A TOPOLOGICAL BASIS PROBLEM

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Joint work with Yinhe Peng

Definition

Let $\mathcal{K}$ be a class of topological spaces. We say that a subclass $\mathcal{B}$ of $\mathcal{K}$ is a **basis** *for $\mathcal{K}$ if every space $X \in \mathcal{K}$ contains a subspace $Y \in \mathcal{B}$.*

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Which classes of topological spaces have manageable bases?

In particular which classes of topological spaces have **finite bases**?

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Remark

It is not difficult to see that in any reasonable **positive** answers of one of these questions one will have to work with a class $\mathcal{K}$ of **regular** spaces.

Examples

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Does the class of **uncountable first countable spaces** have a finite basis?

Remark

The **first countability** assumption is necessary here by the well-known L-space construction of J. Moore (2006).

When considering the Basis Problem we have to restrict to the class of regular and first countable spaces.

Basis as metrization criteria

Given a class $\mathcal{K}$ of topological spaces, let

$$\mathcal{K}^{\perp} = \{X \in \mathrm{Top} : (\forall Y \in \mathcal{K}) Y \not\hookrightarrow X\}$$

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Question

Is there a finite list $Y_1, Y_2, ..., Y_k$ of first countable regular topological spaces so that so that every regular first countable space in

$$\{Y_1, Y_2, ..., Y_k\}^{\perp}$$

is a continuous images of **separable metric spaces**?

Three orthogonal spaces

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Let $D(\aleph_1)$ be the discrete space on $\aleph_1$ points.

Let B be a fixed $\aleph_1$-dense set of reals with the topology induced from $\mathbb{R}$.

Let $B(\rightarrow)$ be the set B with the topology induced by the basis:

$$[x, y) \quad (x, y \in B, x < y).$$

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Remark

Note that $D(\aleph_1)$, $B(\rightarrow)$ and B are pairwise orthogonal.

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Note that $D(\aleph_1)$, $B(\rightarrow)$ and B are pairwise orthogonal.

Theorem (Baumgartner, 1973, 1984)

PFA implies that $D(\omega_1)$ embeds into $B(\rightarrow) \times B(\rightarrow)$.

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$$\{X \in \mathrm{Top} : X^2 \perp D(\aleph_1)\}$$

a continuous images of **separable metric spaces**?

Question (Fremlin 1980's)

Is every compact space in $D(\aleph_1)^{\perp}$ an at most 2-to-1 continuous preimage of a **compact metric space**?

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Question

Is every compact space in $\{D(\aleph_1), B(\to)\}^\perp$ **metrizable**?

Remark

The **split interval** $[0,1] \times \{0,1\}$ is a non-metrizable compact space in $D(\aleph_1)^\perp$.

Applications

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A space X is **perfect** *if closed subsets of X are G_δ.*

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Proposition

Every non metrizable perfect compactum contains an uncountable subspace orthogonal to the class of uncountable metrizable subspace.

Theorem (Gruenhage, 1986, 1990)

Assume PFA and that every uncountable regular first countable space contains a subspace isomorphic to one of the spaces $D(\aleph_1)$, $B(\to)$, B. *Then* $D(\aleph_1) \hookrightarrow X \times Y$ *for every pair* X *and* Y *of nonmetrizable compacta.*

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Theorem (Gruenhage, 1986, 1990)

Assume PFA and that every uncountable regular first countable space contains a subspace isomorphic to one of the spaces $D(\aleph_1)$, $B(\rightarrow)$, B. Then every perfect locally connected compactum is metrizable.

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Theorem (T., 1999)

Every compact set of Baire-class-one functions on a Polish space belonging to $D(\aleph_1)^\perp$ *is an at most 2-to-1 preimage of a compact metric space.*

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Every compact set of Baire-class-one functions on a Polish space belonging to $\{D(\aleph_1), [0,1] \times \{0,1\}\}^\perp$ is **metrizable**.

Separation

Definition

A topological space X is **separated** *if to every point x of X we can assign a neighbourhood U_x in such a way that for all $x \neq y$ in X either $x \notin U_y$ of $y \notin U_x$.*

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Proposition

A first countable space X is a **continuous image of a separable metric space** *iff for every neighbourhoon assignment $U_x (x \in X)$ there is a countable partition $X = \bigcup_{n<\omega} X_n$ such that for all n and $x, y \in X_n$, we have that $x \in U_y$ and $y \in U_x$.*

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For a topological space X, let OGA_X assert that if $G = (X, E)$ is an **open graph** *on X then either G is* **countably chromatic** *or G contains an* **uncountable clique**.

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Remark

1. If Y is a continuous image of X then OGA_X implies OGA_Y.
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Theorem (T. , 1983)

PFA implies that every space in $D(\aleph_1)^\perp$ is hereditarily Lindelöf.

Corollary (PFA)

$X^2 \in D(\aleph_1)^\perp$ implies OGA_X for every regular space X.

Proof.

Let $G = (X, O)$ be a given open graph on X. Since the family $\mathcal{U}$ of sets of the form $U \times V$ for U and V open subsets of X such that $U \times V \subseteq O$ is an open cover of the Lindelöf subspace O of X^2, there is a sequence $U_n \times V_n$ of elements of $\mathcal{U}$ such that $O = \bigcup_{n<\omega} U_n \times V_n$. Pick a second countable topology τ on X such that U_n and V_n belong to τ for all n. Then G is also an open graph on the second countable space (X, τ). □

Theorem (Peng-T., 2022)

Assume PFA and let X be a space of cardinality $\aleph_1$ with weaker metrizable topology. Then OGA_X implies $X \in D(\aleph_1)^{\perp}$.

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Question (PFA)

In which class of spaces $X \in D(\aleph_1)^\perp$ implies $X^2 \in D(\aleph_1)^\perp$?

In other words, in which class of spaces X the three statements $X \in D(\aleph_1)^\perp$, OGA_X and $X^2 \in D(\aleph_1)^\perp$ are equivalent?

For example, is this true in the class of first countable regular spaces?

OGA for powers of Moore's L-space

A **C-sequence** is a sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ such that $C_{\alpha+1} = \{\alpha\}$ and C_α is a cofinal subset of α of order type ω for limit α's.

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The **walk** from β towards a smaller ordinal α is the sequence

$\beta = \beta_0 > \beta_1 > ... > \beta_n = \alpha$ such that $\beta_{i+1} = \min C_{\beta_i} \setminus \alpha$ for each $i < n$.

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The **maximal weight** $\rho_1 : [\omega_1]^2 \to \omega$ is defined recursively by

$$\rho_1(\alpha, \beta) = \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\}$$

with boundary value $\rho_1(\alpha, \alpha) = 0$.
$\rho_{1\beta} : \beta \to \omega$ is defined by $\rho_{1\beta}(\alpha) = \rho_1(\alpha, \beta)$ for $\alpha < \beta$.

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$\rho_{1\beta} : \beta \to \omega$ is defined by $\rho_{1\beta}(\alpha) = \rho_1(\alpha, \beta)$ for $\alpha < \beta$. The **lower trace** $L : [\omega_1]^2 \to [\omega_1]^{<\omega}$ is recursively defined by

$$L(\alpha, \beta) = (L(\alpha, \min(C_\beta \setminus \alpha)) \cup \{\max(C_\beta \cap \alpha)\}) \setminus \max(C_\beta \cap \alpha)$$

with boundary value $L(\alpha, \alpha) = \emptyset$.

The **lower trace oscillation** $osc : [\omega_1]^2 \to \omega$ is defined by, for $\alpha < \beta$,

$$osc(\alpha, \beta) = |Osc(\rho_{1\alpha}, \rho_{1\beta}; L(\alpha, \beta))|,$$

where the **usual oscillation** *Osc* is defined by

$$Osc(s, t; F) = \{\xi \in F \setminus \{\min F\} : s(\xi^-) \leq t(\xi^-) \text{ and } s(\xi) > t(\xi)\}$$

where F is a finite set of ordinals, $s, t : F \to Ord$ are functions from F to ordinals and ξ^- is the greatest element of F below ξ.

