On ordered meet trees

Predrag Tanović

Mathematical Institute SANU, Belgrade

SETTOP 2022

Meet-trees

Definition

A *meet tree* is a partal order $(M, \trianglelefteq)$ in which

(1) predecessors of any element form a chain, and

(2) every pair of elements $x, y \in M$ has infimum, denoted by $x \wedge y$.

A *colored (meet) tree* has (arbitrary) unary predicates added.

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- $C(a) = \{x \in M \mid a \trianglelefteq x\}$ is a *closed cone* centered at a;
- $a \triangleleft x \wedge y$ defines an equivalence relation on $C(a) \setminus \{a\}$; the classes are called *open cones* centered at a and $C_a(x)$ denotes the class of x.

Definition

An *ordered tree* is a structure $(M, \wedge, <)$ satisfying:

(1) $(M, \wedge)$ is a meet tree,

(2) $<$ extends $\lhd$ and linearly orders M, and

(3) All (closed) cones are $<$-convex $(x, y \in C$ implies $[x, y] \subseteq C)$.

A *cor-tree* is a colored ordered tree.

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1. "Geometric" description of definable sets

Which easy-to-visualize definable relations should be added to a colored tree (cor-tree) so that it eliminates quantifiers?

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2. Classification of countable models

For a countable, complete theory T (of colored trees or cor-trees) with $I(T, \aleph_0) < 2^{\aleph_0}$ find a system of invariants which determines countable models of T up to an isomorphism.

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3. The number of countable models

Prove that an $\mathcal{L}_{\omega_1,\omega}$-sentence in the language of cor-trees has either $\leqslant \aleph_0$ or perfectly many countable models.

The case of linear orders

- Rubin 1973. *Theories of linear order.*

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Fact

Let $(M, <)$ be a linear order and $f : M \to M$ a definable function. Then there is a definable decomposition $M = D_1 \cup \ldots \cup D_n$ such that each $f \restriction D_i$ is increasing.

"Geometric" description of colored orders:

Theorem (Ilić, