Some compactness principles and their indestructibility

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Recall the following definition:

Definition

Let κ be a regular cardinal. We say that a κ-tree T is a κ-*Kurepa tree* if it has at least κ^+-many cofinal branches; if we drop the restriction on T being a κ-tree, and require only that T has size and height κ, we obtain a *weak Kurepa tree*. We say that the *Kurepa Hypothesis*, KH(κ), holds if there exists a Kurepa tree on κ; analogously the *weak Kurepa Hypothesis*, wKH(κ), says that there exists a weak Kurepa tree on κ.

The negation of the weak Kurepa hypothesis

Some basic properties:

- If CH holds, then $2^{<\omega_1}$ is a weak Kurepa tree.
- Therefore $\neg$wKH(ω_1) implies $2^\omega > \omega_1$.
- (Mitchell) In the generic extension by Mitchell forcing up to an inaccessible cardinal $\neg$wKH(ω_1) holds.
- (Silver) The inaccessible cardinal is necessary. If $\neg$wKH(ω_1) holds, then ω_2 is inaccessible in L.

The negation of the weak Kurepa hypothesis

Assume $\neg$wKH(ω_1) holds:

- (Baumgartner) If $2^\omega = \omega_2$, then $2^{\omega_1} = \omega_2$; in fact, even $\diamondsuit^+(\omega_2 \cap \text{cof}(\omega_1))$ holds.
- Baumgartner's result can be generalized as follows: if $2^\omega < \aleph_{\omega_1}$, then $2^{\omega_1} = 2^\omega$.
- (Baumgartner) PFA implies $\neg$wKH(ω_1).

Recall the following definition:

Definition

Let κ be a regular cardinal. A κ-tree is called *Aronszajn* if it has no cofinal branches. The tree property holds at κ, $\mathsf{TP}(\kappa)$, if there are no κ-Aronszajn trees.

The tree property

Some basic properties:

- TP(ω) and $\neg$TP(ω_1).
- (Specker) If $\kappa^{<\kappa} = \kappa$ then there exists a κ^+-Aronszajn tree. Therefore $\neg$TP(κ^+).
  - If GCH then $\neg$TP(κ^{++}) for all $\kappa \geq \omega$.
  - TP(κ^{++}) then $2^\kappa > \kappa^+$.
- (Mitchell) In the generic extension by Mitchell forcing up to a weakly compact cardinal TP(ω_2) holds.
- (Silver) The weakly compact cardinal is necessary. If TP(ω_2) holds, then ω_2 is weakly comapact in L.
- (Baumgartner) PFA implies TP(ω_2).

TP(ω_2), $\neg$wKH(ω_1) and the continuum function

Note that in contrast to the $\neg$wKH(ω_1) the tree property does not have effect on the value of 2^{ω_1}. More generally we proved that the tree property has no provable effect on the continuum function below $\aleph_\omega$ except for the restriction that the tree property at κ^{++} implies $2^\kappa > \kappa^+$ for every infinite κ.

Theorem (S., 2021)

Assume there are infinitely many supercompact cardinals and let $f : \omega \to \omega$ be a monotonous function satisfying $f(n) \geq n + 2$, $n < \omega$. Then there is a generic extension $V[G]$ where the tree property holds at each $\aleph_n$, $1 < n < \omega$, and f determines the continuum function in $V[G]$ below $\aleph_\omega$:

$$2^{\aleph_n} = \aleph_{f(n)}.$$

Mitchell forcing

In the first part of the talk, we will discuss primarily the indestructibility over the Mitchell model $V[\mathbb{M}(\omega,\kappa)]$, where κ is weakly compact in V (or just inaccessible), in which the tree property and the negation of the weak Kurepa hypothesis hold at $\kappa = \omega_2 = 2^\omega$ and ω_1, respectively.

The Mitchell forcing is the standard way of obtaining compactness at the double successor of a regular cardinal λ with $2^\lambda > \lambda^+$ (forcing with $\mathbb{M}(\lambda,\mu)$ for a sufficiently large $\mu > \lambda$).

Mitchell forcing, basic properties

Let us assume κ is inaccessible. Mitchell forcing $\mathbb{M} = \mathbb{M}(\omega, \kappa)$ satisfies the following:

- It is κ-Knaster.
- It collapses the cardinals in the open interval (ω_1, κ) to ω_1.
- It forces $2^\omega = \kappa = \omega_2$, the negation of the weak Kurepa hypothesis at ω_1 and the tree property at ω_2 (if κ is moreover weakly compact).

There is a projection from $\mathbb{M}$ to $\text{Add}(\omega, \kappa)$.
The preservation of ω_1 is shown by the existence of a projection from $\text{Add}(\omega, \kappa) \times \mathbb{T}$ to $\mathbb{M}$, where $\mathbb{T}$ is a ω_1-closed forcing.[1]

[1] We call $\mathbb{T}$ the *term forcing*.

TP(ω_2) and $\neg$wKH(ω_1)

- Both TP(ω_2) and $\neg$wKH(ω_1) hold in the Mitchell model $\mathbb{M}(\omega, \kappa)$, where κ is weakly compact.
- But these principles do not imply one another: (i) If κ is inaccessible but not weakly compact in L, then in the Mitchell model $\mathbb{M}(\omega, \kappa)$ we have $\neg$wKH(ω_1), but $\neg$TP(ω_2). (ii) There is also model of TP(ω_2) with an ω_1-Kurepa tree, hence KH(ω_1) and also wKH(ω_1).

Let us discuss a more general result in the following specific form:

Theorem (Honzik, S., 2020)

Suppose κ is weakly compact. The tree property at ω_2 in $V[\mathbb{M}]$ is indestructible under all ccc forcings which live in $V[\mathrm{Add}(\omega,\kappa)]$.

First notice that $V[\mathrm{Add}(\omega,\kappa)] \subseteq V[\mathbb{M}]$ so the statement of the theorem makes sense.

Also note that Unger showed that the tree property in $V[\mathbb{M}]$ is indestructible under all ccc forcings of size at most ω_1 living in $V[\mathbb{M}]$.

We state the theorem for $\mathbb{M}(\omega, \kappa)$ for simplicity, it holds for any $\mathbb{M}(\lambda, \mu)$, $\lambda < \mu$, λ regular, μ weakly compact.
The theorem has many applications, such as:

Ex. 1 Suppose λ is measurable in $V[\mathbb{M}(\lambda, \mu)]$ (this requires some preparation). Then one can apply the theorem with the Prikry forcing or Magidor forcing which is λ^+-cc. The point is that the relevant ultrafilters can be assumed to exist already in $V[\mathrm{Add}(\lambda, \mu)]$ (and they remain ultrafilters in $V[\mathbb{M}(\lambda, \mu)]$ because the quotient is λ^+-distributive). This gives directly the consistency of a singular λ of any desired cofinality with the tree property at λ^{++}, without the need to analyze the quotients of the singularization forcing.

Aplications

Ex. 2 The previous Ex. 1 can be generalized to blow up 2^λ arbitrarily large because the Cohen forcing at λ followed by the Prikry forcing is λ^+-cc.

Ex. 3 The theorem can be used to show the consistency of the statment that there is a singular λ with TP(λ^{++}) and $\mathfrak{u}_\lambda = \lambda^+$ (there exists a uniform ultrafilter with a base of size λ^+).

Branch lemmas

To show the tree property, we use the fact that certain forcing notions do not add new cofinal branches to existing κ-Aronszajn trees:

- (Kurepa) Assume that P is a **ccc** forcing notion. Then P does not add cofinal branches to ω_2-Aronszajn trees.
- (Unger) Suppose $2^\omega \geq \omega_2$. Assume that P and Q are forcing notions such that P is **ccc** and Q is ω_1-**closed**. If T is an ω_2-tree in $V[P]$, then forcing with Q over $V[P]$ does not add cofinal branches to T.

