Geometric Set Theory II

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Pins

Let E be an analytic equivalence relation on a Polish space X. An *E-pin* is a pair (Q, τ) such that

- Q is a partial order
- τ is a Q-name for an element of X and,
- for all generic (G, H) for $Q \times Q$, $V[G, H] \models \tau_G E \tau_H$.

An E-pin represents the same E-equivalence class in all extensions by Q, even though the class may have no members in the ground model.

Note that for any two V-generic filters $G_0, G_1 \subseteq Q$, there exists in some forcing extension an $H \subseteq Q$ such that (G_0, H) and (G_1, H) are both V-generic for $Q \times Q$.

Equivalence of pins

Two pins (Q, τ), (P, σ) are E-equivalent if

$$V[G, H] \models \tau_G E \sigma_H$$

holds for all generic

$$(G, H) \subseteq Q \times P.$$

The corresponding equivalence classes are the *virtual equivalence classes* of E.

κ and λ

It is sometimes possible to prove nonreducibility results between analytic equivalence relations via the association of cardinal invariants.

For any analytic equivalence relation E, we let:

- $\kappa(E)$, the least cardinal κ such that every E-pin is equivalent to one of the form (Q, τ), where $|Q| < \kappa$ (set to ∞ if there is no such κ and $\aleph_1$ if E is pinned)
- $\lambda(E)$, the cardinality of the set of equivalence classes of E-pins (if it exists, otherwise ∞)

Note that $\lambda(E) \leq 2^{\kappa(E)}$.

If E is pinned, then $\kappa(E) = \aleph_1$ and $\lambda(E) \leq 2^{\aleph_0}$.

If E is the product of $\langle E_n : n \in \omega \rangle$, then

$$\kappa(E) \leq (\prod_n \kappa(E_n))^+$$

and

$$\lambda(E) = \prod_n \lambda(E_n).$$

If E is the increasing union of $\{E_n : n \in \omega\}$ then

$$\lambda(E) = \sup_n \lambda(E_n)$$

and

$$\kappa(E) = \sup_n \kappa(E_n).$$

λ and the Friedman-Stanley jump

If E^+ is the Friedman-Stanley jump of E (and E has infintely many classes), then

$$\lambda(E^+) = 2^{\lambda(E)}.$$

Comparing equivalence relations

If $E \leq_{\mathrm{a}} F$ then $\kappa(E) \leq \kappa(F)$ and $\lambda(E) \leq \lambda(F)$.

This shows that:

- If $E \leq_{\mathrm{a}} F$ and F is pinned, then so is E;
- $\mathbb{E}_{\omega_1} \not\leq_{\mathrm{a}} \mathbb{F}_2$;
- for any E with infinitely many classes, $E^+ \not\leq_{\mathrm{a}} E$, where E^+ is the Friedman-Stanley jump of E.

Bounds for Borel relations

Work of Jacques Stern from 1984 shows that if E is a Borel equivalence relation of Borel rank α, then

$$\kappa(E) < \beth_{\alpha}^{+}$$

for every Borel equivalence relation E.

In particular,

$$\kappa(E) < \beth_{\omega_1}$$

for every Borel equivalence relation E.

The rest of this lecture is on Chapters 3 and 4.

In Chapter 3 we study non-mutually generic extensions with the property that $V[H_1] \cap V[H_2] = V$.

In Chapter 4 we study $\subseteq$-descending ω-sequences of models of ZFC.

We relate these situations to the study of virtual equivalence classes, but also prepare for forcing arguments from the second half of the book.

Chapter 3 : Turbulence and Placidity

Disjoint extensions

Solovay showed that whenever H_1 and H_2 are mutually generic filters,

$$V[H_1] \cap V[H_2] = V.$$

Chapter 3 presents a method for finding non-mutually generic filters H_1 and H_2, existing in a common forcing extension, such that

$$V[H_1] \cap V[H_2] = V.$$

P_X and $\dot{x}$

Given a Polish space X, we let P_X be the partial order of nonempty open subsets of X, ordered by inclusion.

This is Cohen forcing for X.

Since P_X is c.c.c., all pins of the form (P_X, τ) are trivial.

Let $\dot{x}$ be the canonical name for the corresponding generic element of X.

We consider the situation where we have Polish spaces X, Y and Z such that P_X adds non-mutually generic filters

$$H \subseteq P_Y$$

and

$$K \subseteq P_Z,$$

with the property that

$$V[H] \cap V[K] = V.$$

Adding generics

First, observe that if

$$f : X \to Y$$

is a continuous open function, then P_X forces that $f(\dot{x}_G)$ will be a P_Y-generic element of Y.

