Geometric Set Theory IV

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Solovay models

Suppose that κ is a strongly inaccessible cardinal, and that

$$G \subseteq \mathrm{Col}(\omega, {<}\kappa)$$

is a V-generic filter. The resulting model

$$\mathrm{HOD}^{V[G]}_{V,\mathcal{P}(\omega)}$$

is a Solovay model (which we will call W).

We study forcing extensions of the Solovay model which recover forms of the Axiom of Choice.

Suslin orders

A preorder $\langle P, \leq \rangle$ is *Suslin* if there is a Polish space X such that

- P is an analytic subset of X
- the ordering $\leq$ is an analytic subset of X^2
- (the incompatibility relation is an analytic subset of X^2)

Virtual conditions

A *virtual condition* for a Suslin order $\langle P, \leq \rangle$ is a pair (Q, τ) such that

- Q is a partial order,
- τ is a Q-name for an element of P, and
- τ realizes to an equivalent P-condition in every V-generic Q-extension.

The last item above is the same as : $Q \times Q$ forces that

$$\tau_{g_0} \leq \tau_{g_1} \wedge \tau_{g_1} \leq \tau_{g_0},$$

where g_0 and g_1 are the left and right filters.

Balanced conditions

Given a Suslin forcing P, a *balanced condition* for P is a virtual condition (Q, τ) such that, for any two V-generic filters

$$G_0, G_1 \subseteq Q$$

existing respectively in mutually generic extensions $V[H_0]$ and $V[H_1]$, and any two conditions

$$p_0 \leq \tau_{G_0} \text{ and } p_1 \leq \tau_{G_1}$$

in $V[H_0]$ and $V[H_1]$ respectively, p_0 and p_1 are compatible in $V[H_0, H_1]$.

Balanced partial orders

A Suslin order is said to be *balanced* if (provably, in ZFC) below each condition there is a balanced virtual condition.

Balance is not in general absolute between V and its forcing extensions.

We say that P is *cofinally balanced below* κ if every partial order in V_κ is regularly embedded in one forcing that P is balanced in V_κ.

Consequences of balance

If P is cofinally balanced then the following hold in $W[G]$:

- every wellordered sequence of elements of W is in W;
- there are no infinite MAD families on ω;
- there are no unbounded linearly ordered subsets of $(\omega^\omega, \leq^*)$ or the Turing degrees.

The finite/countable example

A Suslin partial order is induced by the set of disjoint pairs

$$(a, b) \in [\mathbb{R}]^{<\omega} \times [\mathbb{R}]^{\omega},$$

ordered by containment.

Balanced virtual conditions are given by a finite set of reals and its complement.

A generic Dedekind finite set

A set X is said to be Dedekind finite if there is no injection from ω to X.

If $\mathrm{CC}_{\mathbb{R}}$ holds then every Dedekind finite set is finite.

The disjoint finite/countable order is balanced, so it does not add reals.

It forces $\mathrm{CC}_{\mathbb{R}}$ to fail: the union of the finite parts of the members of the generic filter is a Dedekind finite subset of $\mathbb{R}$ with the property that every real is the sum (similarly, product) of two members of X.

Strong forms of balance

Forcing with a typical balanced partial order explicitly adds a witness to some Σ^2_1 statement.

To show that it does not add a witness to some other Σ^2_1 statement, we show that the partial order satisifes some strong form of balance.

Placid balance

We get a stronger notion of balance if instead of mutual genericity, we require only that

$$V = V[H_0] \cap V[H_1]$$

and that H_0 and H_1 exist in a common forcing extension of V.

Placid balanced conditions

Given a Suslin forcing P, a *balanced condition* for P is a pair (Q, τ) such that

- (Q, τ) is a virtual condition,
- for any two V-generic filters G_0 and G_1 for Q existing respectively in generic extensions $V[H_0]$ and $V[H_1]$ with

$$V[H_0] \cap V[H_1] = V,$$

and any two conditions

$$p_0 \leq \tau_{G_0} \text{ and } p_1 \leq \tau_{G_1}$$

in $V[H_0]$ and $V[H_1]$ respectively, p_0 and p_1 are compatible in $V[H_0, H_1]$.

Examples of placid balanced orders

- Adding a generic function from ω^ω to ω.
- Adding a selector for a countable Borel equivalence relation.
- More generally, when $E \subseteq F$ are Borel equivalence relations and F is placid, forcing with countable sets on which $E = F$ (so, choosing an E-subclass for each F-class).
- Assigning a structure (for instance, a $\mathbb{Z}$-ordering) to each class of a countable Borel equivalence relation.

More examples

- Adding a maximal acyclic subgraph to a Borel graph.
- Adding a Hamel basis to a Polish vector space.
- Adding a countably saturated model of a first order theory, with domain ω^ω.

In placid balanced extensions of W:

- there are no nonprincipal ultrafilters on ω;
- if E and F are analytic equivalence relations such that E is induced by a turbulent Polish group action and F is virtually placid, then there is no injection from the E-classes to the F-classes;
- there are no selectors for non-placid analytic equivalence relations;
- no uncountable field has a transcendence basis over a countable subfield.

Nest balance

If P is a Suslin forcing, a P-nest below a condition $p \in P$ consists of a choice-coherent sequence $\langle M_n \colon n \in \omega \rangle$ of generic extensions of V and a sequence $\langle \bar{p}_n \colon n \leq \omega \rangle$ such that

- for each $n \in \omega$,
  - $2^\omega \cap M_n \supsetneq 2^\omega \cap M_{n+1}$;
  - $\bar{p}_n$ is a balanced virtual condition in M_n below $\bar{p}_{n+1}$;
- $\bar{p}_\omega$ is a balanced virtual conditon in $\bigcap_n M_n$, below p and above each $\bar{p}_n$.

P is nest balanced if it has a nest below each condition.

A nest balanced order cannot inject the $\mathbb{E}_1$ classes into the classes of any orbit equivalence relation induced by a Polish group.

A nested balanced order cannot add a Hamel basis (which would inject $\mathbb{E}_1$ into $\mathbb{F}_2$).

Hamel bases and $\mathbb{F}_2$

If B is a Hamel basis, there is a predicate B' such that, in any ω-model M amenable to B', $M \cap B$ is a Hamel basis.

Given $x \in ({}^{\omega}2)^{\omega}$, the sets

$$L[B', x \restriction [n, \omega)] \cap 2^{\omega}$$

stabilize (to a common set containing a member of the $\mathbb{E}_1$-class of x).

It follows that if there is a Hamel basis, then $|\mathbb{E}_1| \leq |\mathbb{F}_2|$.

Nest balance examples

- When $E \subseteq F$ are Borel equivalence relations and F is pinned and Borel reducible to an orbit equivalence relation, forcing with countable sets on which $E = F$ is nest balanced.

- Assigning a structure (for instance, a $\mathbb{Z}$-ordering) to each class of a countable Borel equivalence relation is nest balanced.

- The partial order of finite acyclic subsets of a Borel graph is nest balanced.

Definable balance

Given a partial order Q, say that P is *definably* Q-balanced if for any $p \in P$ there is a formula ϕ (using parameters from V) which defines a balanced virtual condition below p after forcing with Q.

The partial order for adding a generic function from 2^ω to 2 by countable approximations is definably Q-balanced for all Q: a balanced condition can take value 0 for all reals not in the domain of p.

A Suslin order which is definably Cohen-balanced cannot add an ultrafilter on ω.

Working in $V[K]$ containing a real defining a P-name for an ultafilter, force to add a Cohen real.

The balanced condition definable over $V[K]$ will have to decide whether or not the Cohen-generic real will be in the ultrafilter.

Since the set of Cohen reals over $V[K]$ is closed under complements and finite changes, neither of which change the resulting extension, no Cohen condition can decide this.

Compact Balance

A Suslin order P is *compactly balanced* if there is a suitably definable compact Hausdorff topology T on the set of equivalence classes of balanced virtual conditions.

As a result : there is a way of taking ultrafilter limits of sequences of balanced conditions.

Compactly balanced orders do not add $\mathbb{E}_0$-selectors.

$\mathcal{P}(\omega)/\mathrm{Fin}$ is compactly balanced ($\beta\omega$ is compact).

If $(G, \cdot)$ and $(H, \cdot)$ are Polish groups, with $(H, \cdot)$ divisible and compact, then partial order of countable partial homomorphisms from $(G, \cdot)$ into $(H, \cdot)$ is compactly balanced. (Tychonoff's Theorem)

Compact balance implies nest balance.