Remark. Notice that in the first fact it is sufficient that P is just ccc because the fact is formulated for ω_2-trees - we do not need stronger forms of chain conditions such as ω_1-square-cc ($P \times P$ is ccc) or ω_1-Knaster which are used when dealing with ω_1-trees.This is quite helpful because the plain ccc condition is more stable and is preserved by two-step iterations ($P * \dot{Q}$ is κ-cc iff P is κ-cc and forces that $\dot{Q}$ is κ-cc).

In our proof we make use of this fact.

Sketch of proof

We now give a sketch of proof of the theorem. Let us start with a helpful observation first:

Lemma

Assume that $2^\omega \leq \omega_2$ *and* $\mathbb{P}$ *is ccc. If* $\mathbb{P}$ *adds an* ω_2*-Aronszajn tree, then there exists a regular subforcing* $\bar{\mathbb{P}}$ *of* $\mathbb{P}$ *of size at most* ω_2 *which adds an* ω_2*-Aronszajn tree.*

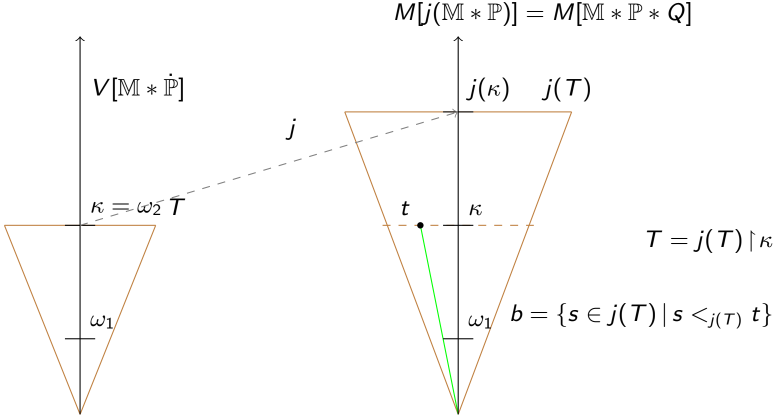
In particular, if no forcing notion of size at most ω_2 which is ccc adds an ω_2-Aronszajn tree, then no ccc forcing adds an ω_2-Aronszajn tree.

Sketch of proof

Suppose $\mathbb{P}$ is a ccc forcing in $V[\mathrm{Add}(\omega,\kappa)]$ of size at most κ and let us fix an $\mathrm{Add}(\omega,\kappa)$-name $\dot{\mathbb{P}}$ for $\mathbb{P}$. We wish to show that the tree property holds in $V[\mathbb{M} * \dot{\mathbb{P}}]$.

- Fix an elementary embedding $j : V \to M$ with critical point κ (for simplicity we assume κ is measurable).
- j restricted to $\mathbb{M} * \dot{\mathbb{P}}$ is a regular embedding into $j(\mathbb{M} * \dot{\mathbb{P}})$ due to $\mathbb{M} * \dot{\mathbb{P}}$ being κ-cc, and one can therefore lift to $j : V[\mathbb{M} * \dot{\mathbb{P}}] \to M[j(\mathbb{M} * \dot{\mathbb{P}})]$.
- Since M is closed under sequences of size κ, the regular embedding is an element of M, and it follows that $M[j(\mathbb{M} * \dot{\mathbb{P}})]$ can be written as $M[\mathbb{M} * \dot{\mathbb{P}} * \dot{Q}]$ for some forcing $\dot{Q}$.

Sketch of proof



The proof finishes by showing that b is in $M[\mathbb{M} * \dot{\mathbb{P}}]$.

Sketch of proof

- Over $M[\mathbb{M} * \dot{\mathbb{P}}]$ there is a projection from the product

$$j(\mathsf{Add}(\omega,\kappa) * \dot{\mathbb{P}})/(\mathsf{Add}(\omega,\kappa) * \dot{\mathbb{P}}) \ \times \ \mathbb{T}_\kappa,$$

  where $\mathbb{T}_\kappa$ is the term forcing in $M[\mathbb{M}]$, onto $\dot{Q}$.
- A crucial step in the proof is to show that the first component of the product is ccc in $M[\mathbb{M} * \dot{\mathbb{P}}]$ and $\mathbb{T}_\kappa$ is ω_1-closed in $M[\mathbb{M}]$.
- After this is shown, the argument is finished by using the fact that a ccc forcing cannot add a cofinal branch to an ω_2-Aronszajn tree, and neither can an ω_1-closed forcing over a ccc forcing.

Two crucial steps of the proof

Let us identify two key steps in the proof. In the next slides we compare these steps with a related result of Todorcevic.

- The quotient $Q = j(\mathbb{M} * \dot{\mathbb{P}})/(\mathbb{M} * \dot{\mathbb{P}})$ is analyzed by means of a projection from a product – in our case $j(\mathrm{Add}(\omega,\kappa) * \dot{\mathbb{P}})/(\mathrm{Add}(\omega,\kappa) * \dot{\mathbb{P}}) \ \times \ \mathbb{T}_\kappa$ – whose first component is ccc and the second is ω_1-closed (in a relevant model). For this product analysis to work, we needed to assume that $\mathbb{P}$ lives in $V[\mathrm{Add}(\omega,\kappa)]$.
- Branch lemmas.

A former result of Todorcevic

Todorcevic showed that the tree property at ω_2 in the Mitchell model $V[\mathbb{M}(\omega, \kappa)]$ is indestructible under any finite-support iteration of length ω_2 of ccc forcing notions which have size less than ω_2 and do not add a new cofinal branch to ω_1-trees.

The proof uses the fact that the quotient analysis is simple because the small forcings of size $< \omega_2$ are not moved by the elementary embedding.

Let us state Devlin's lemma which is crucial for Todorcevic's result:

Lemma (Devlin)

Let λ be a limit ordinal and $\mathbb{P}_\lambda = \langle \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha \,|\, \alpha < \lambda \rangle$ be an iteration with a finite support which preserves ω_1. If T is a tree of height ω_1 in V and b is a cofinal branch through T in $V[\mathbb{P}_\lambda]$ then there is $\alpha < \lambda$ such that b is already in $V[\mathbb{P}_\alpha]$.

A former result of Todorcevic

Todorcevic's result also applies to the negation of the weak Kurepa hypothesis. Our result on the indestructibility of the tree property can also be extended to the weak Kurepa hypothesis (with some new ideas required, since we used the Kurepa lemma that ccc forcings do not add cofinal branches to $\aleph_2$-Aronszajn trees).

The negation of the Kurepa hypothesis

Let us state some simple observations related to the negation of the (weak) Kurepa hypothesis.

- To obtain the negation of the Kurepa hypothesis at ω_1, the Levy collapse of an inaccessible cardinal is enough. (Note that we do not need to violate CH).
- On the other hand to obtain the negation of the weak Kurepa hypothesis at ω_1 we need to violatele CH. The standard method is the Mitchell forcing.

A former result of Jensen and Schlechta

In contrast to $\mathrm{TP}(\omega_2)$ and $\neg\mathrm{wKH}(\omega_1)$, where this is open, there is a model where $\neg\mathrm{KH}(\omega_1)$ holds and it is indestructible under all ccc forcings, by a result of Jensen and Schlechta:

- The model is the Levy collapse of a Mahlo cardinal κ to ω_2.
- Note that $\neg\mathrm{KH}(\omega_1)$ holds already in the Levy collapse of an inaccessible cardinal but for the indestructibility under ccc forcings we need a Mahlo cardinal (this is optimal by Jensen's result that if $\square_{\omega_1}$ holds, then there is a ccc forcing which adds an ω_1-Kurepa tree).
- Note that the proof does not carry over to the Mitchell model since it is crucial that the quotient of the Levy collapse is ω_1-closed and therefore does not add new antichains to ccc forcings).

Some additional indestructibility results

- Chang's conjecture is preserved by all ccc forcings.
- (Honzik, S.) Stationary reflection at $\omega_2 \cap \mathrm{cof}(\omega)$ is preserved by all ccc forcings.
- (Gitik, Krueger) Non approachability at ω_2 is preserved by all σ-centered forcings.
- (Gilton, S.) Club stationary reflection at $\omega_2 \cap \mathrm{cof}(\omega)$ is preserved by all σ-centered forcings (in fact, σ-linked).

Questions

- Is the tree property at ω_2 preserved by all ccc forcing or at least by all σ-centered forcings?
- As above, but with some extra assumptions. For instance, we showed that with PFA (in fact, GMP – Guessing Model Principle, which we review in the next few slides), adding any number of Cohen subsets of ω will not destroy $\mathsf{TP}(\omega_2)$ and that any σ-centered forcing will not destroy $\neg\mathsf{wKH}(\omega_1)$. (Note it is currently unknown whether a single Cohen forcing at ω can destroy $\mathsf{TP}(\omega_2)$ or $\neg\mathsf{wKH}(\omega_1)$.)

Guessing models

Definition

Let $\theta \geq \omega_2$ be a regular cardinal and let $M \prec H(\theta)$ have size ω_1.

1. Given a set $x \in M$, and a subset $d \subseteq x$, we say that
    1. d is *M-approximated* if, for every $z \in M \cap \mathscr{P}_{\omega_1}(M)$, we have $d \cap z \in M$;
    2. d is *M-guessed* if there is $e \in M$ such that $d \cap M = e \cap M$.
2. M is a *guessing model for x* if every M-approximated subset of x is M-guessed.
3. M is a *guessing model* if M is guessing for every $x \in M$.

Guessing model property

Definition

We denote by GMP(θ) the assertion that the set

$$\{M \in \mathscr{P}_{\omega_2}(H(\theta)) \mid M \text{ is a guessing model}\}$$

is stationary in $\mathscr{P}_{\omega_2}(H(\theta))$. We write GMP if GMP($\theta$) holds for every regular $\theta \geq \omega_2$.

Guessing model property

- (Viale, Weiss) In the generic extension by Mitchell forcing up to a supercompact cardinal, GMP holds.
- In the generic extension by Mitchell forcing up to a weakly compact cardinal, GMP(ω_2) holds.
- (Viale, Weiss) PFA implies GMP.
- GMP implies $2^\omega > \omega_1$
- (Lambie-Hanson, S.) GMP implies $2^{\omega_1} = 2^\omega$ if $\mathrm{cf}(2^\omega) \neq \omega_1$, otherwise $2^{\omega_1} = (2^\omega)^+$.
- (Cox, Krueger) GMP(ω_3) implies TP(ω_2).
- (Cox, Krueger) GMP(ω_2) implies $\neg$wKH(ω_1).