To see this, note that if $O \subseteq X$ is open and $D \subseteq P_Y$ is dense open, then $f[O]$ contains some $U \in D$, and

$$f^{-1}[U] \cap O$$

is a condition in P_X below O forcing that U is in the filter generated by $f(\dot{x}_G)$.

A naive attempt

Suppose that $f\colon X \to Y$ and $g\colon X \to Z$ are continuous open functions.

To carry out Solovay's argument for

$$V[G] \cap V[H] = V,$$

it would suffice to have that whenever $O \subseteq X$ is nonempty and open, and W_0 and W_1 are disjoint open subsets of $f[O]$,

$$g[f^{-1}[W_0]] \cap g[f^{-1}[W_1]]$$

is nonempty.

In general this is too much to hope for, however.

Walks

Suppose then that $f\colon X \to Y$ and $g\colon X \to Z$ are continuous and open.

An (f, g)-walk is a sequence

$$\langle x_i : i \leq k \rangle \in X^{<\omega}$$

such that for each $i < k$, either

$$f(x_i) = f(x_{i+1})$$

or

$$g(x_i) = g(x_{i+1}).$$

Independence (I)

We say that f and g are independent if for every nonempty open $O \subseteq X$ there exists a nonempty open

$$U \subseteq f[O]$$

such that for all nonempty open $W_0, W_1 \subseteq U$ there is an (f, g)-walk consisting of points in O which starts in $f^{-1}[W_0]$ and ends in $f^{-1}[W_1]$.

This relation is symmetric in f and g.

Independence (II)

An alternate characterization of independence (due to Andy Zucker) is : f and g are independent if, whenever

$$h \colon Y \to W$$

and

$$k \colon Z \to W$$

(for some Polish space W) are Borel, and

$$\{x \in X : h(f(x)) = k(g(x))\}$$

is nonmeager, there is a point $w \in W$ such that

$$\{x \in X : h(f(x)) = k(g(x)) = w\}$$

is nonmeager.

Theorem

Continuous open functions

$$f : X \to Y$$

and

$$g : X \to Z$$

are independent if and only if P_X forces that

$$V[f(\dot{x}_G)] \cap V[g(\dot{x}_G)] = V.$$

Classification by countable structures

For any countable relational language $\mathcal{L}$, the set of $\mathcal{L}$-structures with domain ω is a Polish space.

Let $E_{\mathcal{L}}$ be the isomorphism relation on this space.

An equivalence relation is said to be classifiable by countable structures if it is Borel reducible to $E_{\mathcal{L}}$, for some $\mathcal{L}$.

A Borel equivalence relation is classifiable by countable structures if and only if it is Borel reducible to a countable iterate of the Friedman-Stanley jump on equality (taking disjoint unions at limit stages).

Walks

Let Γ be a group acting continuously on a Polish space Y.

Given $U \subseteq \Gamma$ and $O \subseteq Y$, a (U, O)-walk is a finite seqence

$$\langle y_i : i \leq k \rangle$$

from O such that, for each $i < k$,

$$y_{i+1} = \gamma_i \cdot y_i$$

for some $\gamma_i \in U$.

The (U, O)-orbit of $y \in O$ is the set of all terminal points of (U, O)-walks starting at y.

Turbulence

Let Γ be a group acting continuously on a Polish space Y.

The action is turbulent at $y \in Y$ if for all open U, O with

$$1 \in U$$

and

$$y \in O,$$

the (U, O)-orbit of y is somewhere dense.

The action is generically turbulent if its orbits are meager and dense, and the action is turbulent at comeagerly many y.

Examples (I)

Let c_0 be the subgroup of $\mathbb{R}^\omega$ consisting of sequences converging to 0, under pointwise addition.

Let $\mathbb{R}^\omega$ have the topology induced by the sup norm.

The action of c_0 on $\mathbb{R}^\omega$ by pointwise addition is (everywhere) turbulent.

Examples (II)

Let I_2 be the set of $x \subseteq \omega$ for which the sum

$$\sum\{\frac{1}{n+1} : n \in x\}$$

is finite.

Letting $\triangle$ be the symmetric differerence operator, $(I_2, \triangle)$ is a Polish group.

The corresponding action $x \cdot y = x \triangle y$ on $\mathcal{P}(\omega)$ is turbulent.

The same holds for asymptotic-density-0 ideal.

(Half of) Hjorth's theorem

Turbulent actions are not classifiable by countable structures.

Forcing with a turbulent action (I)

Let Γ be a Polish group acting continuously on a Polish space Y, and let X be $\Gamma \times Y$.

Let $f \colon X \to Y$ be the second-coordinate projection, and let $g \colon X \to Y$ be given by the group action, i.e., $g(\gamma, y) = \gamma \cdot y$.

Then f and g are continuous and open.

A walk $\langle y_i : i \leq k \rangle$ in our second sense (via $\langle \gamma_i : i < k \rangle$) induces a walk

$$(\gamma_0, y_0), (1, y_1), (\gamma_1, y_1), \ldots, (1, y_k)$$

in our first sense.

Forcing with a turbulent action (II)

If we assume in addition that all obits of the action are dense and meager, then f and g are independent if and only if the action is generically turbulent.

In this case, letting (γ_G, y_G) be V-generic for P_X,

$$V[y_G] \cap V[\gamma_G \cdot y_G] = V.$$

Placidity

Let E be an analytic equivalence relation on a Polish space X.

We say that E is placid if, whenever $V[H_0]$ and $V[H_1]$ are separately generic extensions of V (inside some ambient generic extension) such that

$$V[H_0] \cap V[H_1] = V$$

and $x_0 \in V[H_0]$ and $x_1 \in V[H_1]$ are E-related points in the space X, then they are E-related to some point in V .

Placid implies pinned.

Examples

- Countable Borel equivalence relations are placid.
- Let X be the set of functions from $2^{<\omega}$ to 2^{ω} and let J be the ideal on $2^{<\omega}$ generated by the compatible sets.

  Then mod-J-equivalence is a placid equivalence relation on X.
- $\mathbb{E}_1$ is placid.
- $\mathbb{E}_2$ (summability equivalence) is pinned but not placid.

Virtual placidity

We say that E is virtually placid if, whenever $V[H_0]$ and $V[H_1]$ are separately generic extensions of V (inside some ambient generic extension) such that

$$V[H_0] \cap V[H_1] = V$$

and $\langle Q_0, \tau_0 \rangle \in V[H_0]$ and $\langle Q_1, \tau_1 \rangle \in V[H_1]$ are equivalent E-pins, then they are E-related to some E-pin in V .

Virtual placidity (II)

Equivalently, E is virtually placid if and only if, for any separately generic extensions $V[H_0]$, $V[H_1]$ such that

$$V[H_0] \cap V[H_1] = V$$

and E-related points $x_0 \in V[H_0]$ and $x_1 \in V[H_1]$, x_0 and x_1 are realizations of some virtual E-class in V.

Placid implies virtually placid.

Proposition

An analytic equivalence relation E on a Polish space X is placid if and only if it is pinned and virtually placid.

$\mathbb{F}_2$ is virtually placid but not placid, since it is not pinned.

Ergodicity

Suppose that

- Γ is a Polish group acting continuously in a generically turbulent way on a Polish space X, inducing an equivalence relation E,
- F is a virtually placid equivalence relation on a Polish space Y and
- h is a Borel function from X to Y sending E-equivalent points to F-equivalent points.

Then there is a comeager set $B \subseteq X$ such that $h[B]$ is contained in a single F-class.

Proof sketch

Force with $P_{\Gamma \times X}$, getting generic (γ, x) such that

$$V[x] \cap V[\gamma \cdot x] = V.$$

Since $h(x)Fh(\gamma \cdot x)$, $h(x)$ and $h(\gamma \cdot x)$ are in a virtual equivalence class in V.

Since $P_{\Gamma \times X}$ carries no nontrivial pins, there is a $y \in V$ which is F-equivalent to x and $\gamma \cdot x$.

$h^{-1}[[y]_F]$ is as desired (by the genericity of x it can't be meager).

Closure

The class of virtually placid equivalence relations is closed under:

- Borel almost reduction;
- countable products;
- countable increasing unions;
- The Friedman-Stanley jump.

Corollary

Since every equivalence relation classifiable by countable structures is Borel reducible to a countable iterate of equality via the Friedman-Stanley jump, it follows that every equivalence relation classifiable by countable structures is virtually placid.

This gives another proof of Hjorth's theorem.

Chapter 4 : Nested sequences of models of ZFC

Coherent sequences

We say that a $\subseteq$-decreasing sequence

$$\langle M_n : n \in \omega \rangle$$

of transitive models of ZFC is coherent if, for every ordinal $\lambda \in M_0$ and every natural number n, the sequence

$$\langle M_m \cap V_\lambda : m \in \omega \setminus n \rangle$$

belongs to M_n.