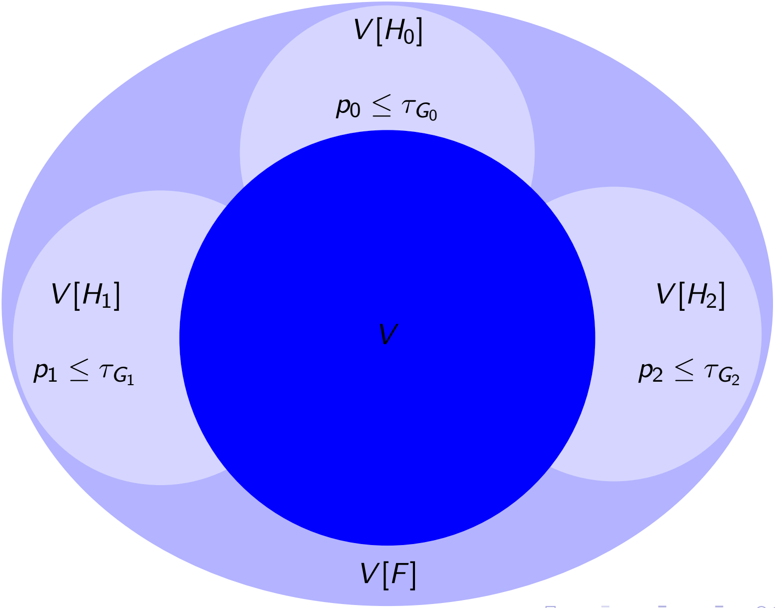
$(3, 2)$-balance

A virtual condition (Q, τ) for P is $(3, 2)$-*balanced* if in any generic extension, whenever

- $\{H_i : i \in 3\}$ are Q-filters such that for every set $a \in [3]^2$ the filters
$$\{H_i : i \in a\}$$
are mutually generic over V,
- for each i, p_i is a condition in $V[H_i]$ below the corresponding realization of τ,

then the set
$$\{p_i : i \in m\}$$
has a common lower bound in P.

$(3, 2)$-balance



$(3, 2)$-balanced orders

- Countable partial selectors for an analytic equivalence relation.
- Countable partial injections from the E-classes to the F-classes, for pinned Borel equivalence relations E and F on Polish spaces.
- The poset of countable linear orders of E-classes, for a pinned Borel equivalence relation E on a Polish space.
- The poset of countable acyclic subsets of a Borel graph on a Polish space.

If P is $(3, 2)$-balanced, then in the P-extension of W

- there are no nonprincipal ultrafilters on ω;
- no uncountable Polish vector space over countable field has a basis;
- there are no discontinuous homomorphisms between Polish groups.

No discontinuous homomorphisms in (3,2)-balanced extensions

Let $\langle P, \leq \rangle$ be a $(3, 2)$-balanced Suslin order, let Γ be a Polish group, and suppose that τ is a P-name for a discontinuous homomorphism from Γ to some Polish group.

Let $V[K]$ be an intermediate ($<\kappa$-sized) forcing extension of V in which (Q, σ) is a (3,2)-balanced virtual condition (for some partial order Q collapsing 2^{ω}).

In any Q-extension there are comeagerly many $V[K]$-generic elements for P_{Γ}.

Since τ is forced to be discontinuous, and since any homomorphism of Γ which is Borel on a somewhere comeager set is continuous, there exists a $z_* \in \Gamma$ in some Q-extension $V[K][G_z]$, such that the τ-value of z_* is not decided in $V[K][G_z]$ by the realization of σ.

Let

$$C = \{(x, y, z) \in G^3 : x \cdot y = z\}$$

and let P_C be the partial order of nonempty open subsets of C, under containment.

There exist generic filters G_x and G_y for Q adding respectively $x_*, y_* \in \Gamma$ such that

$$(x_*, y_*, z_*) \in C$$

and each pair from

$$\{G_x, G_y, G_z\}$$

is mutually generic.

There exist (in W) generic filters

- H_x for $V[K][G_x]$ adding a condition p_x below σ deciding $\tau(x_*)$,
- H_y for $V[K][G_y]$ adding a condition p_y below σ deciding $\tau(y_*)$,
- H_z, H'_z for $V[K][G_z]$ adding conditions p_z and p'_z below σ deciding $\tau(z_*)$ differently,

such that the triples (H_x, H_y, H_z) and (H_x, H_y, H'_z) are each pairwise mutually generic over $V[K]$.

