Ideal topological spaces

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We can say that ideals are folklore in mathematics.

If X is a nonempty set, a family $\mathcal{I} \subset P(X)$ satisfying

(I0) $\emptyset \in \mathcal{I}$,

(I1) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$,

(I2) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$,

is called an **ideal** on X.

$\{\emptyset\}$, $P(X)$,

principal ideal - $A\downarrow$,

finite sets - Fin,

countable sets - $\mathcal{I}_{count}$,

cardinality less than κ - $\mathcal{I}_{<\kappa}$,

closed and discrete sets - $\mathcal{I}_{cd}$,

scattered sets (with T_1) - $\mathcal{I}_{sc}$,

relatively compact sets - $\mathcal{I}_K$,

nowhere dense sets - $\mathcal{I}_{nwd}$,

meager sets - $\mathcal{I}_{mgr}$,

sets of measure zero - $\mathcal{I}_{m0}$.

No need to define topological space $\langle X, \tau \rangle$.

Triplet $\langle X, \tau, \mathcal{I} \rangle$ is called **ideal topological space**.

Short history

The first steps in introducing topological spaces enhanced by an ideal is due to Kuratowski [4, 5] in 1933, who introduced local function as a generalization to closure.

A little bit later ideals in topological spaces were studied by Vaidyanathaswamy [10] (1944).

1958. Freud [2] generalized Cantor-Bendixson theorem using ideal topological space.

1971. Scheinberg [9] applied ideals in the measure theory.

In 1990 Janković and Hamlett [3] wrote a survey paper on the topic of ideal topological spaces.

Today this paper is a starting point, and a pattern for introducing many variations and generalizations of open sets defined by ideals.

Local function

Definition 1 *(Kuratowski 1933)[4] Let $\langle X, \tau, \mathcal{I} \rangle$ be an ideal topological space. Then*

$$A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$$

is called the **local function** *of A with respect to $\mathcal{I}$ and τ.*

For $\mathcal{I} = \{\emptyset\}$ we have that $A^*(\mathcal{I}, \tau) = \mathrm{Cl}(A)$.
For $\mathcal{I} = P(X)$ we have that $A^*(\mathcal{I}, \tau) = \emptyset$.
For $\mathcal{I} = Fin$ we have that $A^*(\mathcal{I}, \tau)$ is the set of ω-accumulation points of A.
For $\mathcal{I} = \mathcal{I}_{count}$ we have that $A^*(\mathcal{I}, \tau)$ is the set of condensation points of A.

The local function has the following properties (see [3]):

(1) $A \subseteq B \Rightarrow A^* \subseteq B^*$;
(2) $A^* = \mathrm{Cl}(A^*) \subseteq \mathrm{Cl}(A)$;
(3) $(A^*)^* \subseteq A^*$;
(4) $(A \cup B)^* = A^* \cup B^*$
(5) If $I \in \mathcal{I}$, then $(A \cup I)^* = A^* = (A \setminus I)^*$.

"Idealized" topology

Definition 2 $\mathrm{Cl}^*(A) = A \cup A^*$ *is a Kuratowski closure operator, i.e.*
(1) $\mathrm{Cl}^*(\emptyset) = \emptyset$, *(2)* $A \subseteq \mathrm{Cl}^*(A)$,
(3) $\mathrm{Cl}^*(A \cup B) = \mathrm{Cl}^*(A) \cup \mathrm{Cl}^*(B)$, *and (4)* $\mathrm{Cl}^*(\mathrm{Cl}^*(A)) = \mathrm{Cl}^*(A)$.
and therefore it generates a topology on X

$\tau^*(\mathcal{I}) = \{A : \mathrm{Cl}^*(X \setminus A) = X \setminus A\}$.

$\tau \subseteq \tau^* \subseteq P(X)$

Set A is closed in τ^* iff $A^* \subseteq A$.

If $\Psi(A) = X \setminus (X \setminus A)^*$, then set $O \in \tau^*$ iff $O \subseteq \Psi(O)$.

$\beta(\mathcal{I}, \tau) = \{V \setminus I : V \in \tau, I \in \mathcal{I}\}$ is a basis for τ^*

$\tau^* = \tau^{**}$

For $\mathcal{I} = \{\emptyset\}$ we have that $\tau^*(\mathcal{I}) = \tau$.

For $\mathcal{I} = P(X)$ we have that $\tau^*(\mathcal{I}) = P(X)$.

If $\mathcal{I} \subseteq \mathcal{J}$ then $\tau^*(\mathcal{I}) \subseteq \tau^*(\mathcal{J})$

If $Fin \subseteq \mathcal{I}$ then $\langle X, \tau^* \rangle$ is T_1 space.

If $\mathcal{I} = Fin$, then $\tau^*_{ad}(\mathcal{I})$ is the cofinite topology on X.

