \mathcal{A} \) — a category of fin generated \( L \)-struct’s with (HP) and (JEP)

\( \mathcal{C} \) — the category of all colimits of \( \omega \)-chains in \( \mathcal{A} \)

**Definition.** A functor \( K : \mathcal{A} \to \mathcal{C} \) is a Katětov functor if

1. \( K \) preserves embeddings, and
2. there exists a natural transformation \( \eta : \text{ID} \to K \) such that for every embedding \( f : A \hookrightarrow B \) in \( \mathcal{A} \) where \( B \) is a 1-point extension of \( A \) there is an embedding \( g : B \hookrightarrow K(A) \) satisfying

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & K(A) \\
\downarrow f & & \downarrow g \\
& & B
\end{array}
\]
Katětov functors

$K(A)$ is “a functorial amalgam” of all 1-point extensions of $A$. 
Why is it hard to construct a Katětov functor by hand?

**Example.** Tournaments.
Example. Tournaments.
Why is it hard to construct a Katětov functor by hand?

**Example.** Tournaments.

How to add edges in a “functorial” way?
Why is it hard to construct a Katětov functor by hand?

**Example.** Tournaments.

\[ T = (V, E) \] — a tournament with \( n \) vertices
\[ T_{\leq n} \] — the tournament with vertices \( V_{\leq n} \) and edges defined by:
  - if \( s \) and \( t \) are seq’s such that \( |s| < |t| \), put \( s \to t \) in \( T_{\leq n} \);
  - if \( s = \langle s_1, \ldots, s_k \rangle \) and \( t = \langle t_1, \ldots, t_k \rangle \) are distinct sequences of the same length, find the smallest \( i \) such that \( s_i \neq t_i \) and then put \( s \to t \) in \( T_{\leq n} \) if and only if \( s_i \to t_i \) in \( T \).

Put \( K(T) = (V^*, E^*) \) where
\[ V^* = V \cup V_{\leq n}, \]
\[ E^* = E \cup E(T_{\leq n}) \cup \{ v \to s : v \in V, s \in V_{\leq n}, v \text{ appears in } s \} \]
\[ \cup \{ s \to v : v \in V, s \in V_{\leq n}, v \text{ does not appear in } s \}. \]
Why is it hard to construct a Katětov functor by hand?

Example. Tournaments.

approx. $2n^n$ new vertices
Katětov functors

\( \mathcal{A} \) — a category of fin generated \( L \)-struct’s with (HP) and (JEP)

\( \mathcal{C} \) — the category of all colimits of \( \omega \)-chains in \( \mathcal{A} \)

**Theorem.** If there exists a Katětov functor \( K : \mathcal{A} \to \mathcal{C} \), then

1. \( \mathcal{A} \) is an amalgamation class,
2. its Fraïssé limit \( F \) can be obtained by the “Katětov construction” starting from an arbitrary \( A \in \mathcal{A} \):

\[
F = \text{colim}(A \hookrightarrow K(A) \hookrightarrow K^2(A) \hookrightarrow K^3(A) \hookrightarrow \cdots),
\]

3. \( F \) is \( \mathcal{C} \)-morphism-homogeneous.
Definition. A structure $F$ is $\mathcal{C}$-morphism-homogeneous if every $\mathcal{C}$-morphism between finitely induced substructures of $F$ extends to a $\mathcal{C}$-endomorphism of $F$. 
$\mathcal{C}$-morphism-homogeneity

$\mathcal{C}$-endomorphism

$\mathcal{C}$-morphism

A Katětov functor exists for the following categories $\mathcal{A}$:

- finite linear orders with order-preserving maps,
- finite graphs with graph homomorphisms,
- finite $K_n$-free graphs with embeddings,
- finite digraphs with digraph homomorphisms,
- finite tournaments with homomorphisms = embeddings.
- finite rational metric spaces with nonexpansive maps,
- finite posets with order-preserving maps,
- finite boolean algebras with homomorphisms,
- finite semilattices/lattices/distributive lattices with embeddings.

A Katětov functor **does not exist** for the category of finite $K_n$-free graphs and graph homomorphisms.
Existence of Katětov functors

\( \mathcal{A} \) — a category of fin generated \( L \)-struct’s with (HP) and (JEP)

\( \mathcal{C} \) — the category of all colimits of \( \omega \)-chains in \( \mathcal{A} \)

**Theorem.** There exists a Katětov functor \( K : \mathcal{A} \to \mathcal{C} \) if and only if \( \mathcal{A} \) is an amalgamation class with the *morphism extension* property.
Morphism extension property

\(\mathcal{C} \) — a category

**Definition.** \(\mathcal{C} \in \mathcal{C} \) has the *morphism extension property in \(\mathcal{C} \) if* for any choice \(f_1, f_2, \ldots\) of partial \(\mathcal{C}\)-morphisms of \(C\) there exist \(D \in \mathcal{C}\) and \(m_1, m_2, \ldots \in \text{End}_\mathcal{C}(D)\) such that \(C\) is a substructure of \(D\), \(m_i\) is an extension of \(f_i\) for all \(i\), and the following *coherence* conditions are satisfied for all \(i, j\) and \(k\):

- if \(f_i = \text{id}_A\), \(A \subseteq C\), then \(m_i = \text{id}_D\),
- if \(f_i\) is an embedding, then so is \(m_i\), and
- if \(f_i \circ f_j = f_k\) then \(m_i \circ m_j = m_k\).