It can't be then that (p_x, p_y, p_z) and (p_x, p_y, p'_z) are both compatible with the equation

$$\tau(x_*) \cdot \tau(y_*) = \tau(z_*).$$

Other forms of balance

- pod balance (preserving nonreducibility to $\mathbb{E}_{K_\sigma}$)
- tethered balance (preserving uniformization properties of W)
- perfect balance (preserving "every Borel equivalence relation with uncoutably many classes has a perfect parwise inequivalent set")
- Bernstein balance (preserving the non-existence of nontrivial probability measures on $\mathcal{P}(\omega)$)
- transcendent balance (in a later paper of Zapletal)

Weak balance (I)

Last time we showed that if $(Q_0, \tau_0) \leq (Q_1, \tau_1)$ are virtual conditions, and (Q_1, τ_1) is balanced, then (Q_0, τ_0) and (Q_1, τ_1) are equivalent.

We say that a virtual condition (Q, τ) is weakly balanced if, whenever (Q', τ') is a virtual condition below (Q, τ), (Q, τ) and (Q', τ') are equivalent.

A Suslin order is weakly balanced if there is a weakly balanced condition below each condition.

Weak balance (II)

In weak balanced extensions there are no new sets of ordinals, but there may be new functions from the ordinals to W.

Certain types of ("improved") MAD families can be added by weakly balanced partial orders, while preserving uncountable chromatic numbers for all Borel hypergraphs.

The weakly balanced partial orders are not closed under products, since it is possible to add either an ω_1-sequence of $\mathbb{E}_0$ degrees with a weakly balanced forcing, an $\mathbb{E}_0$-selector, but not both.

The generic ultrafilter model

Theorem

In the generic (Ramsey) ultrafilter model $W[U]$, the following statements hold:

1. $|\mathbb{E}_0| \not\leq |2^\omega|$;
2. $|\mathbb{E}_1| \not\leq |F|$ *for any orbit equivalence relation* F;
3. *there do not exist tournaments on the quotient spaces of* $\mathbb{E}_2$ *and* $\mathbb{F}_2$;
4. *if* E *is a Borel equivalence relation and* A *is a subset of the* E*-quotient space then either* $|A| \leq \aleph_0$ *or* $|2^\omega| \leq |A|$;
5. *every nonprincipal ultrafilter on* ω *has nonempty intersection with the summability ideal;*
6. *countable-to-one uniformization.*

The generic Hamel basis model

Theorem

In the generic Hamel basis model $W[B]$, the following statements hold:

1. *there is no transcendence basis for any uncountable Polish field;*
2. *there is no nonprincipal ultrafilter on ω;*
3. *countable-to-one uniformization.*

The generic $\mathbb{E}_0$-selector model

Theorem

In the generic $\mathbb{E}_0$-selector model $W[T]$, the following statements hold:

1. $|\mathbb{E}_1| \not\leq |F|$ *for any orbit equivalence relation* F*;*
2. $|E| \not\leq |F|$ *for any equivalence relation* E *generated by a turbulent group action and equivalence relation* F *classifiable by countable structures;*
3. *there is no discontinuous homomorphism of* $(\mathbb{R}, +)$*;*
4. *there is no nonprincipal ultrafilter on* ω*;*
5. *countable-to-one uniformization.*

The generic improved MAD family model

Theorem

In the generic "improved" MAD family model $W[A]$, the following statements hold:

1. *there are no ω_1 sequences of distinct reals;*
2. *there are no nonatomic measures on ω;*
3. *every set of reals is Lebesgue measurable;*
4. *the $\mathbb{E}_0$ classes are not linearly orderable;*
5. *there are no total selectors for $\mathbb{E}_0$;*
6. $2^{\aleph_0} \not\leq |A|$.

Colorings

A natural class of Σ^2_1 statements consists of assertions that various Borel hypergraphs are countably colorable.

Let X be a Polish space and let Γ be a Borel subset of $[X]^{<\omega}$.

The assertion that there exists a function $c \colon X \to \omega$ which is constant on no member of Γ is Σ^2_1.

In this case we say that Γ has countable chromatic number.

The most basic partial order P_Γ for adding such a function consists of countable partial functions with this property (formally, ω-sequences from

$$X \times \omega$$

whose ranges are such functions).

Coloring number

A graph $\Gamma \subseteq [X]^2$ is said (by Erdős and Hajnal, in the ZFC context) to have countable coloring number if there is a (strict) wellordering $\prec$ of X such that, for all $x \in X$, the set

$$\{y \prec x : \{x, y\} \in \Gamma\}$$

is finite.

This was shown by Adams and Zapletal to be equivalent to : for every countable set $a \subseteq X$, the set

$$\{x \in X : \exists^\infty y \in a \{x, y\} \in \Gamma\}$$

is countable.

Examples

Given a countable set D of positive real numbers, the graph consisting of pairs of points in $\mathbb{R}^2$ whose distances are in D (Erdős and Hajnal).