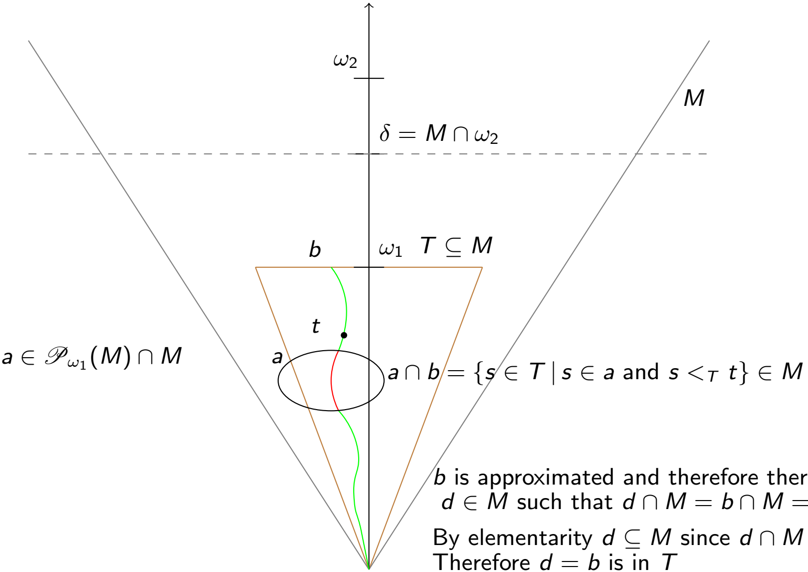
On the complexity of $\neg$wKH and TP

We have seen that $\neg\mathsf{wKH}(\omega_1)$ and $\mathsf{TP}(\omega_2)$ tend to hold together in "natural" models, but they have different syntactical complexities, which makes them different from the point of indestructibility arguments. Suppose we wish to argue by contradiction that these principles hold in a generic extension $V[\mathbb{P}]$, for some $\mathbb{P}$:

- The negation of $\neg\mathsf{wKH}(\omega_1)$ is a Σ_1 statement, and hence assuming $\mathsf{wKH}(\omega_1)$ gives a $\mathbb{P}$-names for a tree, together with a sequence $\langle \dot{b}_\alpha \mid \alpha < \omega_2 \rangle$ of $\mathbb{P}$-names for its cofinal branches. As we will see below, we can use this name to show certain indestructibility results.
- The negation of $\mathsf{TP}(\omega_2)$ is a Σ_2 statement (there is an ω_2 Aronszajn tree): we can still fix a $\mathbb{P}$-name for an ω_2-tree, but we do not have a name which would witness the Π_1 property of not having cofinal branches.

GMP(ω_2) implies ¬wKH(ω_1)

$M \prec H(\omega_2)$ is a guessing model such that $T \in M$, $|M| = \omega_1$ and $\omega_1 \subseteq M$



b is approximated and therefore there is $d \in M$ such that $d \cap M = b \cap M = b$.

By elementarity $d \subseteq M$ since $d \cap M \subseteq T$. Therefore $d = b$ is in T

GMP(ω_2) and $\neg$wKH(ω_1)

We strengthened Cox and Krueger's result and showed that GMP(ω_2) not only implies $\neg$wKH(ω_1), but implies that $\neg$wKH(ω_1) is preserved by all σ-centered forcings.

Theorem (Honzik, Lambie-Hanson, S., 2022)

GMP(ω_2), *and hence also* PFA, *imply* $\neg$wKH(ω_1) *is preserved by all* σ*-centered forcings, i.e. if* V *is a transitive model satisfying* GMP(ω_2), $\mathbb{P} \in V$ *is* σ*-centered, and* G *is* $\mathbb{P}$*-generic over* V*, then* $V[G]$ *satisfies* $\neg$wKH(ω_1). *In particular,* $\neg$wKH(ω_1) *is preserved over models of* GMP(ω_2) *by adding any number of Cohen subsets of* ω.

We will sketch the proof in the next few slides.

Sketch of proof: σ-centered forcings

Definition

Let $\mathbb{P}$ be a forcing. We say that $\mathbb{P}$ is *σ-centered* if $\mathbb{P}$ can be written as the union of a family $\{\mathbb{P}_n \subseteq \mathbb{P} \mid n < \omega\}$ such that for every $n < \omega$:

$$\text{for every } p, q \in \mathbb{P}_n \text{ there exists } r \in \mathbb{P}_n \text{ with } r \leq p, q. \qquad (1)$$

- Some definitions of σ-centeredness require just the compatibility of the conditions, with a witness not necessarily in $\mathbb{P}_n$.
- These two definition are not in general equivalent, but they are equivalent for Boolean algebras.

Sketch of proof: Derived systems

For the rest of this proof, let V denote a transitive model of set theory which satisfies GMP(ω_2).

