# Reflection of stationary sets and GDST

Miguel Moreno
University of Vienna
FWF Meitner-Programm

Young Set Theory Workshop 2022

20 August, 2022

This is a joint work with Gabriel Fernandes and Assaf Rinot.

# Generalised Baire space

Suppose $\kappa$ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

# Generalised Baire space

Suppose $\kappa$ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

The generalised Baire space is the space $\kappa^\kappa$ endowed with the bounded topology,

## Generalised Baire space

Suppose $\kappa$ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

The generalised Baire space is the space $\kappa^\kappa$ endowed with the bounded topology, for every $\eta \in \kappa^{<\kappa}$ the following set

$$N_\eta = \{\xi \in \kappa^\kappa \mid \eta \subseteq \xi\}$$

is a basic open set.

## Generalised Baire space

Suppose $\kappa$ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

The generalised Baire space is the space $\kappa^\kappa$ endowed with the bounded topology, for every $\eta \in \kappa^{<\kappa}$ the following set

$$N_\eta = \{\xi \in \kappa^\kappa \mid \eta \subseteq \xi\}$$

is a basic open set.

The generalised Cantor space is the subspace $2^\kappa$.

# Reductions

For $i < 2$, let $X_i$ be some space from the collection $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$. Let $R_0$ and $R_1$ be binary relations over $X_0$ and $X_1$, respectively.

# Reductions

For $i < 2$, let $X_i$ be some space from the collection $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$. Let $R_0$ and $R_1$ be binary relations over $X_0$ and $X_1$, respectively.

## Definition

A function $f : X_0 \to X_1$ is said to be a *reduction of $R_0$ to $R_1$* iff, for all $\eta, \xi \in X_0$,

$$\eta \; R_0 \; \xi \text{ iff } f(\eta) \; R_1 \; f(\xi).$$

# Reductions

For $i < 2$, let $X_i$ be some space from the collection $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$. Let $R_0$ and $R_1$ be binary relations over $X_0$ and $X_1$, respectively.

## Definition

A function $f : X_0 \to X_1$ is said to be a *reduction of $R_0$ to $R_1$* iff, for all $\eta, \xi \in X_0$,

$$\eta \; R_0 \; \xi \text{ iff } f(\eta) \; R_1 \; f(\xi).$$

The existence of a function $f$ satisfying this is denoted by $R_0 \hookrightarrow R_1$.

## Complexity

The existence of a continuous reduction $f$ is denoted by $R_0 \hookrightarrow_c R_1$. We say that $R_0$ is at most as complex as $R_1$.

## Complexity

The existence of a continuous reduction $f$ is denoted by $R_0 \hookrightarrow_c R_1$. We say that $R_0$ is at most as complex as $R_1$.

In the case $R_0 \hookrightarrow_c R_1$ and $R_1 \not\hookrightarrow_c R_0$, we say that $R_0$ is less complex than $R_0$.

# Complexity

The existence of a continuous reduction $f$ is denoted by $R_0 \hookrightarrow_c R_1$. We say that $R_0$ is at most as complex as $R_1$.

In the case $R_0 \hookrightarrow_c R_1$ and $R_1 \not\hookrightarrow_c R_0$, we say that $R_0$ is less complex than $R_0$.

In generalized descriptive set theory, the complexity of a theory can be study by studying the complexity of the isomorphism relation of the theory.

## Question

In model theory, the complexity of a theory can be measure by the number of models of the theory.

# Question

In model theory, the complexity of a theory can be measure by the number of models of the theory.

Shelah's main gap theorem can be understood as: *Classifiable theories are less complex than non-classifiable theories, in the model theoy complexity.*

# Question

In model theory, the complexity of a theory can be measure by the number of models of the theory.

Shelah's main gap theorem can be understood as: *Classifiable theories are less complex than non-classifiable theories, in the model theoy complexity.*

**Question.** Are classifiable theories less complex than non-classifiable theories, in the generalised descriptive set theory complexity?

# Stationary reflection

Let $\alpha$ be an ordinal of uncountable cofinality. A set $C \subseteq \alpha$ is a club if it is closed and unbounded. A set $S \subseteq \alpha$ is stationary if for all club $C \subset \alpha$, $C \cap S \neq \emptyset$.

# Stationary reflection

Let $\alpha$ be an ordinal of uncountable cofinality. A set $C \subseteq \alpha$ is a club if it is closed and unbounded. A set $S \subseteq \alpha$ is stationary if for all club $C \subset \alpha$, $C \cap S \neq \emptyset$.

## Definition

Let $\alpha \in \kappa$ be an ordinal of uncountable cofinality, and a stationary $S \subseteq \kappa$, we say that $S$ reflects at $\alpha$ if $S \cap \alpha$ is stationary in $\alpha$

# Stationary reflection

Let $\alpha$ be an ordinal of uncountable cofinality. A set $C \subseteq \alpha$ is a club if it is closed and unbounded. A set $S \subseteq \alpha$ is stationary if for all club $C \subset \alpha$, $C \cap S \neq \emptyset$.

## Definition

Let $\alpha \in \kappa$ be an ordinal of uncountable cofinality, and a stationary $S \subseteq \kappa$, we say that $S$ reflects at $\alpha$ if $S \cap \alpha$ is stationary in $\alpha$

If $\kappa$ is a weakly compact cardinal, every stationary subset of $\kappa$ reflects at a regular cardinal $\alpha < \kappa$.

# Equivalence modulo nonstationary

## Definition

For every stationary set $S \subseteq \kappa$ and $\theta \in [2, \kappa]$, the equivalence relation $=^\theta_S$ over the subspace $\theta^\kappa$ is defined via

$$\eta =^\theta_S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\} \text{ is non-stationary.}$$

# Equivalence modulo nonstationary

## Definition

For every stationary set $S \subseteq \kappa$ and $\theta \in [2, \kappa]$, the equivalence relation $=^\theta_S$ over the subspace $\theta^\kappa$ is defined via

$$\eta =^\theta_S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\} \text{ is non-stationary.}$$

## Definition

The quasi-order $\leq^S$ over $\kappa^\kappa$ is defined via

$$\eta \leq^S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \text{ is non-stationary.}$$

# Equivalence modulo nonstationary

## Definition

For every stationary set $S \subseteq \kappa$ and $\theta \in [2, \kappa]$, the equivalence relation $=^\theta_S$ over the subspace $\theta^\kappa$ is defined via

$$\eta =^\theta_S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\} \text{ is non-stationary.}$$

## Definition

The quasi-order $\leq^S$ over $\kappa^\kappa$ is defined via

$$\eta \leq^S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \text{ is non-stationary.}$$

The quasi-order $\subseteq^S$ over $2^\kappa$ is nothing but $\leq^S \cap (2^\kappa \times 2^\kappa)$.

## Progress

Let us denote by $=^\theta_\lambda$ the relation $=^\theta_S$ when $S = \{\alpha < \kappa \mid cf(\alpha) = \lambda\}$.

# Progress

Let us denote by $=^{\theta}_{\lambda}$ the relation $=^{\theta}_{S}$ when $S = \{\alpha < \kappa \mid cf(\alpha) = \lambda\}$.

## Fact (Hyttinen-M)

*The isomorphism relation of any classifiable theory is less complex than $=^{\kappa}_{\lambda}$ for all $\lambda$.*

# Progress

Let us denote by $=^\theta_\lambda$ the relation $=^\theta_S$ when $S = \{\alpha < \kappa \mid cf(\alpha) = \lambda\}$.

## Fact (Hyttinen-M)

*The isomorphism relation of any classifiable theory is less complex than $=^\kappa_\lambda$ for all $\lambda$.*

Under some cardinal arithmetic assumptions the following can be proved:

## Fact (Friedman-Hyttinen-Kulikov)

*Suppose $T$ is a non-classifiable theory. There is a regular cardinal $\lambda < \kappa$ such that $=^2_\lambda$ is at most as complex as the isomorphism relation of $T$.*

# Question

Is $=^\kappa_\lambda$ Borel-reducible to $=^2_\lambda$, i.e. $=^\kappa_\lambda \hookrightarrow_c =^2_\lambda$, for all $\lambda$?

# Comparing $=^\kappa_S$ and $=^2_S$

### Fact (Asperó-Hyttinen-Kulikov-M)

*If every stationary subset of $X$ reflects at stationary many $\alpha \in Y$, then $=^\kappa_X \hookrightarrow_c =^\kappa_Y$.*

# Comparing $=^\kappa_S$ and $=^2_S$

**Fact (Asperó-Hyttinen-Kulikov-M)**

*If every stationary subset of $X$ reflects at stationary many $\alpha \in Y$, then $=^\kappa_X \hookrightarrow_c =^\kappa_Y$.*

**Fact (Friedman-Hyttinen-Kulikov)**

*Suppose $V = L$, and $X \subseteq \kappa$ and $Y \subseteq reg(\kappa)$ are stationary. If every stationary subset of $X$ reflects at stationary many $\alpha \in Y$, then $=^2_X \hookrightarrow_c =^2_Y$.*

# Limitations

- For all regular cardinals $\gamma \leq \lambda < \kappa$, any $X \subseteq S^\kappa_\lambda$, $X$ does not reflect at any $\alpha \in S^\kappa_\gamma$.

## Limitations

- For all regular cardinals $\gamma \leq \lambda < \kappa$, any $X \subseteq S^\kappa_\lambda$, $X$ does not reflect at any $\alpha \in S^\kappa_\gamma$.

- If $\kappa = \lambda^+$ and $\square_\lambda$ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that $Y$ does not reflect at any $\alpha < \kappa$.

# Limitations

- For all regular cardinals $\gamma \leq \lambda < \kappa$, any $X \subseteq S^\kappa_\lambda$, $X$ does not reflect at any $\alpha \in S^\kappa_\gamma$.
- If $\kappa = \lambda^+$ and $\square_\lambda$ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that $Y$ does not reflect at any $\alpha < \kappa$.
- Usual stationary reflection requires large cardinals.

## Capturing clubs

Suppose $S$ is stationary subset of $\kappa$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over $\alpha$.

# Capturing clubs

Suppose $S$ is stationary subset of $\kappa$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over $\alpha$.

## Definition

We say that $\vec{\mathcal{F}}$ *captures clubs* iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is non-stationary;

# Capturing clubs

Suppose $S$ is stationary subset of $\kappa$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over $\alpha$.

## Definition

We say that $\vec{\mathcal{F}}$ *captures clubs* iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is non-stationary;

For any ordinal $\alpha < \kappa$ of uncountable cofinality, denote by $CUB(\alpha)$ the club filter of subsets of $\alpha$. The sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S^\kappa_{\omega_1} \rangle$ define by $\mathcal{F}_\alpha = CUB(\alpha)$, capture clubs.

# Filter reflection

Suppose $X$ and $S$ are stationary subsets of $\kappa$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over $\alpha$.

# Filter reflection

Suppose $X$ and $S$ are stationary subsets of $\kappa$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over $\alpha$.

## Definition

We say that $X$ $\vec{\mathcal{F}}$-reflects to $S$ iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ is stationary

# Filter reflection

Suppose $X$ and $S$ are stationary subsets of $\kappa$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over $\alpha$.

## Definition

We say that $X$ $\vec{\mathcal{F}}$-reflects to $S$ iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary

## Definition

We say that $X$ $\mathfrak{f}$-*reflects to* $S$ iff there exists a sequence of filters $\vec{\mathcal{F}}$ over a stationary subset $S'$ of $S$ such that $X$ $\vec{\mathcal{F}}$-reflects to $S'$.

# Strong forms of filter reflection

## Definition

We say that $X$ strongly $\vec{\mathcal{F}}$-reflects to $S$ iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha\}$ is stationary.

# Strong forms of filter reflection

### Definition

We say that $X$ strongly $\vec{\mathcal{F}}$-reflects to $S$ iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha\}$ is stationary.

### Definition

We say that $X$ *$\vec{\mathcal{F}}$-reflects with $\diamondsuit$ to* $S$ iff $\vec{\mathcal{F}}$ captures clubs and there exists a sequence $\langle Y_\alpha \mid \alpha \in S \rangle$ such that, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y_\alpha = Y \cap \alpha \;\&\; Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ is stationary.

## Strong forms of filter reflection

### Definition

We say that $X$ strongly $\vec{\mathcal{F}}$-reflects to $S$ iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha\}$ is stationary.

### Definition

We say that $X$ *$\vec{\mathcal{F}}$-reflects with $\diamondsuit$ to $S$* iff $\vec{\mathcal{F}}$ captures clubs and there exists a sequence $\langle Y_\alpha \mid \alpha \in S\rangle$ such that, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y_\alpha = Y \cap \alpha \;\&\; Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ is stationary.

We apply the same convention for $X$ *strongly $\mathfrak{f}$-reflects to $S$* and $X$ *$\mathfrak{f}$-reflects with $\diamondsuit$ to $S$*

## Properties

### Fact (Monotonicity)

*For stationary sets $Y \subseteq X \subseteq \kappa$ and $S \subseteq T \subseteq \kappa$. If $X$ $\mathfrak{f}$-reflects to $S$, then $Y$ $\mathfrak{f}$-reflects to $T$;*

# Properties

## Fact (Monotonicity)

*For stationary sets $Y \subseteq X \subseteq \kappa$ and $S \subseteq T \subseteq \kappa$. If $X$ $\mathfrak{f}$-reflects to $S$, then $Y$ $\mathfrak{f}$-reflects to $T$;*

## Fact

*Suppose $X$ strongly $\mathfrak{f}$-reflects to $S$. If $\diamondsuit_X$ holds, then so does $\diamondsuit_S$.*

# Properties

### Fact (Monotonicity)

*For stationary sets $Y \subseteq X \subseteq \kappa$ and $S \subseteq T \subseteq \kappa$. If $X$ $\mathfrak{f}$-reflects to $S$, then $Y$ $\mathfrak{f}$-reflects to $T$;*

### Fact

*Suppose $X$ strongly $\mathfrak{f}$-reflects to $S$. If $\diamondsuit_X$ holds, then so does $\diamondsuit_S$.*

### Fact

*Suppose $V = L$, then for all stationary sets $X, S \subseteq \kappa$, $X$ $\mathfrak{f}$-reflects to $S$.*

# Over the limits

- Usual stationary reflection is a special case of filter reflection.

## Over the limits

- Usual stationary reflection is a special case of filter reflection.
- For all regular cardinals $\gamma \leq \lambda < \kappa$, any $X \subseteq S^\kappa_\lambda$, $X$ does not reflect at any $\alpha \in S^\kappa_\gamma$. $S^\kappa_\lambda$ $\mathfrak{f}$-reflects to $S^\kappa_\gamma$ is consistently true.

# Over the limits

- Usual stationary reflection is a special case of filter reflection.
- For all regular cardinals $\gamma \leq \lambda < \kappa$, any $X \subseteq S^\kappa_\lambda$, $X$ does not reflect at any $\alpha \in S^\kappa_\gamma$. $S^\kappa_\lambda$ $\mathfrak{f}$-reflects to $S^\kappa_\gamma$ is consistently true.
- If $\kappa = \lambda^+$ and $\Box_\lambda$ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that $Y$ does not reflect at any $\alpha < \kappa$. Fake reflection is consistent with $\Box_\lambda$.

# Over the limits

- Usual stationary reflection is a special case of filter reflection.
- For all regular cardinals $\gamma \leq \lambda < \kappa$, any $X \subseteq S^\kappa_\lambda$, $X$ does not reflect at any $\alpha \in S^\kappa_\gamma$. $S^\kappa_\lambda$ $\mathfrak{f}$-reflects to $S^\kappa_\gamma$ is consistently true.
- If $\kappa = \lambda^+$ and $\square_\lambda$ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that $Y$ does not reflect at any $\alpha < \kappa$. Fake reflection is consistent with $\square_\lambda$.
- Fake reflection does not require large cardinals.

# Killing Filter Reflection

**Theorem**

*There exists a cofinality-preserving forcing extension in which, for all stationary subsets $X$, $S$ of $\kappa$, $X$ does not $\mathfrak{f}$-reflect to $S$*

# Killing Filter Reflection

### Theorem

*There exists a cofinality-preserving forcing extension in which, for all stationary subsets $X$, $S$ of $\kappa$, $X$ does not $\mathfrak{f}$-reflect to $S$*

Force a coherent regressive $C$-sequence, then force with $Add(\kappa, \kappa^+)$.

# Forcing Filter Reflection

**Theorem**

*For all stationary subsets $X$ and $S$ of $\kappa$, there exists a $<\kappa$-closed $\kappa^+$-cc forcing extension, in which $X$ $\mathfrak{f}$-reflects to $S$.*

# Forcing Filter Reflection

**Theorem**

*For all stationary subsets $X$ and $S$ of $\kappa$, there exists a $<\kappa$-closed $\kappa^+$-cc forcing extension, in which $X$ $\mathfrak{f}$-reflects to $S$.*

Force with Sakai's forcing

# Stationary Reflection

## Theorem

*If $\kappa$ is strongly inaccessible, then in the forcing extension by $Add(\kappa, \kappa^+)$, for all two disjoint stationary subsets $X, S$ of $\kappa$, the following are equivalent:*

1. *$X$ $\mathfrak{f}$-reflects to $S$;*
2. *every stationary subset of $X$ reflects in $S$.*

## $Add(\omega, \kappa)$

**Theorem**

*Suppose $X$ $\mathfrak{f}$-reflects to $S$ holds. Then $Add(\omega, \kappa)$ forces that $X$ $\mathfrak{f}$-reflects to $S$.*

# Preserving Filter Reflection

## Definition

Let $\kappa = \lambda^+$. A notion of forcing $\mathbb{Q}$ satisfies $\kappa$-stationary-cc if for every sequence $\langle q_\delta \mid \delta < \kappa \rangle$ of conditions in $\mathbb{Q}$ there is a club $D \subseteq \kappa$ and a regressive map $h : D \cap E^\kappa_{\mathrm{cof}(\lambda)} \to \kappa$ such that for all $\gamma, \delta \in dom(h)$, if $h(\gamma) = h(\delta)$ then $q_\gamma$ and $q_\delta$ are comparable.

# Preserving Filter Reflection

## Definition

Let $\kappa = \lambda^+$. A notion of forcing $\mathbb{Q}$ satisfies $\kappa$-stationary-cc if for every sequence $\langle q_\delta \mid \delta < \kappa \rangle$ of conditions in $\mathbb{Q}$ there is a club $D \subseteq \kappa$ and a regressive map $h : D \cap E^\kappa_{\mathrm{cof}(\lambda)} \to \kappa$ such that for all $\gamma, \delta \in dom(h)$, if $h(\gamma) = h(\delta)$ then $q_\gamma$ and $q_\delta$ are comparable.

## Definition

We say that a sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is *$\theta$-complete* if the set $\{\alpha \in S \mid \mathcal{F}_\alpha \textit{ is not } \theta\textit{-complete}\}$ is non-stationary.

# Preserving Filter Reflection

**Theorem**

*Suppose $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a $\theta$-complete sequence. Suppose $\mathbb{P}$ is a forcing notion with $\theta$-cc and $\kappa$-stationary-cc, and $X \subseteq \kappa$ is a stationary set such that $X$ $\vec{\mathcal{F}}$-reflects to $S$. Then $\mathbb{P}$ forces that $X$ $\vec{\mathfrak{f}}$-reflects to $S$.*

# Equivalence Modulo a Filter

## Definition

Let $\mathcal{F}$ be a filter over $\alpha$. For every $\theta \in [2, \kappa]$, the equivalence modulo $\mathcal{F}$, $\sim_{\mathcal{F}}^{\theta}$, over $\theta^{\alpha}$, is defined via

$$(\eta, \xi) \in \sim_{\mathcal{F}}^{\theta} \text{ iff } \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\} \in \mathcal{F}$$

# Equivalence Modulo a Filter

## Definition

Let $\mathcal{F}$ be a filter over $\alpha$. For every $\theta \in [2, \kappa]$, the equivalence modulo $\mathcal{F}$, $\sim^\theta_{\mathcal{F}}$, over $\theta^\alpha$, is defined via

$$(\eta, \xi) \ \in \sim^\theta_{\mathcal{F}} \ \text{iff } \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\} \in \mathcal{F}$$

## Definition

For every $\theta, \gamma \in [2, \kappa]$, $F : \theta^\kappa \to \gamma^\kappa$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ a sequence of filters. We say that $F$ captures $\vec{\mathcal{F}}$ if for all $\alpha \in S$ and $\eta, \xi \in \theta^\kappa$

$$\eta \restriction \alpha \ \sim^\theta_{\mathcal{F}_\alpha} \ \xi \restriction \alpha \text{ iff } F(\eta)(\alpha) = F(\xi)(\alpha).$$

# Characterization of Filter Reflection

**Theorem**

*Let $X, S \subseteq \kappa$ be stationary sets. The following are equivalent:*

1. *$X$ $\mathfrak{f}$-reflects to $S$.*
2. *There is $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ and $F : 2^\kappa \to \kappa^\kappa$ a reduction from $=^2_X$ to $=^\kappa_S$ that captures $\vec{\mathcal{F}}$.*

# Characterization of Strong Filter Reflection

**Theorem**

*Let $X, S \subseteq \kappa$ be stationary sets. The following are equivalent.*

1. *$X$ strongly $\mathfrak{f}$-reflects to $S$.*
2. *There is a reduction $F : 2^\kappa \to 2^\kappa$ from $=^2_X$ to $=^2_S$ such that for all $\alpha \in S$ the set*

$$\{\eta \restriction_\alpha^{-1} [\gamma] \mid \eta \in 2^\kappa \ \& \ F(\eta)(\alpha) = \gamma\}$$

*is a filter.*

# Reductions from Filter Reflection

**Lemma**

*If $X$ strongly $\mathfrak{f}$-reflects to $S$, then for all $\theta \in [2, \kappa]$, $=^{\theta}_{X} \hookrightarrow_{c} =^{\theta}_{S}$.*

# Reductions from Filter Reflection

**Lemma**

*If $X$ strongly $\mathfrak{f}$-reflects to $S$, then for all $\theta \in [2, \kappa]$, $=^{\theta}_{X} \hookrightarrow_{c} =^{\theta}_{S}$.*

**Theorem**

*Let $X, S \subseteq \kappa$ be stationary sets. If $X$ $\mathfrak{f}$-reflects with $\diamondsuit$ to $S$, then*

$$\leq^{X} \hookrightarrow_{c} \subseteq^{S}.$$

# The Main Gap

**Theorem**

*Let $\kappa = \lambda^+$. If cof$(\omega)$ $\mathfrak{f}$-reflects with $\diamondsuit$ to cof$(\omega)$ and cof$(\lambda)$ $\mathfrak{f}$-reflects with $\diamondsuit$ to cof$(\lambda)$, then the isomorphism relation of any classifiable theory is continuous reducible to the isomorphism relation of any non-classifiable theory.*

# The Main Gap

**Theorem**

*Let $\kappa = \lambda^+$ and $X \subseteq \kappa$ a stationary set such that $X \cap cof(\lambda) = \emptyset$. If $X$ strongly $\mathfrak{f}$-reflects to $cof(\lambda)$, then the isomorphism relation of any classifiable theory is continuous reducible to the isomorphism relation of any non-classifiable theory.*

# Martin Maximum

## Theorem

*Suppose Martin's Maximum holds, $\kappa > \omega_2$, $X \subseteq cof(\omega)$ a stationary set and $S \subseteq cof(\omega_1)$. If $\diamondsuit_X$ holds, then $X$ reflects with $\diamondsuit$ to $S$.*

## Martin Maximum

### Theorem

*Suppose Martin's Maximum holds, $\kappa > \omega_2$, $X \subseteq cof(\omega)$ a stationary set and $S \subseteq cof(\omega_1)$. If $\diamondsuit_X$ holds, then $X$ reflects with $\diamondsuit$ to $S$.*

MM implies that for $\kappa = \lambda^+$, $\lambda$ a singular strong limit of uncountable cofinality, it holds

$$=^{\kappa}_{\omega} \hookrightarrow_c =^{2}_{\omega_1}.$$

# Remains Open

Is the following consistently true?
For all $S \subseteq \kappa$ stationary,

$$=^{\kappa}_{S} \not\hookrightarrow_{c} =^{2}_{S}.$$

Thank you