Given a countable set D of positive real numbers converging to 0, the graph consisting of pairs of points from $\mathbb{R}^n$ (for any fixed n) whose distance is in D (Komjath).

Theorem

If Γ has countable coloring number then P_Γ is a balanced Suslin forcing, and (up to equivalence) the balanced virtual conditions are the pairs

$$(\mathrm{Col}(\omega, X), \sigma),$$

where σ is a name for a generic enumeration of some total coloring on X.

Redundancy

A hypergraph $\Gamma \subseteq [X]^{<\omega}$ has redundancy m if every every finite set $b \subseteq X$, the set

$$\{c \in [X]^{\leq m} : b \cup c \in \Gamma\}$$

is countable.

A set $a \subseteq X$ is (m, Γ)-closed if, for each finite $b \subseteq a$, the set above is contained in a.

The m-coloring poset for Γ, P^m_Γ, is the partial order of countable partial colorings whose domains are (m, Γ)-closed.

Example

Let Γ be the graph on $\mathbb{R}^2$ consiting of those sets of size 4 which are the vertices of a square. Then Γ has redundancy 2.

Theorem

Suppose that Γ is a Borel hypergraph of arity 4 and redundancy 2.

Then P^2_Γ is balanced if and only if Γ has countable chromatic number, which follows from the Continuum Hypothesis.

In this case, the balanced virtual conditions are the pairs $(\mathrm{Col}(\omega, X), \sigma)$, where σ is a name for a generic enumeration of some total coloring on X.

(It is not known if ZFC implies that each order P^2_Γ is balanced for graphs of arity 4 and redundancy 2.)

Circular graphs

Let $(G, \cdot)$ be a Polish group and let Φ be a finite set of Borel automorphisms.

The induced circular hypergraph Γ_Φ consists of those sets $\{a, b, c\}$ from G such that, for some $\phi \in \Phi$,

$$\phi(ba^{-1}) = ca^{-1}.$$

There may be a different ϕ for each ordering of (a, b, c).

Example 1

The set

$$\{\{x, x+y, x+2y\} : x, y \in \mathbb{R}\}$$

is a circular hypergraph on

$$(\mathbb{R}, +),$$

as witnessed by the functions $2x$, $-x$ and $x/2$.

Example 2

Φ consisting of the rotations by $\pi/3$ and $-\pi/3$ witnesses that the graph consisting of the vertices of equilateral triangles in $\mathbb{R}^2$ is circular.

More generally, the graph consisting of the verticles of a triangles in $\mathbb{R}^2$ similar to a given triangle is circular.

These two examples were shown by Ceder to have countable chromatic number in ZFC.

Circular graph posets

If Φ witnesses that graph $\Gamma \subseteq [G]^3$ is circular, we let the coloring poset P_ϕ consist of those countable partial colorings

$$p\colon G \to \omega \times \omega$$

such that the domain of p is a subgroup of G closed under each element of Φ and its inverse.

We let $q \leq p$ if $p \subseteq q$ and for every right dom(p)-coset $e \subseteq$ dom(q) distinct from dom(p), the set $q[e]$ has all vertical sections finite.

Theorem

P_Φ is balanced (in fact, placid, nested and (4,3)-balanced) if and only if Γ_Φ has countable coloring number, which follows from the Continuum Hypothesis.

In this case, the balanced virtual conditions are the pairs $(\mathrm{Col}(\omega, X), \sigma)$, where σ is a name for a generic enumeration of some total coloring on X.

(It is not known if ZFC implies that P_Φ is balanced.)

More recent work (I)

Zapletal, Coloring the Distance Graphs (2022).

Let Γ_n be the graph on $\mathbb{R}^n$ connecting points of rational Euclidean distance.

There is a balanced forcing exension of W in which Γ_n has countable chromatic number and the chromatic number of Γ_{n+1} is uncountable.

The proof uses some algebraic geometry (Noetherian topologies, Krull dimension, the Hilbert Basis Theorem).

More recent work (II)

Zapletal, Coloring Triangles and Rectangles (2021).

There is a balanced forcing extension of W in which the hypergraph of rectangles on a given Euclidean space has countable chromatic number, while the hypergraph of equilateral triangles on $\mathbb{R}^2$ does not.

Consistently the chromatic number of Euclidean rectangles in $\mathbb{R}_n$ is countable while that of $\mathbb{R}_{n+1}$ is not.

It is open whether the same can be achieved for equilateral triangles.

More recent work (III)

Zapletal, Krull Dimension in Set Theory (2021).

For every number $n \geq 2$, let Γ_n be the hypergraph on $\mathbb{R}^n$ of arity four consisting of all Euclidean rectangles.

There is a balanced forcing extension of W in which the chromatic number of Γ_n is countable while that of Γ_{n+1} is not.

Is the same possible for parallelograms?

More recent work (IV)

Yuxin Zhou, Coloring Isosceles Triangles in Choiceless Set Theory.

There is a balanced forcing extension of W in which the graph of isosceles triangles in $\mathbb{R}^2$ has countable chromatic number, but the corresponding graph on $\mathbb{R}^3$ does not.

Is this possible for larger n?

More recent work (V)

Zapletal, Trancendental Pairs of Generic Extensions (2021).

Let Γ_n be the hypergraph on $\mathcal{P}(\omega)$ consisting of n-tuples which modulo finite form a partition of ω.

For $n = 3$ and $n = 4$, there is a balanced forcing extension of W in which the chromatic number of Γ_n is countable, yet the chromatic number of Γ_{n+1} is not.

The proof uses a new form of balanced, called transcendental balance.

It is open whether the same can be done for higher n.

Note that all Γ_n have chromatic number 2 if there exists a nonprincipal ultrafilter on ω.

Let Θ_n be the set of nonempty finite $F \subseteq \mathcal{P}(\omega)$ such that $\bigcap F$ and $\omega \setminus \bigcup F$ are both finite.

Zapletal : Θ_4 has uncountable chromatic number in transcendental extensions (e.g., the models from the previous slide).

If Θ_3 has countable chromatic number, must there be a nonprincipal ultrafilter on ω?

If Θ_4 has countable chromatic number, must it be 2-colorable?

Discontinuous Homomorphisms

The existence of a discontinuous homomorphism from $(\mathbb{R}, +)$ to itself implies the existence of an $\mathbb{E}_0$-selector.

Does it imply the existence of a Hamel basis for $\mathbb{R}$?

Does it imply that there is an injection from the $\mathbb{E}_1$-classes to the $\mathbb{F}_2$-classes?

Discontinuous Homomorphisms

One way to show that the answer is negative would be to show that one can force over V to produce a $\subseteq$-decreasing ω-sequence

$$\langle M_0 : n \in \omega \rangle$$

of models of ZFC and a discontinuous homomomorphism

$$h \colon \mathbb{R}^{M_0} \to \mathbb{R}^{M_0}$$

(containing any given countable partial homomorphism) such that

- for each $n \in \omega$, $\mathbb{R}^{M_n} \not\subseteq M_{n+1}$
- for each $n \in \omega + 1$, $h \restriction \mathbb{R}^n \in M_n$

Recall that the analogous statement is impossible for a Hamel basis.

Regularity properties

Sometimes a balanced virtual condition remains a balanced virtual condition after forcing with a certain partial order.

For instance, balanced virtual conditions for $\mathcal{P}(\omega)/\mathrm{Fin}$ are given by ultrafilters, some of which (e.g., Ramsey ultrafilters) are preserved by some forcings (e.g., Sacks forcing).

If a Suslin forcing P has balanced virtual conditions below any conditon which remain balanced after random forcing, then the P-extension of W satisfies the statement that every set of reals is Lebesgue measurable.

Dominating Family Example

Consider for instance the partial order P consisting of countable subsets of ω^ω, with $p \leq q$ if $q \subseteq p$ and no member of $p \setminus q$ is dominated by any member of q.

Balanced virtual conditions are given by pairs of the form $(\mathrm{Col}(\omega, \omega^\omega), \tau)$, where τ is a name for the generic enumeration of some dominating family $F \subseteq \omega^\omega$.

Since ground model dominating families are still dominating after random forcing, every set of reals is Lebesgue measurable in the corresponding extension.

However, every forcing extension of W by a nontrivial balanced Suslin order contains a set without the Baire property.

Questions

Every forcing extension of W by a nontrivial balanced Suslin order contains a set without the Baire property.

What about other other regularity properties? For instance, the Ramsey property?

Does the existence of a set of reals without the Baire property imply the existence of a set without the Ramsey property?

Social Welfare

The following questions are from "Social Welfare Relations and Irregular Sets" and "On non-construcive nature of ethical social welfare orders" by Ram Sewak Dubey and Giorgio Laguzzi.

