Tutorial on Suslin Trees and Related Topics

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Outline

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   - A Brief History of the Suslin Problem
   - A Survey of Aronszajn Trees
   - Homogeneous and Rigid Aronszajn Trees
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   - Suslin Trees as Forcing Notions
   - Coherent Suslin Trees
   - Free Suslin Trees
3. Part 3: Models of $\neg SH$ With Few Suslin Trees
   - Maps Between Suslin Trees
   - Consistency Results

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Mikhail Suslin, Originator of the Suslin Problem

Михаи́л Я́ковлевич Су́слин

Transliteration: Mikhail Yakovlevich Suslin (Souslin)

Mikhail Suslin: Brief Biography

Mikhail Suslin (1894-1919) was a Russian mathematician. He was a student of Nikolai Luzin starting in the 1914-15 academic year and studied descriptive set theory and topology.

In his short mathematical career of around five years, his main mathematical contributions are:

1. introducing the idea of analytic sets in descriptive set theory;
2. asking a question now known as the famous *Suslin problem*, which remained open for around 50 years and eventually led to major advances in set theory.

Mikhail Suslin: Brief Biography

Suslin published a total of three short articles, only one of which appeared during his lifetime.

The Suslin problem was published as *Problem 3* in a list of ten open problems by various mathematicians published in 1920 in the very first issue of Fundamenta Mathematica ([S1920]).

Suslin passed away in 1919 as a result of typhus at the age of 24.

The Suslin Problem

3) Un ensemble ordonné (linéairement) sans sauts ni lacunes et tel que tout ensemble de ses intervalles (contenant plus qu'un élément) n'empiétant pas les uns sur les autres est au plus dénombrable, est-il nécessairement un continu linéaire (ordinaire)?

Problème de M. M. Souslin.

Translation:

Problem 3). Let a linearly ordered set without gaps or jumps have the property that every set of non-overlapping intervals (each containing at least one element) is at most countable. Will this set necessarily be the (usual) linear continuum?

The Suslin Problem

Theorem (Cantor)

Let L be a dense linear order without endpoints which is complete and separable (that is, has a countable dense subset). Then L is isomorphic to the real line $\mathbb{R}$.

Question (Suslin's Problem)

Let L be a dense linear order without endpoints which is complete and has the countable chain condition (that is, every pairwise disjoint family of non-empty open intervals is countable). Is L isomorphic to the real line $\mathbb{R}$ (or equivalently, is L separable)?

Question (Equivalent to Suslin's Problem)

Is every linear order L with the countable chain condition necessarily separable?

The Suslin Hypothesis

A *Suslin continuum* is a complete dense linear order without endpoints which is c.c.c. but not separable.

A *Suslin line* is a linear order which is c.c.c. but not separable.

A Suslin continuum exists iff a Suslin line exists.

Definition

The *Suslin Hypothesis* (SH) is the statement that there does not exist a Suslin line.

Đuro Kurepa

Ђуро Курепа

Đuro Kurepa

One of the earliest and most notable scholars to work on the Suslin problem was the Serbian mathematician Đuro Kurepa (1907-1993).

In his doctoral dissertation of 1935, written under the supervision of M. Fréchet, Kurepa gave the first ever systematic study of trees.

In his dissertation, Kurepa introduced and analyzed many fundamental ideas which are now considered the foundation of the subject.

Kurepa's Contributions

An overview of some of Kurepa's many important contributions to the theory of trees:

(1) Introduced Aronszajn, Suslin, and Kurepa trees ([K1935], [K1937], [K1943]);

(2) In unpublished work Nachman Aronszajn proved the existence of an Aronszajn tree in 1934. Kurepa produced other Aronszajn trees including the first example of a special Aronszajn tree ([K1937]);

(3) Introduced normal trees and lexicographical orderings of trees ([K1935]);

(4) Proved that there is a Suslin line iff there is a Suslin tree ([K1935]);

(5) Proved that any two infinitely branching trees of countable height are isomorphic, and posed the question of whether this is also true for Aronszajn trees ([K1935]);

Kurepa's Contributions

(6) Proved that for any tree *T* of height a regular uncountable cardinal κ such that for some $\lambda < \kappa$, the levels of *T* have size less than λ, *T* has a cofinal branch ([K1935]);

(7) Introduced special Aronszajn trees ([K1937]), and proved the existence of an Aronszajn tree which embeds into the rationals ([K1937]);

(8) Proved that any partial order is the union of countably many antichains iff it embeds into the rationals ([K1940]);

(9) Supervised the PhD dissertation of Stevo Todorčević, who went on to become one of the world's leading set theorists (1979).

Suslin Lines and Suslin Trees

The earliest major result concerning the Suslin problem is the equivalence between the existence of a Suslin line and a Suslin tree.

Theorem

There exists a Suslin line iff there exists a Suslin tree.

This theorem was proven independently by three authors:

1. Kurepa in his 1935 dissertation ([K1935]);
2. Edwin Miller in a 1943 article ([M1943] published posthumously);
3. Wacław Sierpiński in a 1948 article ([S1948]).

Miller's Theorem, Excerpt From His 1943 Article

THEOREM. *In order that there exist a linear order which possesses properties* (1), (2) *and* (4) *without possessing property* (3) *it is necessary and sufficient that there exist a partial order P of power $\aleph_1$ such that*

(a) *if $Q \subset P$ and $\bar{\bar{Q}} = \aleph_1$, then Q contains two comparable elements and two non-comparable elements;*

(b) *if x and y are non-comparable elements of P, then there exists no z in P such that $x < z$ and $y < z$.*

Miller's proof used some ideas from his earlier paper with B. Dushnik, "Partially Ordered Sets", which contains the famous theorem that for every infinite cardinal κ, $\kappa \to (\kappa, \omega)^2$ ([DM1941]).

According to a note by the publisher, Miller passed away two weeks after submitting the article in July 1942.

The Suslin Number

Definition

The *Suslin number* of a topological space is the supremum of the set of cardinalities of any family of pairwise disjoint open sets.

Note that by definition, the Suslin number of a Suslin line (in the order topology) is ω.

Here is another early theorem related to the Suslin problem due to Kurepa:

Theorem (Kurepa [K1950])

Suppose that L is a Suslin line. Then the Suslin number of $L \times L$ (in the product topology) is equal to ω_1.

The Gaifman and Specker Theorem

Kurepa asked whether any two infinitely splitting Aronszajn trees are isomorphic ([K1935]), a problem which he referred to as "premier problème miraculeux." It took almost 30 years to solve.

Theorem (Gaifman and Specker [GS1964])

There exists a family of 2^{ω_1}-many pairwise non-isomorphic infinitely splitting normal Aronszajn trees.

Cohen Invents Forcing

In 1963, Paul Cohen invented the method of forcing, which provided a powerful technique for proving independence results in set theory ([C1966]).

Previously, Kurt Gödel had shown that ZF is consistent with the axiom of choice (AC) and the continuum hypothesis (CH) by developing the idea of the constructible universe L ([G1940]).

In the other direction, Cohen used forcing to construct models of $\mathrm{ZF} + \neg\mathrm{AC}$ and $\mathrm{ZFC} + \neg\mathrm{CH}$. In combination with Gödel's work, these models demonstrated that the axiom of choice does not follow from ZF and the continuum hypothesis does not follow from ZFC.

With the method of forcing now available, a few years later the Suslin problem was finally solved.

A Solution to the Suslin Problem

Suslin's problem was solved by showing that the existence of a Suslin tree can neither be proved nor disproved in the theory ZFC.

The consistency of the negation of Suslin's hypothesis was established independently by Thomas Jech and Stanley Tennenbaum.

Theorem (Jech [J1967], Tennenbaum [T1968])

There exists a forcing poset which forces the existence of a Suslin tree. Therefore, $\neg$SH is consistent with ZFC.

Jech's forcing adds a Suslin tree with countable conditions, and Tennenbaum's forcing adds a Suslin tree with finite conditions.

A Solution to the Suslin Problem

A non-forcing proof of the consistency of ZFC $+ \neg$SH was given by Jensen using Gödel's constructible universe L.

Theorem (Jensen [J1968])

If $\Diamond$ holds, then there exists a Suslin tree. In particular, if $V = L$ then $\neg$SH holds.

The more difficult direction in the independence of the Suslin hypothesis was proved later by Solovay and Tennenbaum, who built a model of ZFC $+$ SH.

Theorem (Solovay and Tennenbaum [ST1971])

There exists a forcing poset which forces Martin's axiom plus $\neg$CH, and in particular, forces that there does not exist a Suslin tree.

Invention of Iterated Forcing and Forcing Axioms

The Solovay and Tennenbaum proof of the consistency of SH involved:

1. developing the new technique of *iterated forcing* (specifically, finite support forcing iterations of c.c.c. forcings), and
2. establishing the consistency of the first *forcing axiom*, Martin's axiom (named after its originator Donald Martin).

These two developments had a transformative effect on the field of set theory. For these and other reasons, such as its impact on the theory of trees, the Suslin problem ranks among the most significant problems in the history of set theory, comparable to Cantor's continuum problem.

Suslin's Hypothesis and the Continuum Hypothesis

A natural question is whether there is any relationship between SH and CH.

The Jech and Tennenbaum models of $\neg$SH satisfy CH. Adding any number of Cohen reals preserves a Suslin tree, so $\neg\text{SH} + \neg\text{CH}$ is consistent as well.

The Solovay and Tennenbaum model of SH satisfies $\neg$CH. It took an ingenious argument of Jensen to prove the consistency of SH + CH.

Theorem (Jensen; Devlin and Johnsbraten [DJ1974])

Assume GCH, $\diamondsuit^*$, and $\square_{\omega_1}$. Then there exists a forcing poset which forces that CH holds and there does not exist a Suslin tree.

Jensen's Proof and Shelah's Proper Forcing

Jensen's model of SH + CH did not use iterated forcing in the way we think of it nowadays, but rather involved defining a sequence of Suslin trees

$$\langle T^\nu : \nu < \omega_2 \rangle,$$

together with projection mappings, and forcing with the direct limit. Given T^ν, a Suslin tree $T^{\nu+1}$ is defined which adds a cofinal branch to T^ν and specializes an Aronszajn tree in V^{T^ν}. No reals are added because forcing with Suslin trees does not add countable sets.

In the 1980's Shelah developed a more general and flexible method for iterating forcing while not adding reals, as part of his theory of proper forcing, and used his method to produce an alternative model of SH + CH ([S1982]).

Outline

Review of Basic Definitions and Notation

Definition

A strict partial order $(T, <_T)$ is *tree-like* if for all $x \in T$, the set $\{y \in T : y <_T x\}$ is linearly ordered by $<_T$.

Definition

A *tree* is a strict partial order $(T, <_T)$ such that for every x, the set $\{y \in T : y <_T x\}$ is well-ordered by $<_T$.

Review of Basic Definitions and Notation

Let T be a tree. For $x \in T$, the order type of $\{y \in T : y <_T x\}$ is the *height of x in T*, denoted $\mathrm{ht}_T(x)$.

For all $\alpha < \mathrm{ht}_T(x)$, we write $x \restriction \alpha$ for the unique $y <_T x$ with height α.

For any ordinal α, $T_\alpha := \{x \in T : \mathrm{ht}_T(x) = \alpha\}$ is the *α-th level of T*. The *height of T* is the least δ such that $T_\delta = \emptyset$.

For any ordinal α, $T \restriction \alpha := \{x \in T : \mathrm{ht}_T(x) < \alpha\}$. More generally, if A is a set of ordinals, then $T \restriction A := \{x \in T : \mathrm{ht}_T(x) \in A\}$.

In these talks we will mostly be interested in trees of height ω_1.

Review of Basic Definitions and Notation

Elements x and y of T are *comparable* if $x \leq_T y$ or $y <_T x$, and otherwise are *incomparable*.

A *chain* is a subset of T consisting of comparable elements, and an *antichain* is a subset of T consisting of incomparable elements.

A *branch* of T is a maximal chain. A branch is *cofinal* if it meets every level of T.

If b is a branch of T and α is less than the order type of b, we will write $b(\alpha)$ for the unique element of b with height α.

Review of Basic Definitions and Notation

Definition

An *ω_1-tree* is a tree of height ω_1 whose levels are countable.

Definition

An *Aronszajn tree* is an ω_1-tree with no cofinal branch (equivalently, no uncountable chain).

Definition

A *Suslin tree* is a tree of height ω_1 with no uncountable chains and no uncountable antichains.

Since the levels of a tree are antichains, every Suslin tree has countable levels and hence is an Aronszajn tree.

Review of Basic Definitions and Notation

Definition

A tree T is *normal* if it satisfies the following properties:

1. T has a *root*, which is the unique element with height 0;
2. every element of T not at the maximal level of the tree (if it exists), has at least two immediate successors;
3. (*unique limits*) if x and y have height δ, where δ is a limit ordinal, then there exists some $\alpha < \delta$ such that $x \upharpoonright \alpha \neq y \upharpoonright \alpha$;
4. if x is in T then there exists an element above x at any higher level of T.

Different authors use somewhat different definitions of normal, but in all variations (4) is always required.

Review of Basic Definitions and Notation

If T is a normal ω_1-tree, then T is Suslin iff T has no uncountable antichains.

For any cardinal λ, a tree T is *λ-ary* if every element of T has exactly λ-many immediate successors.

We are mostly interested in normal ω_1-trees which are either *binary*, which means 2-ary, or *infinitely splitting*, which means ω-ary.

Review of Basic Definitions and Notation

If T is a tree, a *subtree* of T is any subset of T considered as a tree equipped with the ordering inherited from T.

A subset $X \subseteq T$ is *downwards closed* if whenever $x \in X$ and $y <_T x$, then $y \in X$.

If X is a downwards closed subtree of T then the height functions ht_T and ht_X agree on X.

