Geometric Set Theory I

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August 17, 2022

These lectures are based on joint work with Jindřich Zapletal, appearing in the book Geometric Set Theory.

The first half of the book studies equivalence relations on Polish spaces; the second half presents a method for producing independence results in Choiceless set theory.

Our first two lectures will be on the first half. The others will be on the second.

Our first lecture is on Chapter 2.

Geometry is about intersections of points, curves and surfaces. Geometric set theory is about intersections of models of set theory.

– Bjørn Kjos-Hanssen

Part I : Virtual equivalence classes

Polish spaces and analytic sets

A Polish space is a separable, completely metrizable topological space (e.g., $\mathbb{R}^n$, 2^ω, ω^ω).

A subset of a Polish space X is *analytic* if it is a continuous image of ω^ω (equivalently, if it is definable by a formula of the form

$$\exists x \in X \phi(x, y),$$

where the quantifiers in ϕ range over ω and ϕ is allowed an arbitrary element of X as a parameter).

A set is Borel if it is analytic and co-analytic (assuming $CC_{\mathbb{R}}$, the statement that every countable family of nonempty sets of reals has a Choice function).

Borel equivalence relations (I)

- $=_X$, the identity relation on X.
- $\mathbb{E}_0$ is the *Vitali equivalence* on 2^ω, connecting $x, y \in 2^\omega$ if they differ at only finite set of entries.
- $\mathbb{E}_1$ is the equivalence relation on $(2^\omega)^\omega$ connecting x, y if they differ at only finite number of entries.

Borel equivalence relations (II)

- $\mathbb{E}_2$ is the relation on 2^ω connecting x, y if the sum

$$\sum\{\frac{1}{n+1} : x(n) \neq y(n)\}$$

is finite.

- $\mathbb{F}_2$ is the equivalence relation on $(2^\omega)^\omega$ connecting x, y if

$$\mathrm{rng}(x) = \mathrm{rng}(y).$$

- $\mathbb{E}_{K_\sigma}$ is the set of pairs (f, g) from ω^ω which are bounded by the identify function and have bounded difference.

Countable equivalence relations

An equivalence relation E is said to be countable if every E-class is countable.

$=_X$ and $\mathbb{E}_0$ are countable

$\mathbb{E}_1$, $\mathbb{E}_2$, $\mathbb{F}_2$ and $\mathbb{E}_{K_\sigma}$ are not.

Polish group actions

A Polish group consists of a Polish topology on a group for which the group operation and the inverse operation are continuous.

If G is a Polish group and X is a Polish space, a Polish G-action is a continuous map $a\colon G \times X \to X$ for which $a(e, x) = x$ and

$$a(g, a(h, x)) = a(gh, x).$$

Often we write $g \cdot x$ for $a(g, x)$.

Such an action induces an analytic equivalence relation E^G_X on X, where $x E^G_X y$ if there is a $g \in G$ such that $g \cdot x = y$.

Examples

- S_∞ (the group of permutations of ω) acts on a Polish space of the form Y^ω by

$$g \cdot x(n) = x(g(n)).$$

- A group G acts on a Polish space of the form Y^G by

$$g \cdot x(h) = x(g^{-1}h).$$

- $(\mathbb{Z}, +)$ acts on $2^{\mathbb{Z}}$ by $n \cdot x(m) = x(m - n)$.

Orbit equivalence relations

An orbit equivalence relation is a relation induced by a Polish group action.

Feldman-Moore: Countable Borel equivalence relations are orbit relations (via a countable group).

$\mathbb{F}_2$ is an orbit equivalence relation (via permutations of ω).

$\mathbb{E}_2$ and $\mathbb{E}_{K_\sigma}$ are also orbit relations.

$\mathbb{E}_1$ is not.

Reductions

Given equivalence relations E and F on a Polish spaces X and Y, we say that E is Borel-reducible to F (and write $E \leq_{\mathrm{B}} F$) if there is a Borel function $f : X \to Y$ such that

$$xEy \Leftrightarrow f(x)Ff(y)$$

for all $x, y \in X$.

We say that E is almost-reducible to F (and write $E \leq_{\mathrm{a}} F$) if there exist a Borel function $f : X \to Y$ and a countable $C \subseteq X$ such that

$$xEy \Leftrightarrow f(x)Ff(y)$$

for all x, y in

$$X \setminus \bigcup \{[x]_E : x \in C\}.$$

Smooth and essentially countable

An equivalence relation E is said to be smooth if $E \leq_{\mathrm{B}} =_X$ (for any Polish space X).

E is said to be essentially countable if $E \leq_{\mathrm{B}} F$ for some countable Borel equivalence relation F.

Dichotomies

Silver : If E is coanalytic and has uncountably many classes, then

$$=_X \leq_{\mathrm{B}} E.$$

Harrington-Kechris-Louveau : If E is Borel and $E \not\leq_{\mathrm{B}} =_X$, then

$$\mathbb{E}_0 \leq_{\mathrm{B}} E.$$

Structural relations

Let S be a set of structures on ω, in some relational language $\mathcal{L}$.

The isomorphism relation on S is analytic.

Structural relations with wellfoundedness

If R is a binary relation in $\mathcal{L}$, the following relation is also analytic : xEy if either x and y are isomorphic, or R^x and R^y are both illfounded.

1. $\mathbb{E}_{\omega_1}$ is the corresponding relation for linear orders on ω.
2. $\mathbb{HC}$ is the corresponding relation for extensional binary relations on ω.

Reinterpretations

When passing from a model M to an extension $M[G]$, every analytic subset of every Polish space in M has a natural reinterpretation in $M[G]$.

The reinterpretation of an equivalence relation is an equivalence relation.

Mutually generic extensions

Given partial orders P and Q, filters $G \subseteq P$ and $H \subseteq Q$ are said to be *mutually V-generic* if (G, H) is V-generic for the partial order $P \times Q$.

This is equivalent to the assertion that H is $V[G]$-generic.

A classical theorem of Solovay says that if G and H are mutually generic, then

$$V[G] \cap V[H] = V.$$

Proof

Suppose that $(p, q) \in P \times Q$, τ is a P-name for a set of ordinals and σ is a Q-name for a set of ordinals.

If p decides the statement $\check{\alpha} \in \tau$, for each ordinal α, then p forces that $\tau_G \in V$.

If not, there exist $\alpha \in \mathrm{Ord}$ and $p_1, p_2 \leq p$ such that $p_1 \Vdash \check{\alpha} \in \tau$ and $p_2 \Vdash \check{\alpha} \notin \tau$.

Let $q' \leq q$ decide the statement $\check{\alpha} \in \sigma$.

If $q' \Vdash \check{\alpha} \in \sigma$, then $(p_2, q') \leq (p, q)$ forces that $\tau_G \neq \sigma_H$.

If $q' \Vdash \check{\alpha} \notin \sigma$, then $(p_1, q') \leq (p, q)$ forces that $\tau_G \neq \sigma_H$.

Pins

Let E be an analytic equivalence relation on a Polish space X. An E-*pin* is a pair (Q, τ) such that

- Q is a partial order
- τ is a Q-name for an element of X and,
- for all generic (G, H) for $Q \times Q$, $V[G, H] \models \tau_G E \tau_H$.

An E-pin represents the same E-equivalence class in all extensions by Q, even though the class may have no members in the ground model.

Note that for any two V-generic filters $G_0, G_1 \subseteq Q$, there exists in some forcing extension an $H \subseteq Q$ such that (G_0, H) and (G_1, H) are both V-generic for $Q \times Q$.

Trivial pins

When E is an analytic equivalence relation on a Polish space X, $x \in X$ and P is a partial order, then the pair $(P, \check{x})$ is a (trivial) pin.

Some equivalence relations have only trivial pins (up to a suitable notion of equivalence of pins).

These are equivalence relations are said to be pinned.

Nontrivial examples (Borel)

- $\mathbb{F}_2$ (the "same range" equivalence relation for ω-sequences from 2^ω).

  For any (nonempty) set of reals A, $(\mathrm{Col}(\omega, A), \dot{g})$ is an $\mathbb{F}_2$-pin, where $\dot{g}$ is a name for the generic surjection from ω to A.

- The equivalence relation on $\mathcal{P}(\omega)^\omega$ of generating the same filter.

  For any (nonempty) filter F on ω, $(\mathrm{Col}(\omega, F), \dot{g})$ is a pin, where $\dot{g}$ is a name for the generic enumeration of F.

In both cases the examples given characterize all the pins, up to a suitable notion of equivalence.

Nontrivial examples (analytic)

- $\mathbb{E}_{\omega_1}$ (the "isomorphic or both illfounded" equivalence relation on linear orders on ω).

  For any infinite ordinal α, $(\mathrm{Col}(\omega, \alpha), \dot{g})$ is an $\mathbb{E}_{\omega_1}$-pin, where $\dot{g}$ is a name for a generic wellordering of ω in ordertype α.

- $\mathbb{HC}$ (the "isomorphic or both illfounded" equivalence relation on extensional binary relations on ω).

  For any infinite transitive set X, $(\mathrm{Col}(\omega, X), \tau)$ is an $\mathbb{HC}$-pin, where, letting $\dot{g}$ be a generic enumeration of X, τ is a name for the set of pairs $(x, y) \in \omega^2$ for which $\dot{g}(x) \in \dot{g}(y)$.

Again, these are all the pins.

Isomorphism relations

For any theory T in any countable first-order language, the isomorphism relation on the set of models of T with domain ω is analytic.

Every model M of T (of any cardinality) induces a pin via the partial order $\mathrm{Col}(\omega, M)$.

If M has uncountable Scott rank, then the pin induced by M is nontrivial.

For some theories T there are more pins, since an infinitary sentence can be forced to be a Scott sentence in a suitable collapse extension, without having a model in the ground model.

Question : Can this happen for the theory of linear orders?

Equivalent pins

Two pins (Q, τ), (P, σ) are E-equivalent if

$$V[G, H] \models \tau_G E \sigma_H$$

holds for all generic

$$(G, H) \subseteq Q \times P.$$

The corresponding equivalence classes are the *virtual equivalence classes* of E.

Pinned equivalence relations

An E-pin is said to be *trivial* if it is equivalent to a pair of the form $(1, \check{x})$, where 1 is the trivial partial order.

Every E-pin of the form (Q, τ) with Q countable (or even reasonable) is trivial.

An equivalence relation E is *pinned* if every E-pin is trivial.

Nontrivial partial orders

A partial order P is reasonable if, for every ordinal γ and every $f\colon \gamma^{<\omega} \to \gamma$ in a forcing extension by P, there is an $a \subseteq \gamma$ closed under f which is a countable set in the ground model.

Proper forcings are reasonable.

If (P, τ) is a nontrivial pin, then P is not reasonable (in particular, it is not countable).

Examples of pinned relations

- Countable Borel equivalence relations.
- Actions of Polish cli groups (e.g, locally compact topological groups).
- $\mathbb{E}_1$, $\mathbb{E}_2$ and $\mathbb{E}_{K_\sigma}$.

Examples of unpinned relations

- $\mathbb{F}_2$
- $\mathbb{E}_{\omega_1}$
- $\mathbb{HC}$
- The relation on $\mathcal{P}(\omega)^\omega$ of generating the same filter.
- The isomorphism relation for any first-order theory having models of uncountable Scott rank.

Compact metric spaces

Zelinsky: Every orbit equivalence relation of a Polish group action is Borel reducible to the homeomorphism relation on compact metrizable spaces.

Compact Hausdorff spaces give a class of virtual equivalence classes.

Given such a space with a basis of size κ, the space naturally reinterprets (a la Fremlin/Zapletal) as a second countable compact Hausdorff (and thus metrizable) space after forcing with $\mathrm{Col}(\omega, \kappa)$.

Question: Are these all the virtual equivalence classes?

Measure equivalence

Two probability measures on a Polish space X are said to be measure-equivalent if they have the same null sets.

Let E_X be the corresponding relation on the space $P(X)$ of Borel probability measures on X.

$\mathbb{F}_2 <_{\mathrm{B}} E_X$ (that the relation is strict is due to Sofronidis).

Given $\{\mu_n : n \in \omega\}$, $\sum \mu_n 2^{-n-1}$ is in $P(X)$; the E_X-class of the resulting measure does not depend on the enumeration.

This shows that each infinite subset of $P(X)$ induces a virtual E_X-class via $\mathrm{Col}(\omega, P(X))$.

Again: are these all the virtual equivalence classes?

Operations on equivalence relations

Various operations (e.g., products and increasing unions) can be used to generate equivalence relations.

In many cases the pins for the output relation are generated in a canonical way from the pins for the input relations.

Products

Given equivalence relations E_n on X_n ($n \in \omega$), the product relation E on $\prod X_n$ is defined by setting fEg to hold if

$$\forall n \in \omega\, f(n) E_n g(n).$$

If, for each n, (Q_n, τ_n) is an E_n-pin, then $(\prod Q_n, \tau)$ is an E-pin, where τ is a name for the sequence of realizations of the τ_n's.

This characterizes all the E-pins.

It follows that the class of pinned equivalence relations is preserved under products.

Unions

Let E be the the union of an increasing sequence $\langle E_n : n \in \omega \rangle$ of equivalence relations on some Polish space X.

The E-pins are exactly those pairs (Q, τ) which are E_n-pins for some n.

The class of pinned equivalence relations is preserved under increasing unions.

Containment

Suppose that $E \subseteq F$ are equivalence relations over the same Polish space X.

Then every E-pin is an F-pin.

We say that F is countable over E if every F-class is a countable union of E-classes. In this case, every F-pin is an E-pin.

In general this doesn't follow from $E \subseteq F$: let E be $\mathbb{F}_2 \times \mathbb{F}_2$ and F be $\mathbb{F}_2 \times ((2^\omega)^\omega)^2$.

The Friedman-Stanley jump

If E is an equivalence relation on a space X, the Friedman-Stanley jump of E is the relation E^+ on X^ω defined by setting

$$fE^+g$$

to hold if the ranges of f and g represent the same set of E-classes.

For example $\mathbb{F}_2$ is the Friedman-Stanley jump of $=_{2^\omega}$.

An E^+ pin is given by a set $\{(Q_i, \tau_i) : i \in I\}$ of E-pins and a $\prod Q_i \times \mathrm{Col}(\omega, I)$-name for an ω-sequence listing the corresponding realization of the τ_i's.

These represent all the virtual E^+-classes.

The Louveau jump

The Louveau jump of an equivalence relation E on a Polish space X with respect to a filter F on ω is the relation E^F on X^ω given by setting

$$f\,E^F g$$

to hold if $f(n)Eg(n)$ for F-many n.

$\mathbb{E}_1$ is the Louveau jump of $=_{2^\omega}$ with respect to the cofinite (Frechet) filter.

If F is countably generated, then E^F is an increasing union of products of E, so its pins are induced by ω-sequences of E-pins.

In particular, if E is pinned and F is countably generated, then E^F is pinned.

The Coskey-Clemens jump

Given a countable group Γ, the Coskey-Clemens Γ-jump of an equivalence relation E on a Polish space X is the relation $E^{[\Gamma]}$ on X^{Γ} given by setting

$$f\ E^{[\Gamma]} g$$

to hold if there is an $\gamma \in \Gamma$ such that

$$x(\gamma^{-1}\alpha) E g(\alpha)$$

for all $\alpha \in \Gamma$.

This is a countable union of products of E, so its pins are induced by functions from Γ to the set of E-pins.

In particular, if E is pinned, then so is $E^{[\Gamma]}$.

κ and λ

It is sometimes possible to prove nonreducibility results between analytic equivalence relations via the association of cardinal invariants.

For any analytic equivalence relation E, we let:

- $\kappa(E)$, the least cardinal κ such that every E-pin is equivalent to one of the form (Q, τ), where $|Q| < \kappa$ (set to ∞ if there is no such κ and $\aleph_1$ if E is pinned)
- $\lambda(E)$, the cardinality of the set of equivalence classes of E-pins (if it exists, otherwise ∞)

Note that $\lambda(E) \leq 2^{\kappa(E)}$.

$\mathbb{F}_2$ and $\mathbb{E}_{\omega_1}$ again

- Every $\mathbb{F}_2$ pin is equivalent to one of the form $(\mathrm{Col}(\omega, A), \dot{g})$ for some set of reals A, so
$$\kappa(\mathbb{F}_2) = (2^{\aleph_0})^+$$
and
$$\lambda(\mathbb{F}_2) = 2^{2^{\aleph_0}}.$$
- Every $\mathbb{E}_{\omega_1}$ pin is equivalent to one of the form $(\mathrm{Col}(\omega, \alpha), \dot{g})$ for some ordinal α, so
$$\kappa(\mathbb{E}_{\omega_1}) = \lambda(\mathbb{E}_{\omega_1}) = \infty.$$

If E is pinned, then $\kappa(E) = \aleph_1$ and $\lambda(E) = 2^{\aleph_0}$.

If E is the product of $\langle E_n : n \in \omega \rangle$, then

$$\kappa(E) \leq (\prod_n \kappa(E_n))^+$$

and

$$\lambda(E) = \prod_n \lambda(E_n).$$

If E is the increasing union of $\{E_n : n \in \omega\}$ then

$$\lambda(E) = sup_n \lambda(E_n)$$

and

$$\kappa(E) = \sup_n \kappa(E_n).$$

λ and the Friedman-Stanley jump

If E^+ is the Friedman-Stanley jump of E (and E has infintely many classes), then

$$\lambda(E^+) = 2^{\lambda(E)}.$$

Embeddings of relations

Let E be an analytic equivalence relation on a Polish space X, and let F be a Borel equivalence relation on a set Y.

We let E^F be the restriction of $(E \times F)^+$ to the set of ω-sequences from $X \times Y$ whose second coordinates are all F-distinct.

Then $\lambda(E^F) = \lambda(E)^{\lambda(F)}$, as the E^F-virtual classes are respresented by functions from the set of virtual F-classes to the set of virtual E-classes.

Comparing equivalence relations

If $E \leq_{\mathrm{a}} F$ then $\kappa(E) \leq \kappa(F)$ and $\lambda(E) \leq \lambda(F)$.

This shows that:

- If $E \leq_{\mathrm{a}} F$ and F is pinned, then so is E;
- $\mathbb{E}_{\omega_1} \not\leq_{\mathrm{a}} \mathbb{F}_2$;
- for any E with infinitely many classes, $E^+ \not\leq_{\mathrm{a}} E$, where E^+ is the Friedman-Stanley jump of E.

$\aleph_\alpha$ and $\beth_\alpha$

Recall that

- $\aleph_0 = \beth_0 = |\mathbb{N}|$,
- $\aleph_{\alpha+1} = \aleph_\alpha^+$,
- $\beth_{\alpha+1} = 2^{\beth_\alpha}$,
- for limit ordinals γ,

$$\aleph_\gamma = \sup_{\alpha<\gamma} \aleph_\alpha$$

and

$$\beth_\gamma = \sup_{\alpha<\gamma} \beth_\alpha.$$

The Generalized Continuum Hypothesis (GCH) is the statement that $\aleph_\alpha = \beth_\alpha$ for all ordinals α.

For each countable ordinal α there are Borel equivalence relations E_α and F_α for which, provably,

$$\kappa(E_\alpha) = \aleph_\alpha$$

and

$$\kappa(F_\alpha) = \beth_\alpha^+.$$

The relative values of $\aleph_\alpha$ and $\beth_\beta$ can be manipulated by forcing.

This shows for instance that neither of E_3 and F_1 is Borel-reducible to the other.

Bounds for Borel relations

Work of Jacques Stern from 1984 shows that if E is a Borel equivalence relation of Borel rank α, then

$$\kappa(E) < \beth_{\alpha}^{+}$$

for every Borel equivalence relation E.

In particular,

$$\kappa(E) < \beth_{\omega_1}$$

for every Borel equivalence relation E.

Bounds for analytic relations

For analytic equivalence relations E with $\kappa(E) < \infty$ the least measurable cardinal is an upper bound on $\kappa(E)$, but it not known if this can be improved.

It cannot be improved below the least ω_1-Erdös cardinal.

If there is a measurable cardinal, then for any analytic equivalence relation E,

$$\kappa(E) = \infty$$

if and only if

$$\mathbb{E}_{\omega_1} \leq_{\mathrm{a}} E.$$

Absoluteness (Borel)

For Borel equivalence relations E, the property of being pinned is absolute (Π^1_1) between models of ZFC.

E is pinned (in V) if and only if it is pinned in every countable ω-model of a sufficient fragment of ZFC.

Being pinned is not absolute between models of ZF, however.

The restriction of $\mathbb{F}_2$ to sets linearly ordered by any fixed Borel relation (with the property that every uncountable set has an upper bound) is unpinned in ZFC but pinned in many Choiceless models.

Absoluteness (analytic)

If there exists a proper class of Woodin cardinals, then for analytic equivalence relations, the property of being pinned is absolute.

Downwards absoluteness of pinnedness can fail between models of ZFC for analytic equivalence relations.

Open Question 1

By Silver's theorem, every unpinned Borel equivalence relation has at least $2^{\aleph_0}$ many classes.

Question : Must every unpinned Borel equivalence relation have at least $2^{\aleph_1}$ many virtual classes?

This does hold for ground model relations after $\mathrm{Col}(\omega, \kappa)$, when κ is a strongly inaccessible cardinal, even just restricting to pins for $\mathrm{Col}(\omega, \omega_1)$.

This strong form doesn't hold in general for analytic relations: consistently (relative to a strongly inaccessible cardinal, $\mathbb{E}_{\omega_1}$ has less than $2^{\aleph_1}$ many virtual classes on $\mathrm{Col}(\omega, \omega_1)$).

Open Question 2

Suppose that M is a transitive inner model of ZFC containing a code for a Borel equivalence relation E. Must $\kappa(E)^M \leq \kappa(E)$?

The answer is yes if E is almost-reducible to an orbit equivalence relation coded in M such that the reduction is also coded in M.