A social welfare order (SWO) is a total, reflexive transitive binary relation $\preceq$ on a set X of the form Y^ω, for some $Y \subseteq \mathbb{R}$.

For $x, y \in X$, $x \geq y$ is total domination; $x > y$ is strict total domination.

$x \sim y$ means $x \preceq y$ and $y \preceq x$.

SWO properties (I)

An SWO satifies

- AN (for Anonymity) if $\forall x, y \in X$, if there exist $i, j \in \omega$ such that
$$y(j) = x(i),\ x(j) = y(i)$$
and $y(k) = x(k)$ for all $k \in \omega \setminus \{i, j\}$, then $x \sim y$.

- SP (Strong Pareto) if $\forall x, y \in X$, if $x \geq y$ and
$$\exists i \in \omega\, x(i) > y(i),$$
then $x \succ y$.

- IP (Infinite Pareto) if $\forall x, y \in X$, if $x \geq y$ and
$$\exists^{\infty} i \in \omega\, x(i) > y(i),$$
then $x \succ y$.

Questions (I)

SPA is the statement that there exists an SWO on 2^ω satisfying AN and SP.

IPA is the statement that there exists an SWO on 2^ω satisfying AN and IP.

Neither of SPA and IPA holds if every set of reals has the Baire property.

The existence of a nonprincipal ultrafilter on ω implies SPA, which implies IPA.

Are any (or all) of these implications reversible?

Strong Equity

An SWO satifies

- SE (Strong Equity) : $\forall x, y \in X$, if there exist $i, j \in \omega$ such that
$$y(j) > x(j) > x(i) > y(i)$$
while $y(k) = x(k)$ for all $k \in \omega \setminus \{i, j\}$, then $x \succ y$.

Questions (II)

SEA (Strong Equity and Anonimity) is the statement there exists a SWO (on Y^ω for some Y of size at least 4) satisfying SE and AN.

Does SEA imply that there is a non-principal ultrafilter on ω?

Does it imply or follow from the existence of a non-Lebesgue measurable set?

Parity functions

From "Choice and the Hat Game" by Geschke, Lubarsky and Rahn:

A partity function is a function $p\colon 2^\omega \to 2$ such that $p(s) = 1 - p(t)$ whenever s and t disagree in exactly one place.

Does ZF+"there is a parity function" imply the existence of a free ultrafilter on ω or of an $\mathbb{E}_0$-transversal? (No)

Coloring the Hamming graph (I)

The Hamming graph on k^ω (for $k \in \omega$) connects two vertices if they differ at exactly one point.

Let $\chi(k)$ be the coloring number of the Hamming graph on k^ω.

Trivially, $\chi(k) \geq k$.

If there is an $\mathbb{E}_0$-selector, then $\chi(k) = k$ for all k.

If every set of reals has the Baire property, then $\chi(k)$ is uncountable, for all $k \geq 2$.

If $\chi(k)$ is finite, then $\chi(k^n) \leq \chi(k)^n$.

Coloring the Hamming graph (II)

A question left over from Rosendal's, "Continuity of Universally Measurable Homomorphisms":

If $\chi(k)$ is finite for all k (equivalently, some $k \geq 2$), does it follow that

$$\chi(k) = k$$

for infinitely many k (equivalently, some $k \geq 2$)?

One could also ask (for any k) : if $\chi(k)$ is finite, does it follows that $\chi(k) = k$?

Chameleons

From "Flutters and Chameleons", by Bowler, Delhommé, Di Prisco and Mathias.

A k-chameleon is a map from $\mathcal{P}(\omega)$ to $\mathbb{Z}_k$ such that adding one point to the input adds one to the output.

A $\mathbb{Z}$-chameleon is the same thing, mapping into $\mathbb{Z}$.

Does the existence of a 3-chameleon imply the existence of a 5-chameleon?

Does the existence of a $\mathbb{Z}$-chameleon imply the existence of an $\mathbb{E}_0$-selector?

Blocks

The 2-block principle says that for any $c \colon [\omega]^{\omega} \to 2$ there is a product of sets of size 2 on which c is constant.

Di Prisco and Todorcevic showed that the 2-block principle holds in $W[U]$.

It implies that there is no $\mathbb{E}_0$-selector.

Does it imply that there is no $\mathbb{Z}$-chameleon ?

Flectors

A flector on ω chooses between each set and its complement.

$A \sim_k B$ means that $A \mathbb{E}_0 B$ and $A \setminus B$ and $B \setminus A$ have the same size (in the $k = 0$ case) or the same size mod k (in the $k > 0$ case).