The *downward closure* of a set $X \subseteq T$ is the set

$$\{y \in T : \exists x \in X \; y \leq_T x\}.$$

If $X \subseteq T$ and X is a Suslin tree, then the downward closure of X is also a Suslin tree.

Nowhere Suslin Trees

Definition

A tree T of height ω_1 is *nowhere Suslin* if every uncountable subset of T contains an uncountable antichain.

Lemma

If T is an Aronszajn tree, then T is nowhere Suslin iff T has no Suslin subtree.

So for an Aronszajn tree T, T being *not nowhere Suslin* means that T contains a Suslin subtree, not that T itself is Suslin.

Special Trees

Definition

A tree T is *special* if T is the union of countably many antichains.

Being special is equivalent to the existence of a *specializing function*, which is a function $f : T \to \omega$ such that $x <_T y$ implies $f(x) \neq f(y)$.

Note that if T is a special tree, then T does not have an uncountable chain and T has an uncountable antichain. Any subtree of a special tree is also special. Hence, any special tree is nowhere Suslin.

Theorem (Kurepa [K1937])

There exists a special Aronszajn tree.

Trees Embeddable Into Linear Orders

Definition

A map $h : P \to Q$ between partial orders is *strictly increasing* if $x <_P y$ implies $h(x) <_Q h(y)$.

Definition

For a linear order L, a tree T is *L-embeddable* if there exists a strictly increasing map $h : T \to L$.

Theorem (Kurepa [K1940])

A tree is special iff it is $\mathbb{Q}$-embeddable.

Theorem

There exists an ω_1-tree T for which there exists a continuous strictly increasing map $h : T \to \mathbb{Q}$.

Baumgartner's Dissertation of 1970

The study of embeddings of trees into linear orders was initiated by Kurepa.

This topic was explored further in the doctoral dissertation of James Baumgartner in 1970.

We survey some of the results of Baumgartner's dissertation as well as some unpublished results of Fred Galvin and Richard Laver which appear there.

Baumgartner's Dissertation

Theorem (Galvin (unpublished); Baumgartner [B1970])

Let T be a tree of height ω_1. Then T is $\mathbb{R}$-embeddable iff $T \restriction \{\alpha + 1 : \alpha < \omega_1\}$ is special.

Proposition (Baumgartner [B1970])

For a tree T of height ω_1, if $T \restriction \{\alpha + 1 : \alpha < \omega_1\}$ is nowhere Suslin, then T is nowhere Suslin.

If T is $\mathbb{R}$-embeddable, then $T \restriction \{\alpha + 1 : \alpha < \omega_1\}$ is special and hence nowhere Suslin. So by the above proposition, T is nowhere Suslin. Consequently:

Proposition

If T is a tree of height ω_1 which is $\mathbb{R}$-embeddable, then T has no uncountable chains and is nowhere Suslin.

The Tree T^*

Note that for any set X,

$$^{<\omega_1}X := \{f : \exists\alpha < \omega_1 \ f : \alpha \to X\},$$

ordered by strict subset, is a tree of height ω_1.

Definition

Let T^* be the subtree of $^{<\omega_1}\omega$ consisting of injective functions.

Note that T^* has no uncountable chains.

The Tree T^*

Theorem (Laver (unpublished); Baumgartner [B1970])

The tree T^* is not special.

Theorem (Baumgartner [B1970])

A tree T is $\mathbb{R}$-embeddable iff there exists a strictly increasing map of T into T^*.

In particular, T^* itself is $\mathbb{R}$-embeddable.

The Tree T^*

So T^* is an example of a tree of height ω_1 which is $\mathbb{R}$-embeddable but not special. Consequently:

Theorem

The statement that every tree with no uncountable chains is special is disprovable in ZFC.

Of course T^* is not an ω_1-tree because it has levels of size 2^ω.

The Shift Operator $\mathcal{S}$ on Trees

Definition

Let T be a tree of height ω_1. The *shift of T*, denoted by $\mathcal{S}(T)$, is the unique smallest tree satisfying that

$$\mathcal{S}(T) \restriction \{\alpha + 1 : \alpha < \omega_1\} = T.$$

In other words, we shift every element of T up by one level, and add unique limits at limit levels to chains which already had upper bounds in T.

Note that $\mathcal{S}(T)$ also has height ω_1, and if T is an ω_1-tree then so is $\mathcal{S}(T)$.

The Tree $\mathcal{S}(T^*)$

Let us apply the shift operator to the tree T^*.

Consider $\mathcal{S}(T^*)$. Then

$$\mathcal{S}(T^*) \restriction \{\alpha + 1 : \alpha < \omega_1\} = T^*,$$

which:

(a) is not special, and

(b) is nowhere Suslin.

By (a), $\mathcal{S}(T^*)$ is not $\mathbb{R}$-embeddable. By (b), $\mathcal{S}(T^*)$ is nowhere Suslin.

The Trees T^* and $\mathcal{S}(T^*)$

The trees T^* and $\mathcal{S}(T^*)$ are witnesses to the following theorems.

Theorem (Baumgartner [B1970])

There exists a tree of height ω_1 which is $\mathbb{R}$-embeddable and not $\mathbb{Q}$-embeddable.

Theorem (Baumgartner [B1970])

There exists a tree of height ω_1 which is nowhere Suslin but not $\mathbb{R}$-embeddable.

These theorems prove that the implications

$$\text{special} \implies \mathbb{R}\text{-embeddable} \implies \text{nowhere Suslin}$$

cannot be reversed.

Results of Baumgartner's Dissertation

Theorem (Baumgartner [B1970])

The following statements are equivalent:

1. *Every Aronszajn tree is special;*
2. *Every Aronszajn tree is $\mathbb{R}$-embeddable.*

Proof.

(1) $\Rightarrow$ (2): Immediate because being special is equivalent to being $\mathbb{Q}$-embeddable.

(2) $\Rightarrow$ (1): Suppose that T is an Aronszajn tree which does not embed into $\mathbb{Q}$. Then $\mathcal{S}(T)$ is an Aronszajn tree satisfying that $\mathcal{S}(T) \upharpoonright \{\alpha + 1 : \alpha < \omega_1\} = T$ is not special. So $\mathcal{S}(T)$ is not $\mathbb{R}$-embeddable. □

Results of Baumgartner's Dissertation

Theorem (Baumgartner [B1970])

$MA + \neg CH$ implies that every tree with no uncountable chains and size less than 2^ω has a strictly increasing and continuous map into $\mathbb{Q}$, and in particular, that all Aronszajn trees are special.

Theorem (Baumgartner, Malitz, and Reinhardt [BMR1970])

$MA + \neg CH$ implies that every tree-like partial order of size less than 2^ω with no uncountable chains embeds into $\mathbb{Q}$.

As we described above, T^* is a tree of size 2^ω with no uncountable chains which is not special.

Results of Baumgartner's Dissertation

As we have discussed, in ZFC we can prove that the properties of being special, $\mathbb{R}$-embeddable, and nowhere Suslin are distinct for trees of height ω_1. For ω_1-trees, the best we can get is a consistency result.

Theorem (Baumgartner [B1970])

Assume that $V = L[A]$ for some set $A \subseteq \omega_1$. Then there exists an $\mathbb{R}$-embeddable Aronszajn tree which is not special.

Applying the shift operator to a tree as in the above theorem, it follows that under the same assumption there exists a nowhere Suslin Aronszajn tree which is not $\mathbb{R}$-embeddable. In addition, these facts remain true after forcing arbitrarily many Cohen reals ([B1970]).

This concludes the discussion of Baumgartner's dissertation.

The Diamond Principle

The constructibility assumption in the above theorem was soon replaced by the diamond principle and its variations. Starting with Jensen, it was recognized that a great variety of Aronszajn and Suslin trees can be constructed with diamond.

Theorem (Devlin [D1972])

Assume $\diamondsuit$. Then there exist 2^{ω_1} many pairwise non-isomorphic Aronszajn trees which are $\mathbb{R}$-embeddable but not special.

Theorem (Kunen, K., Larson, J., and Steprāns, J. [KLS2012])

Assume $\diamondsuit$. Then for any set $A \subseteq \mathbb{R}$ which contains no perfect subset, there exists a special Aronszajn tree which has no continuous strictly increasing map into A. In particular, this statement holds for $A = \mathbb{Q}$.

Stationary Antichains and Club Antichains

Definition

Let T be an ω_1-tree. An antichain $A \subseteq T$ is a *stationary antichain* if the set

$$\{\mathrm{ht}_T(x) : x \in A\}$$

is a stationary subset of ω_1.

An antichain $A \subseteq T$ is a *club antichain* if the set

$$\{\mathrm{ht}_T(x) : x \in A\}$$

is a club subset of ω_1.

Special Trees Have Stationary Antichains

Lemma

Let T be a special Aronszajn tree. Then T has a stationary antichain.

Proof.

Let $f : T \to \omega$ be a specializing function. For each $\alpha < \omega_1$ choose some $x_\alpha \in T_\alpha$.

By the pressing down lemma, there is a stationary set $X \subseteq \omega_1$ on which the map $\alpha \mapsto f(x_\alpha)$ is constant.

Then

$$\{x_\alpha : \alpha \in X\}$$

is a stationary antichain. □

Results on Stationary Antichains

Theorem (Shelah [S1982])

Assuming $\diamondsuit$, there exists a special Aronszajn tree with no club antichain.

In comparison, MA_{ω_1} implies that every Aronszajn tree is special and has a club antichain.

Theorem (Shelah [S1982])

Assuming $\diamondsuit$, there exists an Aronszajn tree which is $\mathbb{R}$-embeddable, not special, and contains no club antichain.

Theorem (Shelah [S1982])

Assuming $\diamondsuit^*$, there exists an Aronszajn tree which is $\mathbb{R}$-embeddable and contains no stationary antichain.

Almost Suslin Trees

Definition

An ω_1-tree with no stationary antichain is called an *almost Suslin tree*.

Any special Aronszajn tree is not an almost Suslin tree. So MA_{ω_1} implies that there does not exist an almost Suslin tree.

Every Suslin tree is an almost Suslin tree, but the converse is provably false.

Theorem (Devlin and Shelah [DS1979])

If there exists a Suslin tree, then there exists an almost Suslin tree which is not a Suslin tree.

Almost Suslin trees do not have to be Aronszajn trees; see for example Todorčević [T1984, Section 4].

Generic Reals and Suslin Trees

Theorem (Shelah [S1984])

Cohen forcing $Add(\omega)$ forces that there exists a Suslin tree.

Theorem (Todorčević [T2007, page 39])

Cohen forcing $Add(\omega)$ forces that there exists an $\mathbb{R}$-embeddable Aronszajn tree with no stationary antichain.

Generic Reals and Suslin Trees

Laver proved that the same is not true for random reals.

Theorem (Laver [L1987])

Assuming MA_{ω_1}, forcing any number of random reals with the product measure will force that all Aronszajn trees are special.

So SH is consistent with an arbitrarily large continuum of any uncountable cofinality. In particular, SH is consistent with 2^ω being singular.

Special Subtrees

Theorem

Assuming $\diamondsuit^*$, for every stationary and costationary set $S \subseteq \omega_1$, there exists a non-special Aronszajn tree T such that $T \restriction S$ is special and $T \restriction (\omega_1 \setminus S)$ has no stationary antichain.

On the other hand, if T restricted to a club is special, then so is T.

Proposition

Suppose that T is an ω_1-tree, $C \subseteq \omega_1$ is a club, and $T \restriction C$ is special. Then T is special.

Special Subtrees

Proof.

Fix a specializing function $g : T \restriction C \to \omega$.

For each $x \in T \restriction C$, let $\beta_x = \min(C \setminus (\mathrm{ht}_T(x) + 1))$, and fix a bijection

$$h_x : \{y \in T \restriction [\mathrm{ht}_T(x), \beta_x) : x \leq_T y\} \to \omega.$$

Define $f : T \to \omega \times \omega$ as follows. Given $y \in T$, let α_y be the largest element of $C \cap (\mathrm{ht}_T(y) + 1)$, which exists because C is club. Define

$$f(y) := (g(y \restriction \alpha_y), g_{y \restriction \alpha}(y)).$$

Then $y <_T z$ implies $f(y) \neq f(z)$ as is easy to check. □

S-st-Special

Definition (Shelah [S1982])

Let $S \subseteq \omega_1$ be a stationary set of limit ordinals. An ω_1-tree T is *S-st-special* if there exists a function $f : T \restriction S \to \omega_1$ such that:

1. for all $x \in T \restriction S$, $f(x) < \mathrm{ht}_T(x)$;
2. for all $x <_T y$ in $T \restriction S$, $f(x) \neq f(y)$.

If T is $(\omega_1 \cap \mathrm{Lim})$-st-special, then T is special. But if $S \subseteq \omega_1$ is stationary and co-stationary, then T being S-st-special does not imply that $T \restriction S$ is special.

Lemma

If T is S-st-special then T is Aronszajn and has a stationary antichain (and hence is not Suslin).

Back to Suslin's Hypothesis

A natural question is whether SH is equivalent to the statement that all Aronszajn trees are special. All of the early models of SH satisfy that all Aronszajn trees are special (namely, any model of MA_{ω_1} and Jensen's model of CH $+ \neg$SH).

Theorem (Shelah [S1982])

SH does not imply that every Aronszajn is special. Namely, it is consistent that there exists a stationary and costationary set $S \subseteq \omega_1$ such that:

1. *every Aronszajn tree is S-st-special (and hence not Suslin);*
2. *there exists an Aronszajn tree T such that $T \restriction (\omega_1 \setminus S)$ has no stationary antichain (and hence T is not special).*

Back to Suslin's Hypothesis

Shelah's proof used a complicated kind of forcing iteration called an "ω_1-free iteration," which is variation of a countable support forcing iteration. Both Shelah's theorem and the technique he used to prove it were improved by Chaz Schlindwein.

Theorem (Schlindwein [S1993])

Let T be an Aronszajn tree with no stationary antichain. There is a property of a forcing poset called T-proper which implies that the forcing is proper and does not add a stationary antichain to T, and moreover, being T-proper is preserved by any countable support forcing iteration.

Back to Suslin's Hypothesis

Theorem (Schlindwein [S1993])

It is consistent that SH holds and there exists a non-special Aronszajn tree with no stationary antichain.

Other applications of Schlindwein's forcing preservation theorem are given in K. "A forcing axiom for a non-special Aronszajn tree" ([K2020]).

Club Isomorphisms

Recall the result of Gaifman and Specker [GS1964] that there exist 2^{ω_1}-many pairwise non-isomorphic normal Aronszajn trees.

In 1985 Abraham and Shelah introduced the following weakening of the isomorphism relation on trees.

Definition (Abraham and Shelah [AS1985])

Let T and U be trees of height ω_1. Then T and U are *club isomorphic* if there exists a club $C \subseteq \omega_1$ and an isomorphism $f : T \upharpoonright C \to U \upharpoonright C$. In that case, f is called a *club isomorphism*.

The pairwise non-isomorphic Aronszajn trees given in Gaifman and Specker's article are all club isomorphic.

Club Isomorphisms

Lemma

Let T and U be normal ω_1-trees. Then T and U are club isomorphic iff there exists an unbounded set $X \subseteq \omega_1$ such that $T \restriction X$ and $U \restriction X$ are isomorphic.

Sketch.

Let $f : T \restriction X \to U \restriction X$ be an isomorphism. Let $C := X \cup \lim(X)$. Define $f^+ : T \restriction C \to U \restriction C$ extending f as follows.

Let $x \in T \restriction \lim(X)$ have height δ. By the normality of T pick some $z \geq_T x$ in $T \restriction X$. Define $f^+(x) := f(z) \restriction \delta$. Use the normality of T and U to show that this works. □

An Essentially Unique Aronszajn Tree

Abraham and Shelah [AS1985] introduced the hypothesis that there exists an *essentially unique Aronszajn tree*:

Any two normal Aronszajn trees are club isomorphic.

This hypothesis implies that all Aronszajn trees are special. Namely, there exists a special normal Aronszajn tree, and any other normal Aronszajn tree is club isomorphic to it. But if a tree is special on a club of levels, then it is special.

In particular, the hypothesis of an essentially unique Aronszajn tree is a logical strengthening of SH.

An Essentially Unique Aronszajn Tree

Theorem (Abraham and Shelah [AS1985])

Let T and U be normal Aronszajn trees. Then there exists a forcing poset of size ω_1 which is proper and adds a club isomorphism between T and U.

Corollary

The proper forcing axiom implies that any two normal Aronszajn trees are club isomorphic.

In fact, by a forcing iteration theorem of Shelah, any countable support forcing iteration of proper forcings of size ω_1 with length ω_2 is proper and ω_2-c.c. So a model of an essentially unique Aronszajn tree can be obtained without large cardinals.

Club Isomorphisms and Martin's Axiom

Theorem (Abraham and Shelah [AS1985])

MA_{ω_1} does not imply that any two normal Aronszajn trees are club isomorphic.

In particular, the hypothesis that all Aronszajn trees are special does not imply that there is an essentially unique Aronszajn tree.

Theorem (Abraham and Shelah [AS1985])

It is consistent to have Martin's axiom, any two normal Aronszajn trees are club isomorphic, and 2^ω arbitrarily large.

Families of Non-Club-Isomorphic Aronszajn Trees

Recall that neither SH nor the hypothesis that every Aronszajn tree is special have any impact on the value of 2^ω. In contrast:

Theorem (Abraham and Shelah [AS1985])

Suppose the weak diamond principle holds (equivalently, $2^\omega < 2^{\omega_1}$). Then there exists a family of 2^{ω_1} many normal pairwise non-club-isomorphic special Aronszajn trees.

Theorem (Todorčević [T1984])

Suppose that there exists an Aronszajn tree with no stationary antichain. Then there exists a family of 2^{ω_1} many normal pairwise non-club-isomorphic Aronszajn trees.

Outline

Homogeneity Properties of Aronszajn Trees

After Suslin trees were first shown to be consistent by Jech, Tennenbaum, and Jensen in the 1960's, one of the earliest topics studied about Suslin trees was the existence or non-existence of automorphisms.

Definition

An *automorphism* of a tree T is a bijection $f : T \to T$ such that for all $a, b \in T$,

$$a <_T b \iff f(a) <_T f(b).$$

An automorphism is *non-trivial* if it is not the identity function.

Homogeneous and Rigid

Definition

A tree T is *homogeneous* if for all a and b in T of the same height, there exists an automorphism $\sigma : T \to T$ such that $\sigma(a) = b$ and $\sigma(b) = a$.

Definition

A tree T is *rigid* if it has no non-trivial automorphisms.

These ideas were studied by Jensen [J1969] and Jech [J1972], and Jensen's work on this topic appears in the Devlin and Johnsbråten book "The Souslin Problem" [DJ1974].

Rigid and Totally Rigid

Definition

For any tree T and $a \in T$, $T_a := \{b \in T : a \leq_T b\}$.

Definition

A tree T is *totally rigid* if for all distinct a and b of the same height, T_a and T_b are not isomorphic.

Suppose that T is not rigid. Let $\sigma : T \to T$ be an automorphism and a and b distinct elements such that $\sigma(a) = b$. Then $\sigma \restriction T_a$ is an isomorphism between T_a and T_b.

So any totally rigid tree is rigid.

Homogeneous Trees

Lemma

A tree T of height ω_1 is homogeneous iff for any distinct a and b of T of the same height, T_a and T_b are isomorphic.

Lemma

If a tree T of height ω_1 is homogeneous, then for any distinct a and b of T, even of different heights, T_a and T_b are isomorphic.

The Number of Automorphisms of an Aronszajn Tree

Definition

For a tree T, let $\sigma(T)$ denote the cardinality of the collection of all automorphisms of T.

Theorem (Jech [J1972])

Let T be a normal ω_1-tree.

1. *$\sigma(T)$ is either finite or $2^\omega \leq \sigma(T) \leq 2^{\omega_1}$;*
2. *if $\sigma(T)$ is infinite, then $\sigma(T)^\omega = \sigma(T)$;*
3. *if T contains no Suslin subtree and $\sigma(T)$ is infinite, then $\sigma(T)$ has cardinality either 2^ω or 2^{ω_1}.*

The Number of Automorphisms of an Aronszajn Tree

Theorem (Jech [J1972])

Assume CH and $\kappa^\omega = \kappa$. Then there exists a countably closed ω_2-c.c.c. forcing which adds a Suslin tree T such that $|\sigma(T)| = \kappa$.

In other words, it is consistent to have a Suslin tree T such that $\sigma(T)$ has cardinality equal to any prescribed value between 2^ω and 2^{ω_1}.

Theorem (Jensen [J1969]; Devlin and Johnsbråten [DJ1974])

Assuming $\diamondsuit^+$, there exists a homogeneous Suslin tree S such that $\sigma(S) \geq \omega_2$.

Theorem (Todorčević [T1980])

There exists an Aronszajn tree T with $\sigma(T) = 2^\omega$, and there exists an Aronszajn tree U with $\sigma(U) = 2^{\omega_1}$.

Rigid Aronszajn Trees

Jech [J1972] asked whether there exists in ZFC a rigid normal ω_1-tree. This problem was solved independently by Abraham and Todorčević, making use of the construction of Gaifman and Specker.

Theorem (Abraham [A1979])

There exists a rigid normal Aronszajn tree.

Theorem (Todorčević [T1980])

There exist 2^{ω_1} many pairwise non-isomorphic rigid normal Aronszajn trees.

Outline

Outline

$\mathrm{ZFC} + \neg\mathrm{SH}$

For most of the remainder of the talks, we will work in the theory $\mathrm{ZFC} + \neg\mathrm{SH}$. In other words, we will assume that there exists a Suslin tree.

As described earlier, the earliest models of this theory were obtained as follows:

1. By Tennenbaum's forcing with finite conditions;
2. By Jech's forcing with countable initial segments;
3. In the constructible universe L, or from $\diamondsuit$ (Jensen).

We will consider other interesting models of $\neg\mathrm{SH}$ later in the talks.

Suslin Trees as Forcing Notions

One of the most useful facts about Suslin trees which sets them apart from other types of trees is that you can always force with a Suslin tree without collapsing ω_1.

Definition

Given any tree T, let $\mathbb{P}_T$ denote the forcing poset with underlying set T and ordered by $b \leq_{\mathbb{P}_T} a$ if $a \leq_T b$.

Lemma

For any tree T and $a, b \in T$, a and b are comparable in T iff they are compatible in $\mathbb{P}_T$.

Namely, if $c \leq_{\mathbb{P}_T} a, b$ then $a, b \leq_T c$, so a and b are comparable by the definition of a tree.

Suslin Trees Are C.C.C. Forcings

Proposition

Let T be a normal ω_1-tree. Then T is Suslin iff $\mathbb{P}_T$ is c.c.c.

Namely, by the previous lemma, an antichain of the tree T is the same as an antichain of the forcing $\mathbb{P}_T$. T is Suslin iff T has no uncountable antichains iff $\mathbb{P}_T$ has no uncountable antichains iff $\mathbb{P}_T$ is c.c.c.

Proposition

Let T be a normal Suslin tree. If G is a generic filter on $\mathbb{P}_T$, then G is a cofinal branch of T.

Forcing With a Suslin Tree

In particular, if T is a normal Suslin tree, then

$$\Vdash_{\mathbb{P}_T} \text{“}T \text{ is not Aronszajn.”}$$

An important question which will come up later in the talks is:

When you force with a normal Suslin tree T, which other Aronszajn trees in the ground model are no longer Aronszajn in the generic extension?

Dense Open Sets in a Suslin Tree

The following fact about Suslin trees is extremely important and will be used frequently in what follows.

Lemma

Let T be a Suslin tree. Then for any dense open set $D \subseteq \mathbb{P}_T$, there exists some $\beta < \omega_1$ such that

$$T \restriction [\beta, \omega_1) \subseteq D.$$

Proof.

Fix a maximal antichain $A \subseteq D$. Since T is Suslin, A is countable. So there exists some $\beta < \omega_1$ such that $A \subseteq T \restriction \beta$. If $x \in T \restriction [\beta, \omega_1)$, then by the maximality of A, x is above some member of A. Since D is open, $x \in D$. □

Suslin Trees Are Countably Distributive

An easy consequence of this lemma is that for any Suslin tree T, $\mathbb{P}_T$ is a countably distributive forcing poset.

Namely, if $\{D_n : n < \omega\}$ is a family of dense open subsets of $\mathbb{P}_T$, then for each $n < \omega$ we can fix $\beta_n < \omega_1$ such that

$$T \upharpoonright [\beta_n, \omega_1) \subseteq D_n.$$

Fix $\beta < \omega_1$ larger than each β_n. Then

$$T \upharpoonright [\beta, \omega_1) \subseteq \bigcap_n D_n.$$

Hence, $\bigcap_n D_n$ is dense open.

Suslin Algebras

Definition

A *Suslin algebra* is an atomless complete Boolean algebra which is c.c.c. and countably distributive.

Definition

For any normal Suslin tree T, let $\mathbb{B}_T$ denote the Boolean completion of the forcing poset $\mathbb{P}_T$.

Definition

If T is a normal Suslin tree and T is a dense suborder of an atomless complete Boolean algebra $\mathbb{B}$, we say that T is a *Suslinization* of $\mathbb{B}$.

Suslin Algebras

Theorem (Jensen, Devlin and Johnsbråten [DJ1974])

Let $\mathbb{B}$ be an atomless complete Boolean algebra of size ω_1. Then the following are equivalent:

1. *$\mathbb{B}$ is a Suslin algebra;*
2. *$\mathbb{B} = \mathbb{B}_T$ for some normal Suslin tree T (that is, $\mathbb{B}$ has a Suslinization).*

Proposition (Jensen; Devlin and Johnsbråten [DJ1974])

Let S and T be normal Suslin trees. Then the following are equivalent:

1. *$\mathbb{B}_S$ and $\mathbb{B}_T$ are isomorphic;*
2. *S and T are club isomorphic.*

Outline

Trees of Sequences

We can often represent trees of height ω_1 as trees consisting of countable sequences of natural numbers.

Lemma

Let T be an infinitely splitting tree of height ω_1 with a root and unique limits. Then T is isomorphic to a downwards closed subtree U of $({}^{<\omega_1}\omega, \subset)$ such that for all $s \in U$ and $n < \omega$, $s^\frown n \in U$.

Lemma

Let T be a binary tree of height ω_1 with a root and unique limits. Then T is isomorphic to a downwards closed subtree U of $({}^{<\omega_1}2, \subset)$ such that for all $s \in U$ and $i < 2$, $s^\frown i \in U$.

Coherent Trees

An important example of a tree of sequences is a coherent tree.

Definition

Let $2 \leq k \leq \omega$. An ω_1-tree T is *coherent* (on k) if it is a downwards closed subtree of $({}^{<\omega_1}k, \subset)$ such that for any s and t of T of the same height,

$$|\{\beta < \mathrm{dom}(s) : s(\beta) \neq t(\beta)\}| < \omega.$$

A coherent tree T (on k) is *uniform* (or *homogeneously closed*) if for all $s \in T$, if $t : \mathrm{dom}(s) \to k$ is a function and

$$|\{\alpha \in \mathrm{dom}(s) : s(\alpha) \neq t(\alpha)\}| < \omega,$$

then $t \in T$.

Coherent Trees

If $s : \mathrm{dom}(s) \to k$ is a function, where $k \leq \omega$, a *finite variation* of s (relative to k) is any function $t : \mathrm{dom}(s) \to k$ such that

$$|\{\beta \in \mathrm{dom}(s) : s(\beta) \neq t(\beta)\}| < \omega.$$

In a coherent tree, any two elements of the same height are finite variations of each other. A coherent tree is uniform iff it is closed under finite variations.

Lemma

Any uniform coherent tree is normal.

For example, if $s \in T$ and $t \in T$ is on a higher level than s, then the function $s \cup (t \restriction [\mathrm{dom}(s), \mathrm{dom}(t)))$ is a finite variation of t and hence is in T. So s has elements above it at any higher level.

Existence of Coherent Aronszajn Trees

Theorem (Todorčević [T1987], [T2007])

There exists a uniform coherent Aronszajn tree.

Proof (Sketch).

By recursion one can define a sequence of functions $\langle e_\alpha : \alpha < \omega_1 \rangle$ satisfying:

1. each $e_\alpha : \alpha \to \omega$ is injective;
2. each $\omega \setminus \mathrm{ran}(e_\alpha)$ is infinite;
3. for all $\alpha < \beta$, e_α is a finite variation of $e_\beta \restriction \alpha$.

Let T be the tree of sequences of height ω_1 so that for each $\alpha < \omega_1$, T_α is the set of all finite variations of e_α.

A natural example of a uniform coherent Aronszajn tree is $T(\rho_1)$, defined using walks on ordinals (Todorčević [T1987], [T2007]).

Existence of Coherent Suslin Trees

Theorem (Jensen; Devlin and Johnsbråten [DJ1974])

Assuming $\diamondsuit$, there exists a uniform coherent Suslin tree.

Theorem (Todorčević [T1987])

Cohen forcing $Add(\omega)$ forces that there exists a uniform coherent Suslin tree.

For the second theorem, start with a particular uniform coherent Aronszajn tree T (which exists in ZFC), and let $c : \omega \to 2$ be a Cohen real. Then

$$U := \{c \circ s : s \in T\}$$

is a uniform coherent Suslin tree.

An Application of Coherent Trees: Katětov's Problem

Theorem (Katětov [KAT1948])

If the cube of a compact topological space is completely normal, then the space is metrizable.

Katětov asked whether "cube" can be replaced by "square" in this theorem.

This question turned out to be independent of ZFC. For one direction, Nyikos [N1977] proved that under MA_{ω_1}, there is a counterexample, namely, a non-metrizable compact space whose square is completely normal. Gruenhage and Nyikos [GN1993] gave a counterexample from CH.

Katětov's Problem: Suslin's Axiom

In 2001, P. Larson and S. Todorčević introduced *Suslin's axiom*, or $\mathrm{MA}_{\omega_1}(S)$, and used it to prove the other direction of the independence result.

Definition

Suppose that S is a uniform coherent Suslin tree. Then $\mathrm{MA}_{\omega_1}(S)$ is the statement that for any c.c.c. forcing poset $\mathbb{P}$ which forces that S remains Suslin, MA_{ω_1} holds for $\mathbb{P}$.

Theorem (Larson and Todorčević [LT2002])

Assume $\mathrm{MA}_{\omega_1}(S)$ for a uniform coherent Suslin tree S. Then the Suslin tree S forces that every compact space whose square is completely normal is metrizable.

Uniform Coherent Trees Are Homogeneous

Lemma

If T is a uniform coherent ω_1-tree, then T is homogeneous.

Proof (Sketch).

Let a and b be distinct elements of T with the same height. Define $f : T_a \to T_b$ by

$$f(s) := b \cup (s \restriction [\mathrm{dom}(a), \mathrm{dom}(s))).$$

It is easy to check that f is injective and $s \subset t$ iff $f(s) \subset f(t)$. The fact that T is uniform is needed to show that f maps into T_b and that f is surjective. □

The Number of Automorphisms

Theorem (Jensen; Devlin and Johnsbråten [DJ1974])

Let T be a uniform coherent Suslin tree. Then $\sigma(T) = 2^\omega$.

Recall Jech's result which says that for a normal ω_1-tree T, if $\sigma(T)$ is infinite then $2^\omega \leq \sigma(T) \leq 2^{\omega_1}$. Any homogeneous tree T satisfies that $\sigma(T) \geq \omega_1$ (namely, we can find non-trivial automorphisms of T which are the identity on $T \upharpoonright \alpha$, for any $\alpha < \omega_1$).

Proof.

Let T be a uniform coherent Suslin tree. Then T is homogeneous, so $\sigma(T) \geq \omega_1$. By Jech's result, $\sigma(T) \geq 2^\omega$.

The Number of Automorphisms

Let us prove that $\sigma(T) \leq 2^\omega$. It suffices to show that any automorphism $h : T \to T$ is determined by its restriction to $T \restriction \beta$ for some $\beta < \omega_1$.

Claim.

For all $x \in T$ there is $y >_T x$ such that for all $z >_T y$, the value of $h(z)$ is determined by z and $h(y)$.

Suppose the claim is true. Let D be the set of such y. Then by the claim, D is dense, and it is clearly open. So there exists some $\gamma < \omega_1$ such that $T \restriction [\gamma, \omega_1) \subseteq D$. Then h is determined by the restriction of h to T_γ.

The Number of Automorphisms

To prove the claim, we will prove that for all $x \in T$ there is $y >_T x$ such that for all $z >_T y$,

$$\forall \gamma \in [\mathrm{dom}(y), \mathrm{dom}(z))\ z(\gamma) = h(z)(\gamma),$$

and hence $h(z) = h(y) \cup (z \upharpoonright [\mathrm{dom}(y), \mathrm{dom}(z)))$.

Suppose for a contradiction that x is a counterexample. For each $\alpha < \omega_1$, define $D_{x,\alpha}$ to be the set of $z \in T$ such that either z is not above x, or z is above x and for some $\gamma \geq \alpha$,

$$z(\gamma) \neq h(z)(\gamma).$$

By the choice of x, each $D_{x,\alpha}$ is dense open. Since T is Suslin, for each $\alpha < \omega_1$ we can fix some $\gamma_\alpha < \omega_1$ such that $T \upharpoonright [\gamma_\alpha, \omega_1) \subseteq D_{x,\alpha}$.

The Number of Automorphisms

Let C be the club of limit ordinals $\delta < \omega_1$ such that for all $\alpha < \delta$, $\gamma_\alpha < \delta$, and hence $T \restriction [\delta, \omega_1) \subseteq D_{x,\alpha}$.

Fix $\delta \in C$ and $z \in T_\delta$ above x. Then for all $\alpha < \delta$, $z \in D_{x,\alpha}$, so there exists some $\gamma \in (\alpha, \delta)$ such that

$$z(\gamma) \neq h(z)(\gamma).$$

It follows that z and $h(z)$ differ on an infinite set, which contradicts that T is coherent. □

Strongly Homogeneous Trees

Definition

An infinitely splitting normal ω_1-tree T is *strongly homogeneous* if there exists a collection of functions $h_{a,b}$ for any a and b of T of the same height satisfying:

1. each $h_{a,b} : T_a \to T_b$ is an isomorphism and $h_{a,b}$ is the identity if $a = b$;
2. (commutativity) $h_{b,c} \circ h_{a,b} = h_{a,c}$ for all a, b, and c in T of the same height;
3. if $h_{a,b}(c) = d$ then $h_{c,d} = h_{a,b} \restriction T_c$;
4. if c and d are distinct elements of limit height, then there are $a <_T c$ and $b <_T d$ of the same height such that $h_{a,b}(c) = d$.

Strongly Homogeneous Trees

Strongly homogeneous trees appear in Larson [L1999] and Shelah and Zapletal [SZ1999]. Larson [L1999] proved that $\diamondsuit$ implies the existence of a strongly homogeneous Suslin tree.

Note that the property of being strongly homogeneous is upwards absolute between transitive class models of ZFC with the same ω_1.

It turns out that strongly homogeneous trees and uniform coherent trees are the same.

Theorem (Bernhard König [K2003])

Let T be a normal infinitely splitting ω_1-tree. Then T is strongly homogeneous iff T is isomorphic to a uniform coherent tree (on ω).

Strongly Homogeneous and Uniform Coherent Trees

It is not hard to show that any uniform coherent tree T is strongly homogeneous using the isomorphisms $h_{a,b} : S_a \to S_b$ given by

$$h_{a,b}(s) := b \cup (s \restriction [\mathrm{dom}(a), \mathrm{dom}(s))).$$

For example, to prove property (4) consider a limit ordinal $\delta < \omega_1$ and distinct s and t in T_δ. Then s and t differ on a finite set, so we can find some $\beta < \delta$ such that

$$\forall \gamma \in [\beta, \delta)\ s(\gamma) = t(\gamma).$$

Then

$$h_{s \restriction \beta, t \restriction \beta}(s) = (t \restriction \beta) \cup (s \restriction [\beta, \delta)) = (t \restriction \beta) \cup (t \restriction [\beta, \delta)) = t.$$

The other direction is more difficult and we omit it.

Further Topics to Explore on Coherent Trees

(1) Larson [L1999] and Larson and Todorčević [LT2001] relativized Woodin's $\mathbb{P}_{\max}$ theory to create maximal models in which there exists a coherent Suslin tree.

(2) The forcing axioms MA(S) and PFA(S) have been widely studied in recent years in both set theory and topology and is a topic of contemporary interest. See Todorčević's "Forcing with a coherent Souslin tree" [TPFAS] for more information.

(3) See Chapter 4 of Todorčević's book "Walks on Ordinals and Their Characteristics" [T2007] and Todorčević's article "Lipschitz Maps on Trees" [T2007] for a detailed analysis of coherent Aronszajn trees.

(4) Justin Moore made use of coherent Aronszajn trees in his proof of the consistency of a five element basis for the class of uncountable linear orders ([M2006]).

Outline

Rigidity and Homogeneity for Suslin Trees

The topics of rigidity and homogeneity take on additional meaning for Suslin trees due to the fact that we often consider a Suslin tree as a forcing notion.

Let T be a normal Suslin tree and let a and b be elements of T with the same height. If T_a and T_b are isomorphic, then forcing a cofinal branch above a also adds a cofinal branch above b. In other words,

$$a \Vdash_{\mathbb{P}_T} \text{“}T_b \text{ is not Aronszajn.”}$$

In general, for any distinct elements a and b of T, we can ask, for example, whether any of the following are satisfied:

1. $a \Vdash_{\mathbb{P}_T}$ “T_b is Aronszajn.”
2. $a \Vdash_{\mathbb{P}_T}$ “T_b is special.”
3. $a \Vdash_{\mathbb{P}_T}$ “T_b is Suslin.”

Products of Trees

These kinds of questions can be analyzed in terms of products of trees.

Definition

For trees S and T, let $S \otimes T$ be the set of all pairs (a, b) such that for some ordinal γ, $a \in S_\gamma$ and $b \in T_\gamma$. Order $S \otimes T$ componentwise by $(a_0, b_0) <_{S \otimes T} (a_1, b_1)$ if $a_0 <_S a_1$ and $b_0 <_T b_1$.

This definition can be easily generalized from products of two trees to products of finitely many trees.

Products of Trees

Assume that S and T are normal ω_1-trees. Then:

1. $S \otimes T$ is a normal ω_1-tree;
2. $S \otimes T$ is a cofinal subset of the partial order $S \times T$ ordered componentwise;
3. If one of S or T is Aronszajn, then so is $S \otimes T$;
4. If $S \otimes T$ has no uncountable antichain, then neither do S and T, but not necessarily conversely.

Similar statements hold for any finite product of trees.

$T \otimes T$ Is Not Suslin

Theorem (Kurepa [K1950])

If T is a normal ω_1-tree, then $T \otimes T$ is not Suslin.

Proof.

By recursion define a sequence of triples

$$\langle (a_i, b_i, c_i) : i < \omega_1 \rangle,$$

where for each $i < \omega_1$, b_i and c_i are distinct immediate successors of a_i, and for all $i < j < \omega_1$, $\mathrm{ht}_T(a_i) + 1 < \mathrm{ht}_T(a_j)$. Then $\{(b_i, c_i) : i < \omega_1\}$ is an uncountable antichain of $S \otimes S$. □

It is possible, however, that certain subtrees of $T \otimes T$ are Suslin.

Derived Trees

Definition

Let T be an ω_1-tree. Let $1 \leq n < \omega$. A *derived tree of dimension n* (or an *n-derived tree* for short) is a tree of the form

$$T_{a_0} \otimes \cdots \otimes T_{a_{n-1}}$$

where $a_0, \ldots, a_{n-1}$ are distinct elements of T of the same height.

A derived tree of dimension 1 is just a tree of the form T_a where $a \in T$.

Note that T is Suslin iff every derived tree of dimension 1 is Suslin.

Free Suslin Trees

Definition

Let $1 \leq n < \omega$. A Suslin tree T is *n-free* if all of its n-derived trees are Suslin.

Definition

A Suslin tree T is *free* if for all $1 \leq n < \omega$, T is n-free.

It is easy to show that $n + 1$-free implies n-free.

In the literature, free trees go by several different names. They are also called *full trees* or just *Suslin trees whose derived trees are Suslin*.

The Existence of Free Suslin Trees

The forcings of Jech [J1967] and Tennenbaum [T1968] for adding a Suslin tree both add free Suslin trees, although this is not stated in their articles.

Theorem (Jensen; Devlin and Johnsbråten [DJ1974])

Assuming $\diamondsuit$, there exists a free Suslin tree.

This theorem is not stated explicitly in [DJ1974], but the rigid Suslin tree constructed there is free. The definition of free was introduced by Jensen in unpublished notes which came later.

Also, there is a free Suslin tree after forcing with $\mathrm{Add}(\omega)$, as we will see later.

$\Diamond$ Implies a Free Suslin Tree

Theorem (Jensen; Devlin and Johnsbråten [DJ1974])

Assuming $\Diamond$, there exists a free Suslin tree.

Proof (Sketch for the case of a 2-free Suslin tree).

Fix a diamond sequence $\langle s_\alpha : \alpha < \omega_1 \rangle$. Given any tree T with underlying set ω_1, we interpret each $s_\alpha \subseteq \alpha$ as coding:

1. a pair (a, b) of distinct elements of $T \restriction \alpha$ of the same height;
2. an antichain A_α of $(T \restriction \alpha)_a \otimes (T \restriction \alpha)_b$.

Given any pair (a, b) of distinct elements of T of the same height and any maximal antichain $A \subseteq T_a \otimes T_b$, there exists a club $C \subseteq \omega_1$ such that for all $\delta \in C$, $A \restriction \delta$ is a maximal antichain of $(T \restriction \delta)_a \otimes (T \restriction \delta)_b$.

$\Diamond$ Implies a Free Suslin Tree

The plan is to build a normal ω_1-tree T by recursion which satisfies the following property at any limit stage $\delta < \omega_1$:

Suppose that s_δ codes a maximal antichain A_δ of $(T \upharpoonright \delta)_a \otimes (T \upharpoonright \delta)_b$. Then for any pair of distinct elements c and d of T_δ such that $a <_T c$ and $b <_T d$, there is some member of A_δ below (c, d).

To see that this is enough, given any maximal antichain A of a 2-derived tree $T_a \otimes T_b$, find some $\delta < \omega_1$ such that:

1. $A \upharpoonright \delta$ is a maximal antichain of $(T \upharpoonright \delta)_a \otimes (T \upharpoonright \delta)_b$, and
2. (a, b) and $A \upharpoonright \delta$ are coded by s_δ.

Then by the above property, every member of $T_a \otimes T_b$ of height δ is above some member of $A \upharpoonright \delta$. Hence, $A \upharpoonright \delta = A$ so A is countable.

$\Diamond$ Implies a Free Suslin Tree

The base case and successor stages of the construction of T are straightforward.

At any limit stage $\delta < \omega_1$, we will choose cofinal branches $\langle b_n : n < \omega \rangle$ so that every member of $T \restriction \delta$ belongs to some b_n. Then we put a unique upper bound on level δ above each such branch.

The difference between this construction and constructions of non-free trees is that we will define all of the branches at the same time in ω many steps, so that at any given stage, we have put only finitely many elements into each b_n.

Assume that s_δ codes a pair (a, b) of distinct elements of $T \restriction \delta$ of the same height and a maximal antichain A_δ of $(T \restriction \delta)_a \otimes (T \restriction \delta)_b$.

$\Diamond$ Implies a Free Suslin Tree

Enumerate $T \restriction \delta$ as $\langle x_n : n < \omega \rangle$. Fix $\langle \delta_n : n < \omega \rangle$ cofinal in δ. Fix a bijection $f : \omega \to \omega \times \omega$.

At stage stage 0 of the construction, we put x_n into b_n for each $n < \omega$.

Suppose $n < \omega$ and we have completed stages 0 through n. We now describe stage $n + 1$. Let $f(n) = (k, m)$. At this stage we consider the branches b_k and b_m. If $k = m$, then do nothing, so assume that $k \neq m$.

Let y and z be the last elements that have been added to b_k and b_m in stages 0 through n of the construction.

First, choose y' and z' above y and z in $T \restriction \delta$ with the same height, where that height is at least δ_n. Put y' in b_k and z' in b_m.

$\diamondsuit$ Implies a Free Suslin Tree

If (y', z') is not in $(T \restriction \delta)_a \otimes (T \restriction \delta)_b$, then we are done with stage $n+1$.

Suppose that (y', z') is in $(T \restriction \delta)_a \otimes (T \restriction \delta)_b$. Using the fact that A_δ is a maximal antichain of $(T \restriction \delta)_a \otimes (T \restriction \delta)_b$, we can find some (y^+, z^+) above (y', z') which is also above some member of A_δ. Now put y^+ in b_k and z^+ in b_m.

This completes the construction. Now for each pair (c, d) of height δ such that $a <_T c$ and $b <_T d$, c and d are the unique upper bounds of some branches b_k and b_m. Fix n such that $f(n) = (k, m)$. Then at stage $n+1$ we put something below (c, d) which is in A_δ as required. $\square$

When the Product of Suslin Trees Is a Suslin Tree

Theorem (Product Lemma)

Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are c.c.c. forcing posets. Then the following are equivalent:

1. $\mathbb{P} \times \mathbb{Q}$ *is c.c.c.;*
2. $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ *is c.c.c."*

Suppose that S and T are normal Suslin trees. Then $\mathbb{P}_S$ and $\mathbb{P}_T$ are c.c.c. forcings.

Since $S \otimes T$ is a cofinal subset of $S \times T$, $\mathbb{P}_{S \otimes T}$ is a dense subset of $\mathbb{P}_S \times \mathbb{P}_T$. So $\mathbb{P}_{S \otimes T}$ is c.c.c. iff $\mathbb{P}_S \times \mathbb{P}_T$ is c.c.c.

When the Product of Suslin Trees Is a Suslin Tree

Theorem

Let S and T be normal Suslin trees. Then the following are equivalent:

1. *$S \otimes T$ is a Suslin tree;*
2. $\Vdash_{\mathbb{P}_S}$ *"T is Suslin";*
3. $\Vdash_{\mathbb{P}_T}$ *"S is Suslin".*

Proof.

$$S \otimes T \text{ is Suslin } \iff \mathbb{P}_{S\otimes T} \text{ is c.c.c. } \iff \mathbb{P}_S \times \mathbb{P}_T \text{ is c.c.c.}$$
$$\iff \Vdash_{\mathbb{P}_S} \text{“}\mathbb{P}_T \text{ is c.c.c.”} \iff \Vdash_{\mathbb{P}_S} \text{“}T \text{ is Suslin”}.$$

This proves (1) iff (2). For (1) iff (3), $S \otimes T$ is Suslin iff $T \otimes S$ is Suslin because these trees are isomorphic. □

Freeness in Terms of Forcing

In particular:

Theorem

Let T be a normal Suslin tree. Then the following are equivalent:

1. *T is 2-free;*
2. *For any distinct elements a and b of T with the same height, $a \Vdash_{\mathbb{P}_T}$ "T_b is Suslin".*

In fact, if T is 2-free, then (2) is true for any incomparable a and b in T.

Similar results hold for n-free and free Suslin trees. For example, T is 3-free iff for any distinct elements a, b, and c of T with the same height, $a \Vdash_{\mathbb{P}_T} T_b \otimes T_c$ is Suslin.

2-Free Suslin Trees Are Totally Rigid

Corollary

Let T be a normal Suslin tree. If T is 2-free, then T is totally rigid.

Namely, for any distinct elements a and b of T with the same height, T_a and T_b are not isomorphic, otherwise forcing with $\mathbb{P}_T$ below a would add a cofinal branch to T_b. In fact, being 2-free implies an even stronger form of rigidity.

Proposition

Suppose that T is a normal Suslin tree, a and b are elements of T with the same height, and there exists a strictly increasing function defined on an uncountable subset of T_a mapping into T_b. Then T is not 2-free.

In particular, if T is a normal 2-free Suslin tree, then for any club $C \subseteq \omega_1$, $T \upharpoonright C$ is rigid.

Rigid Suslin Algebras

Theorem

Let $\mathbb{B}$ be a Suslin algebra. Then the following are equivalent:

1. *$\mathbb{B}$ is rigid (has no non-trivial automorphisms);*
2. *for every Suslinization T of $\mathbb{B}$, for every club $C \subseteq \omega_1$, $T \upharpoonright C$ is rigid.*

Corollary

If T is a normal 2-free Suslin tree then $\mathbb{B}_T$ is rigid.

2-Free Trees Are Forcing Minimal

Theorem

Let T be a normal 2-free Suslin tree. Then $\mathbb{P}_T$ is forcing minimal. That is, if G is a generic filter on $\mathbb{P}_T$, then G is minimal over the ground model in the sense that for any transitive model M of ZFC with

$$V \subseteq M \subseteq V[G],$$

either $M = V$ or $M = V[G]$.

For a very rough sketch of the proof, suppose that $\dot{A}$ is a $\mathbb{P}_T$-name for a set of ordinals not in the ground model. Using 2-freeness, we can prove that for unboundedly many levels $\delta < \omega_1$, any two elements x and y of T_δ disagree about whether or not some ordinal is in $\dot{A}$. Thus, for any generic branch G, if we know $\dot{A}^G$, we can recover G. (We will see more details of similar arguments later.)

Simple Boolean Algebras

Definition

An atomless complete Boolean algebra is *simple* if it has no atomless complete subalgebras other than itself and the trivial subalgebra.

Theorem (McAloon (1971))

Any atomless complete Boolean algebra is simple iff it is rigid and forcing minimal.

Corollary

Suppose T is a normal 2-free Suslin tree. Then $\mathbb{B}_T$ is simple.

Entangled Sets of Reals

We will give another characterization of freeness similar to entangled sets of reals.

Definition (Abraham and Shelah [AS1981])

Let $1 \leq n < \omega$. An uncountable set of reals $X \subseteq \mathbb{R}$ is *n-entangled* if whenever

$$\{(a_{\beta,0}, \ldots, a_{\beta,n-1}) : \beta < \omega_1\}$$

is a family of pairwise disjoint injective n-tuples from X and $h : n \to 2$ is a function, then there exist $\beta < \gamma < \omega_1$ such that for all $i < n$,

$$a_{\beta,i} < a_{\gamma,i} \iff h(i) = 1.$$

A set of reals X is *entangled* if it is n-entangled for all $1 \leq n < \omega$.

Entangled Sets of Reals and Rigidity

Theorem (Abraham and Shelah [AS1981])

CH implies the existence of an entangled set of reals. MA_{ω_1} implies that every uncountable set of reals is not n-entangled for some n.

Entangledness for sets of reals is a form of rigidity. For example, if X is 2-entanged, then for any disjoint uncountable sets $A, B \subseteq X$, A and B are not isomorphic.

In particular, if there exists a 2-entangled set of reals, then not all ω_1-dense sets of reals are isomorphic (that is, Baumgartner's axiom fails).

Freeness and Entangledness

Freeness for Suslin trees is equivalent to a property similar to entangledness for sets of reals.

Theorem (K. (2020))

Let T be a Suslin tree and $1 \leq n < \omega$. Then T is n-free iff for any n-derived tree $U = T_{c_0} \otimes \cdots \otimes T_{c_{n-1}}$, whenever

$$\{(a_{\beta,0}, \ldots, a_{\beta,n-1}) : \beta < \omega_1\}$$

is a family of disjoint members of U and $h : n \to 2$ is a function, then there exist $\beta < \gamma < \omega_1$ such that for all $i < n$,

$$a_{\beta,i} <_T a_{\gamma,i} \iff h(i) = 1.$$

Strongly Homogeneous and Free Trees

Free Suslin trees and strongly homogeneous Suslin trees are on opposite sides of the spectrum with regards to homogeneity and rigidity. So the following result is surprising.

Theorem (Larson [L1999])

Suppose that there exists a strongly homogeneous Suslin tree T. Then there exists a free Suslin tree S, a club $C \subseteq \omega_1$, and a strictly increasing surjection $f : T \restriction C \to S \restriction C$.

So if there exists a uniform coherent Suslin tree, then there exists a free Suslin tree. (We will see later that the converse of this statement is false in general.) In particular, Cohen forcing $\mathrm{Add}(\omega)$ adds a free Suslin tree.

Strongly Homogeneous and Free Trees

Larson's proof was generalized by Gido Scharfenberger-Fabian (using his concept of optimal matrices of partitions) as follows.

Theorem (G. Sharfenberger-Fabian [SF2010])

Suppose that T is a strongly homogeneous Suslin tree. Let $2 \leq n < \omega$. Then there exist free Suslin trees $S_0, \ldots, S_{n-1}$ such that

$$T \cong S_0 \otimes \cdots \otimes S_{n-1}.$$

n-Free Does Not Imply $n+1$-Free

Scharfenberger-Fabian proved that the property n-free does not imply $n+1$-free for any $1 \leq n < \omega$, which answered a problem of Fuchs and Hamkins [FH2009].

Theorem (G. Sharfenberger-Fabian [SF2010])

Suppose that there exists a strongly homogeneous Suslin tree. Then for every $1 \leq n < \omega$, there exists an n-free Suslin tree which is not $n+1$-free.

Theorem (K. [K2022])

Suppose that T is a normal free Suslin tree. Then for every $1 \leq n < \omega$, there exists a c.c.c. forcing poset which forces that T is n-free but every $n+1$-derived tree is special.

Self-Specializing Suslin Trees

The main types of Suslin trees we have considered so far are strongly homogeneous and free trees. Let us consider a third type of tree.

Definition

A Suslin tree T is *self-specializing* if $\mathbb{P}_T$ forces that $T \setminus \dot{G}$ is special.

According to Shelah and Zapletal [SZ1999] (stated without proof), a self-specializing Suslin tree exists assuming $\Diamond$ and after forcing ω_1 many Cohen reals.

Self-specializing trees are defined in terms of forcing, but there is a natural combinatorial property which implies self-specializing.

The Property That All 2-Derived Trees Are Special

Lemma

Suppose that T is a normal Suslin tree. If every 2-derived tree of T is special, then T is self-specializing.

Proof.

For every pair a and b of distinct elements of T with the same height, fix a specializing function $f_{a,b} : T_a \otimes T_b \to \omega$. Let G be a generic filter on $\mathbb{P}_T$. Define $f : T \setminus G \to \omega$ as follows. For any $a \in T \setminus G$, let β_a be the least ordinal such that $a \restriction \beta_a \notin G$. Define $f(a) = f_{G(\beta_a), a \restriction \beta_a}(G(\mathrm{ht}_T(a)), a)$. If $a <_T b$ are in $T \setminus G$, then $\beta_a = \beta_b$ and $a \restriction \beta_a = b \restriction \beta_b$. So

$$f(a) = f_{G(\beta_a), a \restriction \beta_a}(G(\mathrm{ht}_T(a)), a) \neq f_{G(\beta_b), b \restriction \beta_b}(G(\mathrm{ht}_T(b)), b).$$

□

Self-Specializing Suslin Trees

In particular, if T is 2-free, then there exists a c.c.c. forcing extension in which T is a self-specializing Suslin tree.

For a partial converse to the lemma, if T is a Suslin tree which forces that there exists a strictly increasing and continuous map of $T \setminus \dot{G}$ into $\mathbb{Q}$, then every 2-derived tree of T is special.

In general, if T is a self-specializing Suslin tree, then every 2-derived tree is $\mathbb{R}$-embeddable.

Self-Specializing Suslin Trees

Lemma

Self-specializing Suslin trees are totally rigid and not 2-free.

Proof.

Suppose that $\mathbb{P}_T$ forces that $T \setminus \dot{G}$ is special. Then $\mathbb{P}_T$ forces that $T \setminus \dot{G}$ does not have a cofinal branch. This implies that T is totally rigid.

If a and b are distinct elements of T of the same height, then a forces in $\mathbb{P}_T$ that $T_b \subseteq T \setminus \dot{G}$, and hence that T_b is special. So a forces in $\mathbb{P}_T$ that T_b is not Suslin. Therefore, $T_a \otimes T_b$ is not Suslin. □

The Unique Branch Property

Definition

A normal Suslin tree T has the *unique branch property* (or UBP) if forcing with T produces exactly one cofinal branch, in other words,

$$\Vdash_{\mathbb{P}_T} \text{"}T \setminus \dot{G} \text{ is Aronszajn."}$$

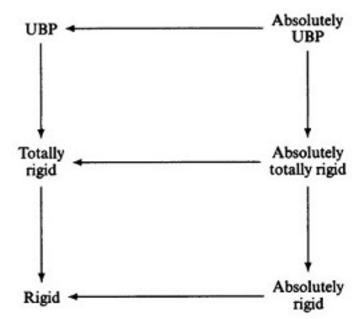
Clearly, 2-free Suslin trees and self-specializing Suslin trees have the unique branch property. The unique branch property is a form of rigidity.

For any property P of a tree, we call a Suslin tree T *absolutely P* if T satisfies P and $\mathbb{P}_T$ forces that $T \setminus \dot{G}$ has property P.

Separating Degrees of Rigidity

Theorem (Fuchs and Hamkins [FH2009])

The following implication diagram is complete, in the sense that no other implications are provable in ZFC other than the ones explicitly displayed in the diagram. Counterexamples to the missing implications all exist assuming $\Diamond$.



Outline

Outline

Goal

The goal for the remainder of the talks is to construct models of ZFC by forcing satisfying statements of the form:

1. There exist Suslin trees, but not many different ones, and
2. Every nowhere Suslin tree is special, or there are not many different nowhere Suslin trees.

"Different" here refers to not being club isomorphic. For example, in these models the number of club-isomorphism-types of different kinds of trees is small or can be described precisely.

Goal

The first theorem of this type was proven by Abraham and Shelah [AS1985] using Jensen's method of producing a model of CH $+ \neg$SH.

Theorem (Abraham and Shelah [AS1985])

Assume $V = L$ (or just $\diamondsuit^*$, $\square_{\omega_1}$, and GCH). Then there exists a forcing extension satisfying:

1. *GCH;*
2. *there exists a free Suslin tree R and a special Aronszajn tree U;*
3. *for every normal Suslin tree S, there exists a derived tree of R which club embeds into S;*
4. *every normal nowhere Suslin Aronszajn tree club embeds into U, and hence is special.*

We will give a complete proof of an easier variation of this theorem.

Strictly Increasing and Level Preserving Maps

Definition

Let S and T be trees and $f : S \to T$.

1. f is *strictly increasing* if $x <_S y$ implies $f(x) <_T f(y)$ for all x and y in S;
2. f is *level preserving* if $\mathrm{ht}_T(f(x)) = \mathrm{ht}_S(x)$ for all $x \in S$.

If $f : S \to T$ is strictly increasing, we can define a level preserving map $f^* : S \to T$ by

$$f^*(x) := f(x) \upharpoonright \mathrm{ht}_S(x).$$

This definition makes sense because f being strictly increasing implies that $\mathrm{ht}_T(f(x)) \geq \mathrm{ht}_S(x)$. It is easy to check that f^* is strictly increasing. So oftentimes without loss of generality we can assume that the strictly increasing maps we are working with are level preserving.

Suslin Subtrees

Lemma

Let S be a Suslin tree, T an Aronszajn tree, and suppose that there exists a strictly increasing map $f : S \to T$. Then T contains a Suslin subtree (in other words, T is not nowhere Suslin).

Proof.

Let $U := f[S]$ be the range of f. Then U is an uncountable subtree of T. Note that U cannot have an uncountable branch because T is Aronszajn.

If $A \subseteq U$ is an antichain of U, then the fact that f is strictly increasing implies that $f^{-1}(A)$ is an antichain of S. Since S is Suslin, $f^{-1}(A)$ is countable, so A is countable. □

Embeddings

Another type of map between trees is an embedding, which is an isomorphism between one tree and a subtree of another.

Definition

Let S and T be trees. A function $f : S \to T$ is an *embedding* if f is injective and for all x and y in S,

$$x <_S y \iff f(x) <_T f(y).$$

Conflict in terminology: We previously talked about $\mathbb{Q}$-embeddable and $\mathbb{R}$-embeddable trees. In our current terminology, those words indicate strictly increasing maps into $\mathbb{Q}$ and $\mathbb{R}$, not embeddings.

Strictly Increasing Maps Versus Embeddings

In general, strictly increasing maps are not necessarily embeddings.

For example, if T is a strongly homogeneous Suslin tree, then there exists a surjective strictly increasing map of T onto a free Suslin tree. But a strongly homogeneous Suslin tree cannot be isomorphic to a free tree.

The issue distinguishing strictly increasing maps from embeddings is injectivity.

Strictly Increasing Maps Versus Embeddings

Lemma

Suppose that S and T are normal ω_1-trees and $f : S \to T$ is strictly increasing. Then f is an embedding iff f is injective.

Proof.

($\Rightarrow$) Immediate.
($\Leftarrow$) We need to show that for all x and y in S, if $f(x) <_T f(y)$ then $x <_S y$. Suppose that $f(x) <_T f(y)$ but $x \not<_S y$. Let $x' := y \restriction \mathrm{ht}_S(x)$. Then $x' \neq x$.

Now $f(x') <_T f(y)$ since f is strictly increasing. Since f is injective, $f(x') \neq f(x)$. But then $f(x')$ and $f(x)$ are both below $f(y)$, and hence they are equal because T is a tree, which is a contradiction. □

Club Embeddings

Definition

Let S and T be normal ω_1-trees. A *club embedding* of S into T is an embedding of the form $f : S \restriction C \to T$ for some club $C \subseteq \omega_1$. If there exists such a function, then *S club embeds into T*.

The following is easy to check.

Lemma

Let S and T be normal ω_1-trees. If S club embeds into T, then there exists a club $D \subseteq \omega_1$ and a club embedding $g : S \restriction D \to T \restriction D$ which is level preserving.

Strictly Increasing Maps and Adding a Cofinal Branch

Proposition (Essentially Abraham and Shelah [AS1985])

Let S be a normal Suslin tree and T a normal Aronszajn tree. Then the following are equivalent:

1. $\Vdash_{\mathbb{P}_S}$ *"T has a cofinal branch";*
2. *there exists a club $C \subseteq \omega_1$ and a strictly increasing and level preserving map $f : S \restriction C \to T \restriction C$;*
3. *there exists an unbounded set $X \subseteq \omega_1$ and a strictly increasing map $g : S \restriction X \to T$.*

Proof.

(2 implies 3) and (3 implies 1) are easy.

Strictly Increasing Maps and Adding a Cofinal Branch

(1 implies 2) Suppose that S forces that $\dot{b}$ is a cofinal branch of T. Note that if $x \in S$ decides $\dot{b}(\alpha)$, then x decides $\dot{b}(\beta)$ for all $\beta < \alpha$.

We claim that if $\delta < \omega_1$ is a limit ordinal and $x \in S$ decides $\dot{b} \upharpoonright \delta$, then x decides $\dot{b}(\delta)$. Otherwise there are distinct y and z in S above x in S and distinct elements c and d in T_δ such that $y \Vdash_{\mathbb{P}_S}$ "$\dot{b}(\delta) = c$" and $z \Vdash_{\mathbb{P}_S}$ "$\dot{b}(\delta) = d$.".

But then c and d have the same set of predecessors in T, namely the set

$$\{z \in T : \exists \gamma < \delta \ x \Vdash_{\mathbb{P}_S} \text{“}\dot{b}(\gamma) = z\text{”}\}.$$

This contradicts that T has unique limits since it is normal.

Strictly Increasing Maps and Adding a Cofinal Branch

For each $\alpha < \omega_1$, let D_α be the dense open set of $x \in S$ such that x decides $\dot{b}(\alpha)$. By the previous claim, if $\delta < \omega_1$ is a limit ordinal and $x \in \bigcap\{D_\alpha : \alpha < \delta\}$, then $x \in D_\delta$.

Since S is a Suslin tree, for each $\alpha < \omega_1$ we can fix $\gamma_\alpha < \omega_1$ such that

$$S \restriction [\gamma_\alpha, \omega) \subseteq D_\alpha.$$

So every member of S with height at least γ_α decides $\dot{b}(\beta)$ for all $\beta \leq \alpha$.

Let C be the club of limit ordinals $\delta < \omega_1$ such that for all $\alpha < \delta$, $\gamma_\alpha < \delta$.

Strictly Increasing Maps and Adding a Cofinal Branch

Consider $\delta \in C$ and $x \in S \restriction C$. For all $\alpha < \delta$, $\gamma_\alpha < \delta$, so $x \in D_\alpha$. So $x \in \bigcap\{D_\alpha : \alpha < \delta\}$, and therefore $x \in D_\delta$.

Define $f : S \restriction C \to T \restriction C$ as follows. Let $\delta \in C$ and $x \in T_\delta$. Define $f(x) := c$ where c is the unique element of T_δ such that $x \Vdash_{\mathbb{P}_S}$ "$\dot{b}(\delta) = c$."

If $x <_S y$ in $T \restriction C$, then y forces that $f(x)$ and $f(y)$ are both in $\dot{b}$, and hence that $f(x) <_T f(y)$. So f is strictly increasing and level preserving. □

Strictly Increasing Maps

Lemma

Let S and T be normal ω_1-trees, where T is Aronszajn, and suppose that $f : S \to T$ strictly increasing and level preserving. Then for all distinct elements x and y of S with the same height, there exist x' and y' in S with the same height such that $x <_S x'$, $y <_S y'$ and $f(x') \neq f(y')$.

Proof.

Suppose for a contradiction that x and y in S are a counterexample. Note that for all x_0 and x_1 in S above x of the same height, $f(x_0) = f(x_1)$. Namely, by the normality of S fix y_0 above y of the same height as x_0 and x_1. Then by the choice of x and y, $f(x_0) = f(y_0) = f(x_1)$.

Strictly Increasing Maps

Let ξ be the height of x and y. For all $\beta < \omega_1$ above ξ, let z_β be the unique value of $f(x_0)$ for any $x_0 >_S x$ with height β. This is possible since S is normal.

We claim that $\{z_\beta : \xi < \beta < \omega_1\}$ is a chain, which contradicts that T is Aronszajn.

Consider $\beta < \gamma < \omega_1$ above ξ. By the normality of S, we can fix $x_\beta \in S_\beta$ above x and $x_\gamma \in S_\gamma$ above x_β. Since f is strictly increasing, $z_\beta = f(x_\beta) <_T f(x_\gamma) = z_\gamma$. □

Going From Strictly Increasing Maps to Embeddings

Lemma

Let S be a normal Suslin tree and let T be a normal Aronszajn tree. Suppose that $f : S \to T$ is strictly increasing and level preserving. If there exists an unbounded set $A \subseteq \omega_1$ such that f is injective on $S \restriction A$, then there exists a club $C \subseteq \omega_1$ such that $f : S \restriction C \to T \restriction C$ is a level preserving embedding.

Proof.

Let $C := A \cup \lim(A)$. It suffices to show that for all $\delta \in \lim(A)$, f is injective on S_δ. Consider distinct x and y in S_δ. Since S is normal and δ is a limit point of A, fix $\beta \in A \cap \delta$ such that $x \restriction \beta$ and $y \restriction \beta$ are distinct. Then $f(x \restriction \beta) \neq f(y \restriction \beta)$, which implies that $f(x) \neq f(y)$. □

Strictly Increasing Maps on 2-Free Trees Are Embeddings

Theorem (Abraham and Shelah [AS1985])

Suppose that S is a normal 2-free Suslin tree and T is a normal Aronszajn tree. Assume that $f : S \to T$ is strictly increasing and level preserving. Then there exists a club $C \subseteq \omega_1$ such that $f : S \restriction C \to T \restriction C$ is an embedding. So S club embeds into T.

Contrast this with the situation of a strongly homogeneous tree which has a strictly increasing map onto a free tree, but does not club embed into the free tree.

Strictly Increasing Maps on 2-Free Trees Are Embeddings

Proof.

Let $f : S \to T$ be strictly increasing and level preserving. By the previous lemma, it suffices to show that there are unboundedly many $\delta < \omega_1$ such that f is injective on S_δ.

Suppose for a contradiction that there exists some $\xi < \omega_1$ such that for all α with $\xi < \alpha < \omega_1$, f is not injective on S_α. For each such α, choose distinct elements x_α and y_α of S_α such that $f(x_\alpha) = f(y_\alpha)$.