Suppose $\mathbb{P}$ is a σ-centered forcing notion. For contradiction assume that there is a $\mathbb{P}$-name $\dot{T}$ for a weak Kurepa tree in $V[\mathbb{P}]$ at ω_1. We will use $\dot{T}$ to define in V a certain generalization of a tree, called a *derived system*. We review this concept on the next slide.

Sketch of proof: Well-behaved strong (ω_1, ω_1)-systems

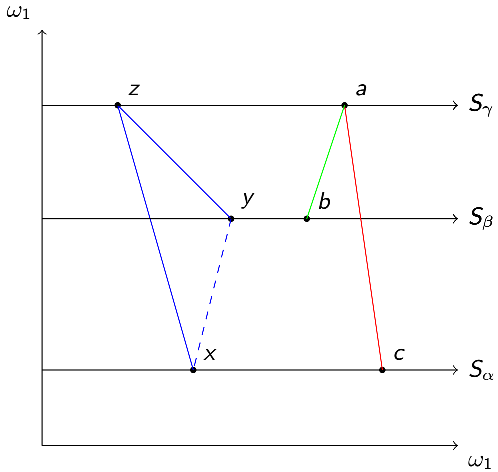
Let $D \subseteq \omega_1$ be unbounded in ω_1. For each $\alpha \in D$, let $S_\alpha \subseteq \{\alpha\} \times \omega_1$ and let $S = \bigcup_{\alpha \in D} S_\alpha$. Moreover, let I be an index set of cardinality at most ω and $\mathcal{R} = \{<_i \mid i \in I\}$ a collection of binary relations on S. We say that $\langle S, \mathcal{R} \rangle$ is an (ω_1, ω_1)-*system* if the following hold for some D:

1. For each $i \in I$, $\alpha, \beta \in D$ and $\gamma, \delta < \omega_1$; if $(\alpha, \gamma) <_i (\beta, \delta)$ then $\alpha < \beta$.
2. For each $i \in I$, $<_i$ is irreflexive and transitive.
3. For each $i \in I$, and $\alpha < \beta < \gamma$, $x \in S_\alpha$, $y \in S_\beta$ and $z \in S_\gamma$, if $x <_i z$ and $y <_i z$, then $x <_i y$.
4. For all $\alpha < \beta$ there are $y \in S_\beta$ and $x \in S_\alpha$ and $i \in I$ such that $x <_i y$.

We call a (ω_1, ω_1)-system $\langle S, \mathcal{R} \rangle$ a *strong* (ω_1, ω_1)-*system* if the following strengthening of item (iv) holds:

4'. For all $\alpha < \beta$ and for every $y \in S_\beta$ there are $x \in S_\alpha$ and $i \in I$ such that $x <_i y$.

Sketch of proof: Well-behaved strong (ω_1, ω_1)-systems



Sketch of proof: Cofinal branches through systems

A *branch* in a system is a subset B of S such that for some $i \in I$, and for all $a \neq b \in B$, $a <_i b$ or $b <_i a$. A branch B is *cofinal* if for every $\alpha < \omega_1$ there is $\beta \geq \alpha$ and some $b \in B$ on level β.

Sketch of proof: Systems derived from σ-centered forcing notions

Let $\mathbb{P}$ be a σ-centered forcing which forces that $\dot{T}$ is a tree of height and size ω_1. Let us write $\mathbb{P} = \bigcup_{n<\omega} \mathbb{P}_n$. The derived system has domain $\omega_1 \times \omega_1$, and is equipped with binary relations $<_n$ for $n < \omega$, where

$$x <_n y \Leftrightarrow (\exists p \in \mathbb{P}_n)\ p \Vdash x \dot{<}_T y. \tag{2}$$

Let us now put all these concepts together and argue that $\neg$wKH(ω_1) holds in $V[\mathbb{P}]$.