What happens in balanced forcing extensions adding a $\mathbb{E}_0$-invariant (or $\sim_k$-invariant) flector?

Flitters

A flitter is a collection F of subsets of ω such that whenever $A, B \in F$, $A \triangle B$ is coinfinite.

A flitter which is also a flector is an ultraflitter (maximal flitter) - these are also the same as being a $\mathbb{E}_0$-closed flector.

A flitter is disjointed if its members are pairwise almost equal or almost disjoint.

A $\mathbb{Z}$-selector is a selector for the family of countably infinite disjointed flitters.

Does the existence of a $\mathbb{Z}$-selector imply the existence of a $\mathbb{Z}$-chameleon ?

MAD families

What happens when you generically add a MAD family by countable approximations?

Is this forcing weakly balanced?

Recall that balanced forcing cannot add MAD families, although weakly balanced forcing can certain kinds of MAD families (and in these models every set of reals is Lebesgue measurable).

More generally : develop the theory of weakly balanced partial orders.

Independent families

An independent family is a set $F \subseteq [\omega]^\omega$ such that, for any two disjoint nonempty sets $A, B \subseteq F$,

$$\bigcup A \setminus \bigcup B$$

is infinite.

Is there a (weakly) balanced forcing extension adding a maximal independent family?

Densely maximal independent families

V. Fischer : An independent family F is said to be densely maximal if, for any

$$X \in [\omega]^\omega \setminus F$$

and any two disjoint nonempty sets $A, B \subseteq F$, there exist disjoint $A', B' \subseteq F$ such that $A \subseteq A'$, $B \subseteq B'$ and one of

$$(\bigcup A' \setminus \bigcup B') \cap X$$

and

$$(\bigcup A' \setminus \bigcup B') \setminus X$$

is finite.

Models of arithmetic

(Placid) balanced forcing can add a countably saturated model on $\mathcal{P}(\omega)$ of any first-order theory having infinite models.

Is there a first-order theory for which the existence of a saturated model contradicts the Axiom of Determinacy?

For instance, Peano Arithmetic?

Banach Limits

A Banach limit is a finitely additive probability measuring on $\mathcal{P}(\omega)$ giving finite sets measure 0.

A well known question asks whether the existence of a Banach limit implies the existence of a non-Lebesgue-measurable set of reals.

There are many ways to generically add Banach limits over Solovay models.

Some of these are balanced, and some add nonprincipal ultrafilters on ω.

I don't know if any of them fail to add an ultrafilter or fail to add an $\mathbb{E}_0$-selector.

Blackwell strategies

A strategy for player *I* (*II*) in a Blackwell game on ω consists of a function assigning to each finite sequence from ω of even (odd) length a probability measure on ω.

A pair of strategies, σ for *I* and τ for *II* gives rise to a measure $\mu_{\sigma,\tau}$ on ω^ω.

The corresponding outer measure is $\mu^+_{\sigma,\tau}$ and the corresponding inner measure is $\mu^-_{\sigma,\tau}$.

Blackwell games

In the Blackwell game on ω with payoff $A \subseteq \omega^\omega$, we can think of the object of a strategy σ for player I (respectively, τ for II) to be to make $\mu_{\sigma,\tau}(A)$ as large (small) as possible.

Given a strategy σ for player I, $\mathrm{val}^I_\sigma(A)$ is the infimum of the values $\mu^-_{\sigma,\tau}(A)$ where τ ranges over all strategies for II.

Given a strategy τ for player I, $\mathrm{val}^{II}_\tau(A)$ is the supremum of the values $\mu^+_{\sigma,\tau}(A)$ where σ ranges over all strategies for I.

Finally, $\mathrm{val}_I(A)$ is the supremum of $\mathrm{val}^I_\tau(A)$ over all strategies for player I and $\mathrm{val}_{II}(A)$ is the infinimum of $\mathrm{val}^{II}_\tau(A)$ over all strategies τ for player I.

The set A is said to be Blackwell-determined if $\mathrm{val}_I(A) = \mathrm{val}_{II}(A)$.

Blackwell determinacy

Blackwell determinacy for ω (BlAD) is the statement that all $A \subseteq \omega^\omega$ are Blackwell determined.

BlAD implies that every subset of ω^ω is Lebesgue measurable.

Martin showed that BlAD follows from AD, but the reverse implication is open.

Does Blackwell determinacy imply AD?

Does it imply that all sets of reals have the Baire property?

Does it hold in the dominating family model above (if κ is a limit of Woodin cardinals)?