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and $w_\beta(\alpha) = 1$ otherwise.
Let $\mathcal{L} = \{w_\beta : \beta < \omega_1\}$ viewed as subspace of $\mathbb{T}^{\omega_1}$.

Theorem (Moore, 2006)

1. $\mathcal{L}$ *is hereditarily Lindelöf.*
2. $\mathcal{L}$ *is not separable.*
3. *Every continuous function from* $\mathcal{L}$ *into a metric space has countable range.*
4. $D(\omega_1) \hookrightarrow \mathcal{L}^2$.

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Theorem (Peng-T., 2022)

PFA implies $OCA_{\mathcal{L}^n}$ for any $n < \omega$.

Inner topologies

Definition

For a topological space (X, τ) *and a collection* $\mathcal{C} \subset P(X)$*, the* **inner topology** $(X, \tau_{in}^{\mathcal{C}})$ *induced by* $\mathcal{C}$ *is the topology with base*

$$\{\{x\} \cup O_{in}^{\mathcal{C}} : x \in O, O \in \tau\}$$

where $O_{in}^{\mathcal{C}} = \bigcup\{C \in \mathcal{C} : C \subset O\}$.

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where $O_{in}^{\mathcal{C}} = \bigcup\{C \in \mathcal{C} : C \subset O\}$*.*

Remark

Note that the inner topology is a topology that is finer than the original topology. If we take $\mathcal{C}$ to be $\{\{x\} : x \in X\}$ or any **network**, then $\tau_{in}^{\mathcal{C}} = \tau$.

Remark

For a subspace $Y \subset X$, by $(Y, \tau_{in}^{\mathcal{C}})$ denotes the corresponding subspace of $(X, \tau_{in}^{\mathcal{C}})$.
In other words, the topology of $(Y, \tau_{in}^{\mathcal{C}})$ is generated by

$$\{(\{x\} \cup O^{I,\mathcal{C}}) \cap Y : O \in \tau, x \in O \cap Y\}.$$

It may happen that $(Y, \tau_{in}^{\mathcal{C}}) \neq (Y, \tau_{in}^{\mathcal{C}_Y})$ where $\mathcal{C}_Y = \{C \cap Y : C \in \mathcal{C}\}$ and $\tau_{in}^{\mathcal{C}_Y}$ can be viewed as the inner topology of Y induced by $\mathcal{C}_Y$.
However, if we enlarge $\mathcal{C}$ to $\mathcal{C}' = \mathcal{C} \cup \{C \cap Y : C \in \mathcal{C}\}$, then

$$(Y, \tau_{in}^{\mathcal{C}'}) = (Y, \tau_{in}^{\mathcal{C}'_Y}).$$

Lemma

If an inner topology induced by some family $\mathcal{C}$ is hereditarily Lindelöf, then $|O \setminus O_{in}^{\mathcal{C}}| \leq \aleph_0$ for any open set O.

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Lemma

If $f : X \to Y$ is a continuous surjection and $(X, \tau_{in}^{\mathcal{C}})$ is hereditarily Lindelöf for some $\mathcal{C} \subset P(X)$, then $(Y, \tau_{in}^{f(\mathcal{C})})$ is hereditarily Lindelöf where $f(\mathcal{C}) = \{f[C] : C \in \mathcal{C}\}$.

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Theorem (Peng-T., 2022)

Assume PFA. Let (X, τ) *be a regular space that admits a countably generated hereditarily Lindelöf inner topology, then either*

1. (X, τ) *is a continuous image of a separable metric space, or*
2. $B(\rightarrow)$ *embeds into* (X, τ)*.*

Theorem (Peng-T., 2022)

Assume PFA. Suppose that X is first countable. regular space admitting a countably generated hereditarily Lindelöf inner topology. Then X contains a subspace isomorphic to B or a subspace isomorphic to $B(\rightarrow)$.