Given a coherent sequence of models $\langle M_n : n \in \omega \rangle$, a sequence $\langle v_n : n \in \omega \rangle$ is coherent if for every $n \in \omega$,

$$\langle v_m : m \in \omega \setminus n \rangle \in M_n.$$

The trivial coherent sequence: each M_n is V.

Generic coherent sequences

A coherent sequence $\langle M_n : n \in \omega \rangle$ is said to be M-generic if M is a model of ZFC contained in each M_n, and M_0 is a generic extension of M.

This implies that all models M_n are generic extensions of M and that M_n is a generic extension of M_m whenever $n \leq m$.

A natural way to produce V-generic coherent sequences (with $M = V$) is to force with a product $\prod_{n \in \omega} P_n$, and let M_n be

$$V[\langle G_m : n \leq m < \omega \rangle].$$

Projections

Coherent sequences of models are most often formed as generic extensions of the constant sequence $\langle M_n : n \in \omega \rangle$ using the following definition and theorem.

A projection from a poset Q to a poset P is a pair of order-preserving functions $\pi\colon Q \to P$ and $\xi\colon P \to Q$ such that

- $\pi \circ \xi$ is the identity on P;
- whenever $\pi(q) \leq p$ then $q \leq \xi(p)$;
- whenever $p \leq \pi(q)$ then there is a $q' \leq q$ such that $\pi(q') \leq p$.

The π-image of a generic filter on Q is then a generic filter on P.

Examples

(1) For any posets P and R, the maps

$$\pi(p, r) = p$$

and

$$\xi(p) = (p, 1_R)$$

form a projection from $P \times R$ to P.

(2) When $f : X \to Y$ is continuous and open,

$$\pi(O) = f[O]$$

and

$$\xi[U] = f^{-1}[U]$$

form a projection from P_X to P_Y.

Coherent sequences of posets

If $\bar{M} = \langle M_n : n \in \omega \rangle$ is a coherent sequence of models of ZFC, an $\bar{M}$-coherent sequence of posets is a sequence

$$\langle P_n, \pi_{nm}, \xi_{nm} : n \leq m \in \omega \rangle$$

such that

- For all $n \leq m$ the maps $\pi_{nm} : P_n \to P_m$ and $\xi_{mn} : P_m \to P_n$ form a projection of P_n to P_m;
- for all $k \leq n \leq m$, $\pi_{km} = \pi_{nm} \circ \pi_{kn}$ and $\xi_{mk} = \xi_{nk} \circ \xi_{mn}$;
- for each n, the functions $\pi_{nn} = \xi_{nn} = \mathrm{id}_{P_n}$;
- for every number $k \in \omega$, the sequence

  $$\langle P_n, \pi_{nm}, \xi_{nm} : k \leq n \leq m \in \omega \rangle$$

  belongs to the model M_k.

In particular, every commutative sequence of projections

Coherent posets theorem

Let $\bar{M} = \langle M_n : n \in \omega \rangle$ be a coherent sequence of models of ZFC and

$$\langle P_n, \pi_{nm}, \xi_{nm} : n \leq m \in \omega \rangle$$

be a $\bar{M}$-coherent sequence of posets. Let $G \subseteq P_0$ be a filter generic over M_0, and for each $n \in \omega$ let

$$G_n = \xi_{n0}^{-1}[G].$$

Then the sequence

$$\langle M_n[G_n] : n \in \omega \rangle$$

is a coherent sequence of models of ZFC.

Theorem

If $\langle M_n : n \in \omega \rangle$ is a coherent V-generic sequence of models of ZFC, then

$$M_\omega = \bigcap_n M_n$$

is a class in all models M_n, and it is a model of ZF + DC.

Example

Let $\mathbb{G}$ be the graph on 2^ω which connects two points if they disagree at an odd (finite) number of points.

This graph has uncountable Borel chromatic number, and chromatic number 2 in the presence of an $\mathbb{E}_0$-selector (e.g., in ZFC).

Let $c\colon \omega_1 \times \omega$ be a Cohen-generic map, and for each $n \in \omega$ let

$$c_n = c \restriction \omega_1 \times (\omega \setminus n).$$

Let $M_n = V[c_n]$. In the model

$$M_\omega = \bigcap_n M_n$$

the chromatic number of $\mathbb{G}$ is greater than 2, so the Axiom of Choice fails.

Choice-coherent sequences

Let $\langle M_n : n \in \omega \rangle$ be an inclusion decreasing sequence of transitive models of ZFC.

We say that the sequence is choice-coherent if it is coherent and for every ordinal $\lambda \in M_0$ there is a well-ordering $\leq_\lambda$ of

$$V_\lambda \cap M_0$$

such that its intersection with each model M_n belongs to M_n.

Theorem

If $\bar{M} = \langle M_n : n \in \omega \rangle$ is a coherent V-generic sequence then $\bar{M}$ is choice-coherent if and only if

$$M_\omega = \bigcap_{n\in\omega} M_n$$

is a model of ZFC.

Diagonal distributivity

Let $\langle P_n, \pi_{nm}, \xi_{mn} : n \leq m \in \omega \rangle$ be a coherent sequence of posets.

The diagonal game is the following inifnite game between Players I and II.

In round n Player I plays $p_n \in P_n$ and Player II responds with $q_n \leq p_n$. Additionally, $p_{n+1} \leq \pi_{nn+1}(q_n)$.

In the end, Player II wins if there is a condition $r \in P_0$ such that $\pi_{0n}(r) \leq q_n$ holds for all $n \in \omega$.

The sequence is diagonally distributive if Player I has no winning strategy in the diagonal game.

Example

Suppose that $\langle Q_m : m \in \omega \rangle$ are arbitrary posets, and let

$$P_n = \prod_{m \geq n} Q_m$$

be the countable support product with the natural projection maps from P_n to P_m for $n \leq m$.

Player II has a simple winning strategy in the diagonal game in this setup: set $q_n = p_n$.

Theorem 1

Let $\langle M_n : n \in \omega \rangle$ be a choice-coherent sequence of models of ZFC. Let

$$\langle P_n, \pi_{nm}, \xi_{mn} : n \leq m \in \omega \rangle$$

be a coherent sequence of posets which is diagonally distributive in M_0. Let $G \subseteq P_0$ be a filter generic over M_0, and let for each $n \in \omega$ let $G_n = \xi_{n0}^{-1}$. Then the sequence

$$\langle M_n[G_n] : n \in \omega \rangle$$

is choice-coherent, and the models $\bigcap_n M_n$ and $\bigcap_n M_n[G_n]$ contain the same ω-sequences of ordinals

Theorem 2

Let $\langle M_n : n \in \omega \rangle$ be a choice-coherent sequence generic over V and let E be an orbit equivalence relation with code in

$$M_\omega = \bigcap_n M_n.$$

If a virtual E-class is represented in each M_n, then it is represented in M_ω.

The same conclusion holds for analytic equivalence relations that are almost-reducible to an orbit equivalence relation.

$\mathbb{E}_1$ (I)

The conclusion of the previous theorem fails for $\mathbb{E}_1$ on $(2^\omega)^\omega$.

To see this, let Q be the full support product of ω-many copies of P_{2^ω}, and for each $n \in \omega$ let Q_n be the product of the copies of P_{2^ω} indexed by natural numbers $\geq n$.

The posets Q_n for $n \in \omega$ form a coherent sequence.

Let $G \subseteq Q$ be a generic filter, and for each $n \in \omega$ let $G_n \subseteq Q_n$ be the restriction of Q to conditions in Q_n.

$\mathbb{E}_1$ (II)

Then $\langle V[G_n] : n \in \omega\rangle$ is a choice-coherent sequence of models, and all reals in $\bigcap_n V[G_n]$ are in V.

In $V[G_n]$, let $x_n \in X$ be the sequence defined by letting $x_n(m)$ be the zero sequence if $m < n$ and the mth generic real otherwise.

The points x_n all represent the same $\mathbb{E}_1$-class, which is not represented in V and therefore not represented in $\bigcap_n V[G_n]$

Corollary

Theorem (Kechris-Louveau). $\mathbb{E}_1$ is not Borel reducible to any orbit equivalence relation.

Necessity of virtual classes

$\mathbb{F}_2$ (which is an orbit equivalence relation) shows that is necessary to consider virtual E-classes as opposed to just E-classes in the statement of Theorem 2.

To see this, start with the trivial coherent sequence (where each model is V) and let P_n be the countable support product of ω-many copies of the poset $\mathrm{Col}(\omega, 2^\omega)$.

This induces a choice-coherent sequence of models $\langle V[G_n] : n \in \omega\rangle$ such that the model $\bigcap_n V[G_n]$ contains only ground model ω-sequences of ordinals.

Each model $V[G_n]$ contains a surjection from ω to $2^\omega \cap V$, and all of these surjection are $\mathbb{F}_2$-related.

There is no $\mathbb{F}_2$- equivalent of them in the intersection model, which has the same reals as V.

However, the reals of V induce a virtual $\mathbb{F}_2$-class related to these enumerations which is in V, so also in the intersection model.