- Suppose $\mathbb{P} = \bigcup_{n<\omega} \mathbb{P}_n$ is a σ-centered forcing.
- Assume for contradiction that $\dot{T}$ is forced by the weakest condition in $\mathbb{P}$ to be a weak ω_1-Kurepa tree.
- Let $S(\dot{T})$ be the derived system with respect to $\dot{T}$.
- Since $S(\dot{T})$ is a derived system for a weak ω_1-Kurepa tree $\dot{T}$, it can be checked that $S(\dot{T})$ has at least ω_2-many cofinal branches.
- To finish the proof, we argue that GMP(ω_2) implies that every well-behaved strong (ω_1, ω_1)-system has at most ω_1-cofinal branches. But this is a contradiction since $S(\dot{T})$ is supposed to have at least ω_2-many cofinal branches.

Preservation of the tree property

The previous argument for preservation of $\neg\mathsf{wKH}(\omega_1)$ by σ-centered forcings does not seem to generalize to $\mathsf{TP}(\omega_2)$. Using a system derived from a name $\dot{T}$ for an ω_2-tree, we initially only managed to show that over models of $\mathsf{GMP}(\omega_3)$, $\mathsf{TP}(\omega_2)$ is preserved by adding a single Cohen subset of ω.

Then we learned that Menachem Magidor showed earlier (unpublished) that the principle GMP (and hence $\mathsf{TP}(\omega_2)$) is preserved by a single Cohen forcing over PFA.

We generalized Magidor's construction to any number of Cohen subsets of ω, and showed:

Preservation of GMP by Cohen forcing

Theorem (Honzik, Lambie-Hanson, S., 2022)

Suppose GMP *holds. Then* GMP *is preserved by adding any number of Cohen subsets of* ω*. In particular, over models of* PFA, $\neg$wKH(ω_1) *and* TP(ω_2) *are preserved by adding any number of Cohen subsets of* ω*, and* $\neg$wKH(ω_1) *is preserved moreover by all* σ*-centered forcings (so for instance by all ccc forcings of size* ω_1*).*

Preservation of GMP by Cohen forcing

The key ingredient for the preservation is the following theorem which asserts that a guessing model in V is still a guessing model in the generic extension by Cohen forcing.

Theorem (Honzik, Lambie-Hanson, S., 2022)

Let $\chi < \theta$ be infinite regular cardinals with and let $\mathbb{P} := \mathrm{Add}(\omega, \chi)$. Suppose that $M \prec H(\theta)$ is a guessing model such that $|M| = \omega_1 \subseteq M$ and $\mathbb{P} \in M$. Then, in $V[\mathbb{P}]$, $M[\mathbb{P}]$ is a guessing model.

Preservation of GMP by Cohen forcing

It is instructive to note that GMP(ω_2), unlike TP(ω_2), is an existential statement, so it is easier to show the indestructibility for it.

Open questions

(1) Can our theorem – that $\mathsf{TP}(\omega_2)$ is preserved by all ccc forcings in $V[\mathsf{Add}(\omega,\kappa)]$ – be extended to all ccc forcing notions in $V[\mathbb{M}(\omega,\kappa)]$? Or more generally, is there a model V^* over which $\mathsf{TP}(\omega_2)$ is indestructible under all ccc forcings?[2]

(2) More specifically, neither our result nor Todorcevic's result applies to an iteration of ω_1-Suslin trees of length ω_2. Can either of these results be extended to this forcing?

(3) Analogous questions can be asked for the negation of the weak Kurepa hypothesis and other principles.

[2] Recall, as we remarked earlier, that some very basic questions related to TP are still open: for instance, it is still open whether single Cohen forcing at ω can destroy the tree property at ω_2.

(4) Is there a principle P such that over all models which satisfy P, all ccc forcings preserve $\neg$wKH(ω_1)? (In particular, P implies $\neg$wKH(ω_1)).

(5) Analogous questions can be asked for the tree property and other principles.

Note that such a principle exists for $\neg$KH(ω_1), namely Chang's Conjecture, CC. In this case, we actually know that CC itself is preserved by all ccc forcings.

In our result we showed that GMP is an example of such P for $\neg$wKH(ω_1) if we restrict ourselves to σ-centered forcings.