Strictly Increasing Maps on 2-Free Trees Are Embeddings

By the pressing down lemma, fix a stationary set $X \subseteq \omega_1$ of ordinals greater than ξ and distinct elements x and y of S_ξ such that for all $\alpha \in X$, $x_\alpha \upharpoonright \xi = x$ and $y_\alpha \upharpoonright \xi = y$.

Let D be the set of (c, d) in $S_x \otimes S_y$ such that $f(c) \neq f(d)$. By a previous lemma, D is dense open in $S_x \otimes S_y$.

Since S is 2-free, $S_x \otimes S_y$ is Suslin. So we can find some $\delta < \omega_1$ such that every member of $S_x \otimes S_y$ with height at least δ is in D. Now choose some $\alpha \in X$ with $\alpha \geq \delta$. Then $(x_\alpha, y_\alpha) \in D$, so $f(x_\alpha) \neq f(y_\alpha)$, contradicting the choice of x_α and y_α. □

Strictly Increasing Maps on 2-Free Trees Are Embeddings

Corollary

Let S be a normal 2-free Suslin tree and let T be a normal Aronszajn tree. Suppose that

$$\Vdash_{\mathbb{P}_S} \text{“}T \text{ has a cofinal branch.”}$$

Then S club embeds into T.

Proof.

By a previous lemma, there exists a club $C \subseteq \omega_1$ and a strictly increasing and level preserving map $f : S \upharpoonright C \to T \upharpoonright C$. Now apply the previous theorem to f, $S \upharpoonright C$, and $T \upharpoonright C$. □

A Generalization to Derived Trees

A similar but more elaborate argument proves the following theorem.

Theorem (Abraham and Shelah [AS1985])

Let S be a normal Suslin tree and let T be a normal Aronszajn tree. Suppose that for some derived tree S^* of S, there is a strictly increasing map from S^* into T, and moreover, S^* has the smallest dimension of any such derived tree.

Let n be the dimension of S^*. Assume that S is $2n$-free.

Then for any strictly increasing and level preserving map $f : S^* \to T$, there exists a club $C \subseteq \omega_1$ such that $f : S^* \restriction C \to T \restriction C$ is a level preserving embedding.

A Generalization to Derived Trees

Corollary

Let S be normal Suslin tree and let T be a normal Aronszajn tree. Suppose that $n > 0$, S^* is an n-derived tree of S,

$$\Vdash_{\mathbb{P}_{S^*}} \text{“}T \text{ has a cofinal branch,”}$$

and moreover, n is the least positive integer for which there exists such a derived tree. Assume that S is $2n$-free. Then S^* club embeds into T.

Outline

R and $\mathcal{D}_R$

For the remainder of the talks, R will denote a normal free Suslin tree and $\mathcal{D}_R$ will denote the collection of all derived trees of R.

Going forward we will adopt the abbreviation of writing T for the forcing poset $\mathbb{P}_T$. For example, we will talk about dense open subsets of T instead of $\mathbb{P}_T$, and write $\Vdash_T$ instead of $\Vdash_{\mathbb{P}_T}$.

R and $\mathcal{D}_R$

Lemma

$\mathcal{D}_R$ is closed under the operation which maps $T \in \mathcal{D}_R$ and $x \in T$ to T_x.

Proof.

If $T \in \mathcal{D}_R$ then for some $n > 0$ and distinct $a_0, \dots, a_{n-1}$ of R of the same height,

$$T = R_{a_0} \otimes \cdots \otimes R_{a_{n-1}}.$$

Consider $x \in T$. Then $x = (b_0, \dots, b_{n-1})$ for some $b_0, \dots, b_n$ in R of the same height such that $a_0 \leq_R b_0, \dots, a_{n-1} \leq_R b_{n-1}$. Then

$$T_x = T_{(b_0,\dots,b_n)} = R_{b_0} \otimes \cdots \otimes R_{b_{n-1}} \in \mathcal{D}_R.$$

□

R and $\mathcal{D}_R$

Corollary

Let $T \in \mathcal{D}_R$ and let U be a normal Aronszajn tree. Suppose that $x \in T$ and

$$x \Vdash_T \text{“}U \text{ is not Aronszajn.”}$$

Then there exists some $T^* \in \mathcal{D}_R$ such that

$$\Vdash_{T^*} \text{“}U \text{ has a cofinal branch.”}$$

Proof.

If $x \Vdash_T$ “U has a cofinal branch” then $\Vdash_{T_x}$ “U has a cofinal branch.” But $T_x \in \mathcal{D}_R$. □

Disjoint Unions of Suslin Trees

Suppose that $\langle T_n : n < \omega \rangle$ is a sequence of normal Suslin trees. Then we can form a tree by taking a disjoint union

$$\bigsqcup_n T_n,$$

which is essentially the tree consisting of the T_n's placed side-by-side.

Formally, this tree has underlying set equal to $\bigcup_n \{n\} \times T_n$ together with a root 0, and ordered by $(k, x) <_{\bigsqcup_n T_n} (m, y)$ if $k = m$ and $x <_{T_m} y$.

It is easy to prove that $\bigsqcup_n T_n$ is also a normal Suslin tree.

Disjoint Unions of Derived Trees

In any model in which R is a normal free Suslin tree, the following trees will also be normal Suslin trees:

1. R;
2. Every tree in $\mathcal{D}_R$;
3. Every countable disjoint union of trees in $\mathcal{D}_R$.

Note that there are trees as described in (3) which are not club isomorphic to trees in $\mathcal{D}_R$. For example, if we choose T_n to be an n-derived tree of R for each $1 \leq n < \omega$, then $\bigsqcup_n T_n$ is not itself club isomorphic to a derived tree of R.

In fact, if X and Y are distinct infinite subsets of ω, then $\bigsqcup_{n \in X} T_n$ and $\bigsqcup_{n \in Y} T_n$ are not club isomorphic. So there are 2^ω many distinct club-isomorphism-types of normal Suslin trees.

Does There Have to Be a Free Suslin Tree?

We will prove that it is consistent that every normal Suslin tree is club isomorphic to either a derived tree of R or a countable disjoint union of derived trees of R.

This is the minimum collection of club-isomorphism-types of normal Suslin trees in the presence of a free Suslin tree.

Question (Shelah and Zapletal [SZ1999])

Is it consistent that there exists a Suslin tree, but there does not exist a free Suslin tree?

Specializing an Aronszajn Tree

Definition

Let T be a tree of height ω_1. Define $\mathbb{Q}(T)$ to be the forcing poset whose conditions are finite functions $p : \mathrm{dom}(p) \subseteq T \to \omega$ such that $x <_T y$ in $\mathrm{dom}(p)$ implies $p(x) \neq p(y)$, ordered by reverse inclusion.

Theorem (Baumgartner [B1970])

Let T be a tree of height ω_1. Then T has no cofinal branch iff $\mathbb{Q}(T)$ is c.c.c. In that case, $\mathbb{Q}(T)$ forces that T is special.

Note that since $\mathbb{Q}(T)$ is defined by finite conditions, it is absolute between inner models of ZFC with the same ω_1.

Preserving a Suslin Tree

Theorem

Let S be a normal Suslin tree and U a tree of height ω_1 with no cofinal branch. Then

$$\Vdash_S \text{"}U \text{ has no cofinal branch"} \iff \Vdash_{\mathbb{Q}(U)} \text{"}S \text{ is Suslin."}$$

Proof.

By the product lemma,

$$\Vdash_{\mathbb{Q}(U)} \text{"}S \text{ is Suslin"} \iff \Vdash_{\mathbb{Q}(U)} \text{"}S \text{ is c.c.c."} \iff$$
$$\mathbb{Q}(U) \times S \text{ is c.c.c.} \iff S \times \mathbb{Q}(U) \text{ is c.c.c.} \iff$$
$$\Vdash_S \text{"}\mathbb{Q}(U)^V \text{ is c.c.c."} \iff \Vdash_S \text{"}\mathbb{Q}(U)^{V^S} \text{ is c.c.c."}$$
$$\iff \Vdash_S \text{"}U \text{ has no cofinal branch."}$$

□

Preserving a Suslin Tree

Corollary

Let U be a normal Aronszajn tree, and suppose that for every $T \in \mathcal{D}_R$,

$$\Vdash_T \text{“}U \text{ is Aronszajn.”}$$

Then

$$\Vdash_{\mathbb{Q}(U)} \text{“}R \text{ is a free Suslin tree.”}$$

The same result is true if we replace $\mathbb{Q}(U)$ with Shelah's proper forcing poset for specializing U without adding reals. But the argument is much more complex.

Finite Support Forcing Iterations Preserve Suslin Trees

Theorem

Let T be a Suslin tree. Then the property of a forcing poset that it is c.c.c. and forces that T is Suslin is preserved by any finite support forcing iteration.

Proof.

We will prove the theorem by induction on the length of the iteration. The successor case is immediate. So let δ be a limit ordinal and consider a finite support forcing iteration

$$\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\gamma : \beta \leq \delta, \gamma < \delta \rangle$$

of c.c.c. forcings, where we assume that for all $\beta < \delta$, $\mathbb{P}_\beta$ forces that T is Suslin. We will prove that $\mathbb{P}_\delta$ forces that T is Suslin.

Finite Support Forcing Iterations Preserve Suslin Trees

Suppose for a contradiction that $q \in \mathbb{P}_\delta$ and

$$q \Vdash_\delta \text{“}\dot{A} = \{\dot{x}_i : i < \omega_1\} \text{ is an uncountable antichain of } T.\text{”}$$

For each $i < \omega_1$, fix $q_i \leq_\delta q$ and $x_i \in T$ such that $q_i \Vdash_\delta$ “$\dot{x}_i = x_i$.” Note that if $i < j < \omega_1$ and q_i and q_j are compatible in $\mathbb{P}_\delta$, then x_i and x_j are incomparable in T.

Applying the Δ-system lemma to the domains of the q_i's, fix an uncountable set $X \subseteq \omega_1$, an ordinal $\beta < \delta$, and a finite set $r \subseteq \beta$ such that for all $i < j$ in X, $\mathrm{dom}(q_i) \cap \mathrm{dom}(q_j) = r$.

Note that for all $i < j$ in X, q_i and q_j are compatible in $\mathbb{P}_\delta$ iff $q_i \restriction \beta$ and $q_j \restriction \beta$ are compatible in $\mathbb{P}_\beta$.

Finite Support Forcing Iterations Preserve Suslin Trees

Since $\mathbb{P}_\beta$ is c.c.c., by a standard argument we can find $r \leq_\beta q \restriction \beta$ such that

$$r \Vdash_\beta \text{“}\dot{Z} := \{i < \omega_1 : q_i \restriction \beta \in \dot{G}_\beta\} \text{ is uncountable.”}$$

Let H be a generic filter on $\mathbb{P}_\beta$ with $r \in H$. Let $Z := \dot{Z}^H$.

Working in $V[H]$, consider $i < j$ in Z. Then $q_i \restriction \beta$ and $q_j \restriction \beta$ are in H and so are compatible in $\mathbb{P}_\beta$. Therefore, q_i and q_j are compatible in $\mathbb{P}_\delta$. Hence, x_i and x_j are incomparable in T.

If follows that $\{x_i : i \in Z\}$ is an uncountable antichain of T in $V[H]$, which contradicts the inductive hypothesis which implies that $\mathbb{P}_\beta$ forces that T is Susin. □

A Consistency Result on Few Suslin Trees

Theorem

Assume that R is a normal free Suslin tree and GCH holds. Then there exists a c.c.c. forcing poset which forces:

1. $2^\omega = 2^{\omega_1} = \omega_2$;
2. *R is a free Suslin tree;*
3. *for every normal Suslin tree S, there exists a derived tree of R which club embeds into S;*
4. *for every normal Aronszajn tree U, if U does not contain a Suslin subtree, then U is special.*

A Consistency Result on Few Suslin Trees

Proof.

We define by recursion a finite support forcing iteration

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2,\ \beta < \omega_2 \rangle$$

of c.c.c. forcings. Our inductive hypothesis is that each $\mathbb{P}_\alpha$ forces that R is a free Suslin tree.

At stage $\alpha < \omega_2$, we consider a $\mathbb{P}_\alpha$-name $\dot{U}_\alpha$ for a normal Aronszajn tree satisfying that for every derived tree T of R,

$$\Vdash_{\mathbb{P}_\alpha * T} \text{“}\dot{U}_\alpha \text{ is Aronszajn.”}$$

Let $\dot{\mathbb{Q}}_\alpha := \mathbb{Q}(\dot{U}_\alpha)$. Then in $V^{\mathbb{P}_\alpha}$, for every derived tree T of R, $\dot{\mathbb{Q}}_\alpha$ forces that T is Suslin, and hence $\dot{\mathbb{Q}}(\alpha)$ forces that R is a free Suslin tree.

A Consistency Result on Few Suslin Trees

By standard bookkeeping we can arrange that every normal Aronszajn tree in $V^{\mathbb{P}_{\omega_2}}$ is considered at some stage $\alpha < \omega_2$.

Let G be a generic filter on $\mathbb{P}_{\omega_2}$. Then in $V[G]$, R is a free Suslin tree and $2^\omega = 2^{\omega_1} = \omega_2$.

Consider a normal Aronszajn tree U in $V[G]$. Then there are two possibilities.

Case 1: For every derived tree T of R, $\Vdash_T$ "U is Aronszajn."

By downwards absoluteness, the above statement is also true in any intermediate model, and in particular, in the model $V[G_\alpha]$ such that $U = \dot{U}_\alpha^{G_\alpha}$. Then $\mathbb{Q}_\alpha = \mathbb{Q}(U)$, so U is special in $V[G]$.

Consistency Result

Case 2: For some derived tree T of R and for some $x \in T$,

$$x \Vdash_T \text{“}U \text{ has a cofinal branch.”}$$

But then T_x is a derived tree of R as well, and T_x forces that U has a cofinal branch.