If $\mathcal{I} = \mathcal{I}_{m0}$ - ideal of the sets of measure zero, then τ^*-Borel sets are precisely the Lebesgue measurable sets. (Scheinberg 1971)[9]

For $\mathcal{I} = \mathcal{I}_{nwd}$ then $A^* = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ and $\tau^*(\mathcal{I}_{nwd}) = \tau^\alpha$. ($\alpha$-open sets, $A \subseteq \mathrm{Int}(Cl(Int(A)))$ - (Njåstad 1965)[6])

$\tau \sim \mathcal{I}$

Definition 3 *(Njástad 1966)[7] Let* $\langle X, \tau, \mathcal{I} \rangle$ *be an ideal topological space. We say* τ *is compatible with the ideal* $\mathcal{I}$*, denoted* $\tau \sim \mathcal{I}$ *if the following holds for every* $A \subseteq X$*: if for every* $x \in A$ *there exists a* $U \in \tau(x)$ *such that* $U \cap A \in \mathcal{I}$*, then* $A \in \mathcal{I}$*.*

$\tau \sim \mathcal{I}$ implies $\beta = \tau^*$
$\tau \sim \mathcal{I}$ iff $A \setminus A^* \in \mathcal{I}$, for each A.

Theorem 1 $\langle X, \tau \rangle$ *is hereditarily Lindelöf iff* $\tau \sim \mathcal{I}_{count}$

Theorem 2 $\tau \sim \mathcal{I}_{nwd}$

Theorem 3 $\tau \sim \mathcal{I}_{mgr}$

Theorem 4 *Let* $\langle X, \tau, \mathcal{I} \rangle$ *be an ideal topological space. The following are equivalent.*
(a) $\mathcal{I} \sim \tau$ *and* $Fin \subseteq \mathcal{I}$*.*
(b) Scattered sets in $\langle X, \tau^* \rangle$ *are in* $\mathcal{I}$*.*
(c) Discrete sets in $\langle X, \tau^* \rangle$ *are in* $\mathcal{I}$*.*

Cantor-Bendixson

Theorem 5 *(Cantor-Bendixson). A second countable (moreover, hereditarily Lindelof) space can be represented as the union of two sets, one of which is perfect (closed without isolated points) and the other countable.*

Theorem 6 *(Freud 1958)[2] Let* $\langle X, \tau, \mathcal{I} \rangle$ *be an ideal topological space such that* $\mathcal{I} \sim \tau$ *and* $Fin \subseteq \mathcal{I}$. *If a set* A *is closed with respect to* *, *then* A *is the union of a set which is perfect with respect to* τ *and a set in* $\mathcal{I}$.

$X = X^*$

Theorem 7 *(Samules 1975)[8] Let* $\langle X, \tau, \mathcal{I} \rangle$ *be an ideal topological space. Then* $X = X^*$ *iff* $\tau \cap \mathcal{I} = \{\emptyset\}$.

τ_s is the family of regular open sets $(U = \mathrm{Int}(\mathrm{Cl}(U)))$ in τ

Theorem 8 *(Janković Hamlett 1990)[3] Let* $\langle X, \tau \rangle$ *be a space with an ideal* $\mathcal{I}$ *on* X*. If* $X = X^*$ *then* $\tau_s = {\tau_s}^*$.

It was observed in (Bourbaki 1966)[1] that some important topological properties are shared by $\langle X, \tau \rangle$ and $\langle X, \tau_s \rangle$. Some of these properties, so called semiregular properties, are: Hausdorffness, property of a space being Urysohn $(T_{2\frac{1}{2}})$, connectedness, extremal disconnectedness, H-closedness, light compactness (every locally finite collection of open subsets is finite), pseudocompactness, ...

Theorem 9 *Semiregular properties are shared by* $\langle X, \tau \rangle$ *and* $\langle X, \tau^* \rangle$ *if* $X = X^*$

A space $\langle X, \tau \rangle$ is said to a **Baire space** if the intersection of every countable family of open dense sets in $\langle X, \tau \rangle$ is dense.

$\langle X, \tau \rangle$ is a Baire space iff $X = X^*(\mathcal{I}_{mgr})$.

Space is anticompact iff the only compact sets are finite.

If $\langle X, \tau \rangle$ is Hausdorf, then $\tau^*(\mathcal{I}_{count})$, $\tau^*(\mathcal{I}_{sc})$ and $\tau^*(\mathcal{I}_K)$ are anticompact.
If $Fin \subseteq \mathcal{I}$ and $\tau \sim \mathcal{I}$ then also $\tau^*(\mathcal{I})$ is anticompact (Janković Hamlett 1990)[3].

Theorem 10 *(Samuels 1971)[8] If $X = X^*$ and Y is regular, then $f : \langle X, \tau \rangle \to Y$ is continuous iff $f : \langle X, \tau^* \rangle \to Y$ is continuous.*

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