We say that \(\mathcal{C}\) has the *morphism extension property* if every \(\mathcal{C} \in \mathcal{C} \) has the morphism extension property in \(\mathcal{C} \).
Existence of Katětov functors for algebras

$L$ — algebraic language

$\mathcal{V}$ — a variety of $L$-algebras understood as a category of $L$-algebras with embeddings

$\mathcal{A}$ — the full subcategory of $\mathcal{V}$ spanned by all finitely generated algebras in $\mathcal{V}$

$\mathcal{C}$ — the full subcategory of $\mathcal{V}$ spanned by all countably generated algebras in $\mathcal{V}$

**Theorem.** There exists a Katětov functor $K : \mathcal{A} \to \mathcal{C}$ if and only if $\mathcal{A}$ is an amalgamation class.
The Importance of Being Earnest Functor

**Theorem.** Let $K : \mathcal{A} \to \mathcal{C}$ be a Katětov functor and let $F$ be the Fraïssé limit of $\mathcal{A}$. Then for every object $C$ in $\mathcal{C}$:

- $\text{Aut}(C) \hookrightarrow \text{Aut}(F)$;
- $\text{End}_C(C) \hookrightarrow \text{End}_C(F)$.

**Proof (Idea).** Take any $f : C \to C$. Then:

$$
\begin{array}{cccccc}
C & \xrightarrow{\eta} & K(C) & \xrightarrow{\eta} & K^2(C) & \xrightarrow{\eta} & \ldots & \sim & F \\
\downarrow f & & \downarrow K(f) & & \downarrow K^2(f) & & & \downarrow f^* \\
C & \xleftarrow{\eta} & K(C) & \xleftarrow{\eta} & K^2(C) & \xleftarrow{\eta} & \ldots & \sim & F
\end{array}
$$
Theorem. Let $K : A \to C$ be a Katětov functor and let $F$ be the Fraïssé limit of $A$. Then for every object $C$ in $C$:

- $\Aut(C) \hookrightarrow \Aut(F)$;
- $\End_C(C) \hookrightarrow \End_C(F)$.

Moreover, if $K$ is locally finite (that is, $K(A)$ is finite whenever $A$ is finite), then the above embeddings are countinuous w.r.t. the topology of pointwise convergence.
Corollary. For the following Fraïssé limits $F$ we have that $\text{Aut}(F)$ embeds all permutation groups on a countable set:

- $\mathbb{Q}$,
- the random graph [Henson 1971],
- Henson graphs [Henson 1971],
- the random digraph,
- the rational Urysohn space [Uspenskij 1990],
- the random poset,
- the countable atomless boolean algebra,
- the random semilattice,
- the random lattice,
- the random distributive lattice.
Corollary. For the following Fraïssé limits $F$ we have that $\text{End}(F)$ embeds all transformation monoids on a countable set:

- $\mathbb{Q}$,
- the random graph [Bonato, Delić, Dolinka 2010],
- the random digraph,
- the rational Urysohn space,
- the random poset [Dolinka 2007],
- the countable atomless boolean algebra.
The Importance of Being Earnest Functor

$\mathcal{C}$ — a locally finite category of $L$-struct’s and all $L$-hom’s

$\mathcal{A}$ — the full subcategory of $\mathcal{C}$ consisting of all finite struct’s in $\mathcal{C}$

**Theorem.** Assume that there exists a locally finite Katětov functor $K : \mathcal{A} \rightarrow \mathcal{C}$. Then the following are equivalent for a $\mathcal{C} \in \mathcal{C}$:

1. $\mathcal{C}$ is locally $K$-closed;
2. $\mathcal{C}$ is algebraically closed in $\mathcal{C}$;
3. $\mathcal{C}$ is a retract of $\text{Flim}(\mathcal{A})$. 
\( \mathcal{A} \) — a category of fin generated \( L \)-struct’s with (HP) and (JEP)

\( \mathcal{C} \) — the category of all colimits of \( \omega \)-chains in \( \mathcal{A} \)

**Theorem.** Assume that there exists a Katětov functor \( K : \mathcal{A} \to \mathcal{C} \) and that \( \mathcal{C} \) has retractive natural (JEP). Let \( F \) be the Fraïssé limit of \( \mathcal{A} \). Then:

1. \( \text{End}_\mathcal{C}(F) \) is *strongly distorted*,
2. the *Sierpiński rank* of \( \text{End}_\mathcal{C}(F) \) is at most 5,
3. if \( \text{End}_\mathcal{C}(F) \) is not finitely generated then it has the *semigroup Bergman property*. 
Corollary. For the following Fraïssé limits $F$ we have that $\text{End}(F)$ has the semigroup Bergman property:

- random graph,
- random digraph,
- rational Urysohn sphere (the Fraïssé limit of the category of all fin met spaces with distances in $[0, 1]_\mathbb{Q}$),
- random poset,
- random boolean algebra (the Fraïssé limit of the category of all finite boolean algebras).