Taking a derived tree S of R of smallest dimension which forces that U has a cofinal branch, by a previous result S club embeds into U.

□

Subtrees of Suslin Trees

Let us discuss in more detail the properties satisfied in the above model. We will need a lemma about subtrees of Suslin trees.

Lemma

Suppose that T is a Suslin tree. Let $X \subseteq T$ be uncountable. Then there exists some $x \in T$ such that X is dense below x in T.

Proof.

If not, then for all $x \in T$ there exists $y >_T x$ such that there is nothing in X above y. Let D be the set of such y. Then D is dense open in T.

Since T is Suslin, we can fix $\delta < \omega_1$ such that $T_\delta \subseteq D$. Then every member of T_δ has nothing above it in X. It follows that $X \subseteq T \upharpoonright \delta$, which contradicts that X is uncountable. □

Few Suslin Trees

Proposition

Suppose that:

1. *R is a normal free Suslin tree;*
2. *for every normal Suslin tree S, there exists some derived tree of R which club embeds into S.*

Then every Suslin tree is club isomorphic to a countable disjoint union of derived trees of R.

Under the above assumptions, there are ω_1 many derived trees of R, and hence $\omega_1^\omega = 2^\omega$ many countable disjoint unions of such trees. As we saw previously, there are 2^ω many non-club-isomorphic such trees. So there are exactly 2^ω many club-isomorphism-types of normal Suslin trees. This is the minimal number under the assumption of a free Suslin tree.

Few Suslin Trees

Proof.

Let S be a normal Suslin tree. The following is easy to check.

Claim.

Suppose that $x \in S$ and $f : T \upharpoonright C \to S_x \upharpoonright C$ is a club isomorphism for some derived tree T of R. Then for any $\delta \in C$ and $y \in S_x \upharpoonright C$, there is a club isomorphism between some derived tree of R and S_y.

Define D as the set of $y \in S$ such that there exists a derived tree T of R and a club isomorphism $f : T \upharpoonright C_y \to S_y \upharpoonright C_y$.

Let us show that it suffices to prove that D is dense. Then the upward closure E of D is dense open. So there exists some $\beta < \omega_1$ such that $S_\beta \subseteq E$.

Few Suslin Trees

Fix $\delta \in \bigcap\{C_y : y \in D \cap (S \restriction (\beta+1))\}$. By the claim, for all $z \in S_\delta$ there exists a club isomorphism

$$f_z : T(z) \restriction C_z \to S_z \restriction C_z$$

for some derived tree $T(z)$ of R and club C_z with $\delta \in C_z$.

Let $C := \bigcap\{C_z : z \in S_\delta\}$. Let $f := \bigcup\{f_z \restriction (T(z) \restriction C) : z \in S_\delta\}$. Then

$$f : \bigsqcup\{T(z) : z \in S_\delta\} \restriction C \to S \restriction C$$

is a club isomorphism.

Few Suslin Trees

To show that D is dense, consider $x \in S$. Then S_x is Suslin. By assumption there exists a derived tree T of R and a club embedding $f : T \restriction C \to S_x \restriction C$.

The range of f is an uncountable subset of S_x, so by the lemma we can find $y >_S x$ such that the range of f is dense below y. Without loss of generality, $\mathrm{ht}_S(y) \in C$.

It easily follows that the range of f contains $S_y \restriction C$. So f gives a club isomorphism between the derived tree $T_{f^{-1}(y)}$ of R and S_y. □

Few Suslin Trees With CH

Theorem (Abraham and Shelah [AS1985])

Assume GCH, $\diamondsuit^*$, and $\square_{\omega_1}$. Then there exists a forcing poset which forces:

1. *GCH;*
2. *there exists a normal free Suslin tree R and a special Aronszajn tree U;*
3. *for any Aronszajn tree T, either some derived tree of R club embeds into T, or else T club embeds into U (and hence is special).*

This theorem was proven using Jensen's technique for forcing a model of SH + CH. Starting with a free Suslin tree R, a sequence of Suslin trees $\langle T^\nu : \nu < \omega_2 \rangle$ is defined so that forcing with $T^{\nu+1}$ adds a cofinal branch to T^ν and specializes an Aronszajn tree A in V^{T^ν} which remains Aronszajn after forcing with any derived tree of R.

Another Model With Few Suslin Trees and CH

Note that in a model of CH and few Suslin trees, there are $2^\omega = \omega_1$ many club-isomorphism-types of normal Suslin trees.

A different proof with few Suslin trees and CH was given later by Abraham and Shelah [AS1993] using the method of proper forcing, iterations of proper forcings without add reals, and the following preservation theorem:

Theorem (Abraham and Shelah [AS1993]; Miyamoto [M1993])

Let S be a Suslin tree. Then the property of a forcing poset being proper and preserving the fact that S is Suslin is preserved by any countable support forcing iteration.

Using Suslin Trees to Solve a Problem of Woodin

Theorem (Woodin)

Assume that there exists a measurable Woodin cardinal. Then every Σ_1^2 set of reals is determined. Hence, there cannot exists a Σ_1^2 well-ordering of the reals.

Woodin asked whether this theorem could be improved by replacing Σ_1^2 by Σ_2^2.

Abraham and Shelah [AS1993] used their method of constructing models with few Suslin trees (in an indirect way) to prove a negative answer.

Using Suslin Trees to Solve a Problem of Woodin

Rather than using a single free Suslin tree as in our previous examples, Abraham and Shelah [AS1993] made use of a sequence $\langle R_i : i < \omega_1 \rangle$ of free Suslin trees.

Every subset of ω_1 determines a *pattern*, and it is possible to obtain a model in which each R_i is a free Suslin tree, and the pattern determines which finite products of the R_i's are free and which finite products are special. All normal Aronszajn trees are either special or contain a club isomorphic copy of some derived tree of some R_i.

Under CH, if the subset of ω_1 under consideration codes a well-ordering of the reals, then the pattern of the free trees as just described determines a Δ^2_2-well ordering of the reals.

Few Suslin Trees and a Unique Nowhere Suslin Aronszajn Tree

Theorem (K. 2022)

The following is consistent:

1. *There exists a normal free Suslin tree R;*
2. *For every normal Suslin tree, there exists a derived tree of R which club embeds into it;*
3. *Any two normal Aronszajn trees, neither of which contains a Suslin subtree, are club isomorphic.*

In contrast to the preceding models, the above statement is not consistent with CH. Namely, CH implies the existence of 2^{ω_1} many pairwise non-club-isomorphic special Aronszajn trees.

Forcing Two Aronszajn Trees to Be Club Isomorphic

Definition (Abraham and Shelah [AS1985])

Let T and U be normal Aronszajn trees. Define the forcing poset $\mathbb{Q}(T, U)$ to consist of all pairs (x, f) satisfying:

1. x is a finite set of countable limit ordinals;
2. f is an injective function whose domain is a finite downwards closed subset of $T \restriction x$ mapping into U;
3. f is strictly increasing and level preserving.

The ordering of $\mathbb{Q}(T, U)$ is defined by $(y, g) \leq (x, f)$ if $x \subseteq y$ and $f \subseteq g$.

Compatibility Lemma

Lemma (Compatibility Lemma)

Suppose that T and U are normal Aronszajn trees. Let $Y \subseteq \omega_1$ be a stationary set of limit ordinals. Assume that

$$\{(x_\alpha, f_\alpha) : \alpha \in Y\}$$

is a set of conditions in $\mathbb{Q}(T, U)$ such that

$$\forall \alpha \in Y \ \alpha \in x_\alpha.$$

Then there exist $\alpha < \beta$ in Y such that (x_α, f_α) and (x_β, f_β) are compatible.

$\mathbb{Q}(T, U)$ Is Proper

Theorem

$\mathbb{Q}(T, U)$ is proper.

Proof.

Fix a large enough regular cardinal θ and let N be a countable elementary substructure of $H(\theta)$ containing $\mathbb{Q}(T, U)$. Let $(x, f) \in N \cap \mathbb{Q}(T, U)$. We claim that

$$q_N := (x \cup \{N \cap \omega_1\}, f)$$

is $(N, \mathbb{Q}(T, U))$-generic. Let $D \in N$ be a dense open subset of $\mathbb{Q}(T, U)$. Consider a condition $(y, g) \leq q_N$, and we will show that (y, g) is compatible with some member of $N \cap D$. Assume for a contradiction that it is not. Without loss of generality, assume that $(y, g) \in D$.

$\mathbb{Q}(T, U)$ Is Proper

Working in N, we define by recursion a set $Y \subseteq \omega_1$ and a sequence

$$\langle (x_\alpha, f_\alpha) : \alpha \in Y \rangle$$

satisfying:

1. for all $\alpha \in Y$, $\alpha \in x_\alpha$;
2. for all $\alpha \in Y$, $(x_\alpha, f_\alpha) \in D$;
3. for all $\alpha < \beta$ in Y, (x_α, f_α) and (x_β, f_β) are incompatible;
4. Y is stationary.

This will contradict the compatibility lemma.

$\mathbb{Q}(T, U)$ Is Proper

If the recursion stops before ω_1 many steps, then by elementarity it would stop at some ordinal in $N \cap \omega_1$. So we would have some $\gamma \in N \cap \omega_1$ where $Y \subseteq \gamma$, the sequence

$$\langle (x_\alpha, f_\alpha) : \alpha \in Y \rangle$$

is in N, but it is not possible to add anything further. But

$$(x_{N\cap\omega_1}, f_{N\cap\omega_1}) := (y, g)$$

can be added, and we have a contradiction.

When defining Y, we arrange that whenever $Y \cap \beta$ is determined, the next element of Y is as small as possible. By a similar argument as above, this guarantees that $N \cap \omega_1 \in Y$. But then it follows that Y is stationary. □

Club Isomorphisms and Preserving a Suslin Tree

Proposition

Suppose that S is a normal Suslin tree. Let T and U be normal Aronszajn trees such that

$$\Vdash_S \text{"}T \text{ and } U \text{ are Aronszajn."}$$

Then $\Vdash_{\mathbb{Q}(T,U)}$ "S is Suslin."

Proof.

Suppose that there is a condition $p = (x, f) \in \mathbb{Q}(T, U)$ such that

$$p \Vdash_{\mathbb{Q}(T,U)} \text{"}\dot{A} = \{\dot{b}_\alpha : \alpha < \omega_1\} \text{ is an uncountable antichain of } S.\text{"}$$

We will find some $a \in S$ which forces in S that either T or U is not Aronszajn.

Club Isomorphisms and Preserving a Suslin Tree

For each limit ordinal $\alpha < \omega_1$ above $\max(x)$, fix some

$$(x_\alpha, f_\alpha) \leq (x \cup \{\alpha\}, f)$$

which decides $\dot{b}_\alpha$ as some $b_\alpha \in S$. Note that if (x_α, f_α) and (x_β, f_β) are compatible, then b_α and b_β are incomparable in S.

By an argument which we will omit, there exists some $a \in S$ such that

$$a \Vdash_S \text{“}\{\alpha < \omega_1 : b_\alpha \in \dot{G}_S\} \text{ is stationary in } \omega_1.\text{”}$$

We claim that a forces in S that either T or U is not Aronszajn.

Club Isomorphisms and Preserving a Suslin Tree

Let G_S be a generic filter on S which contains a. Then G_S is a cofinal branch of S. Let Y be the stationary set $\{\alpha < \omega_1 : b_\alpha \in G_S\}$.

Consider the collection

$$\{(x_\alpha, f_\alpha) : \alpha \in Y\}.$$

Then Y is stationary and for all $\alpha \in Y$, $\alpha \in x_\alpha$.

Now $\mathbb{Q}(T, U)$ is the same forcing in both V and $V[G_S]$. If T and U are still Aronszajn in $V[G_S]$, then we can apply the compatibility lemma to find $\alpha < \beta$ in Y such that (x_α, f_α) and (x_β, f_β) are compatible in $\mathbb{Q}(T, U)$. As noted above, this implies that b_α and b_β are incomparable in S. But this is impossible because b_α and b_β are both in the branch G_S. □

Consistency Result

Now we describe the consistency proof. We start with a model V in which GCH holds and there exists a free Suslin tree R.

We define by recursion a countable support forcing iteration

$$\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\alpha : \beta \leq \omega_2,\ \alpha < \omega_2 \rangle.$$

At each stage $\alpha < \omega_2$, we consider names for normal Aronszajn trees $\dot{T}_\alpha$ and $\dot{U}_\alpha$ such that every derived tree of R forces in $V^{\mathbb{P}_\alpha}$ that $\dot{T}_\alpha$ and $\dot{U}_\alpha$ are Aronszajn. Then we let $\dot{\mathbb{Q}}_\alpha = \mathbb{Q}(\dot{T}_\alpha, \dot{U}_\alpha)$.

Consistency Result

By the previous theorem, in $V^{\mathbb{P}_\alpha}$ the forcing $\dot{\mathbb{Q}}_\alpha$ forces that every derived tree of R is still Suslin, and hence R is still a free Suslin tree. By the forcing iteration theorem, $\mathbb{P}_{\omega_2}$ is proper and preserves the fact that R is free as well. Also $\mathbb{P}_{\omega_2}$ is ω_2-c.c.

Let G be a generic filter on $\mathbb{P}_{\omega_2}$. Then in $V[G]$, R is a free Suslin tree. If T and U are normal Aronszajn trees neither of which has a derived tree of R which club embeds into it, then T and U are club isomorphic.

Some Open Problems

Question (Shelah and Zapletal [SZ1999])

Does the existence of a Suslin tree imply that there exists a free Suslin tree?

Question

Is it consistent to have a model of $\neg SH$ with fewer than 2^ω many club-isomorphism-types of Suslin trees?

Question (Abraham and Shelah [AS1985])

Is it consistent to have a unique Suslin tree up to club isomorphism?

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