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Assume PFA.
Let (X, τ) be a first countable regular space. Then either,

1. *For any countable collection $\mathcal{C}$ of subsets of X, the space $(X, \tau_{in}^{\mathcal{C}})$ is σ-discrete, or*
2. *X contains a subspace isomorphic to B or a subspace isomorphic to $B(\rightarrow)$.*

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Theorem (Peng-T., 2022)

Assume PFA. Let X be a first countable regular hereditarily Lindelöf space of size $\aleph_1$ with a countably generated hereditarily Lindelöf inner topology. Then there is a partition $X = \bigcup_{n<\omega} X_n$ such that each X_n is either metrizable or isomorphic to a subspace of $\mathbb{R}(\rightarrow)$.

Basis problem for compact space

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Theorem (Peng-T., 2022)

Assume PFA. Suppose that X is a regular space, M a regular space with a countably generated hereditarily Lindelöf inner topology and $f : X \to M$ is a finite-to-one perfect mapping. Then every uncountable subset of X contains a subspace homeomorphic to B or a subspace homeomorphic to $B(\to)$.

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Corollary (PFA)

If a compact space X admits a finite-to-one map to a metric space, then every uncountable subset of X contains an subspace isomorphic to B or a subspace isomorphic to $B(\to)$.

Perfect preimages

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Theorem (Peng-T., 2022)

Assume PFA. Suppose that X is a regular hereditarily Lindelöf perfect preimage of a metric space. Then there is a 2-to-1 perfect map from X to a subspace of $[0,1]^{\omega_1}$. In particular, X is first countable and there is an increasing union $X = \bigcup_{\alpha<\omega_1} X_\alpha$ such that each X_α admits a 2-to-1 perfect map to a metric space.

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Theorem (Peng-T., 2022)

Assume PFA. Suppose that X is a regular hereditarily Lindelöf perfect preimage of a metric space. Then the following are equivalent:
(i) X is homeomorphic to a subspace of $[0,1]^{\omega_1}$. Hence $w(X) \leq \omega_1$ and X is an increasing union of ω_1 metrizable subspaces.
(ii) Every subspace of size continuum contains an uncountable metrizable subspace.
(iii) For any $Y \in [X]^{\omega_2}$, there is $Z \in [Y]^{\omega_1}$ orthogonal to $\mathbb{R}(\to)$.

Corollary (PFA)

1. *Every uncountable compact space K contains an uncountable subset that is either metrizable or a subspace of $\mathbb{R}(\rightarrow)$*
2. *Moreover, if a compact space K is orthogonal to $D(\aleph_1)$, then every subset of K of size continuum contains an uncountable subset that is either homeomorphic to B or to $B(\rightarrow)$.*

Corollary (PFA)

1. *Every uncountable compact space K contains an uncountable subset that is either metrizable or a subspace of $\mathbb{R}(\rightarrow)$*
2. *Moreover, if a compact space K is orthogonal to $D(\aleph_1)$, then every subset of K of size continuum contains an uncountable subset that is either homeomorphic to B or to $B(\rightarrow)$.*

Corollary (PFA)

If an uncountable regular space X is a perfect preimage of a metric space, then X contains a subspace homeomorphic to one of the three spaces $D(\aleph_1)$, $B(\rightarrow)$, B.

Open set-mapping principle

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For a regular space (X, τ), *we let* OSM_X *asserts that for any open set-mapping* $N : X \to \tau$, *there is a partition of* X *into countably many pieces* X_n *such that for each* n *and* x, y *in* X_n, *either* $x \in N(y)$ *or* $y \in N(x)$.

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Proposition

OSM_X holds for every continuous image of a separable metric space.

Proposition

$OSM_{\mathbb{R}(\to)}$ holds.

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Proposition

$OSM_{\mathcal{L}}$ fails for Moore's L-space $\mathcal{L}$.

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Theorem (Peng-T., 2022)

Assume PFA. If X is a regular space with countably generated hereditarily Lindelöf inner topology then OSM_X holds.

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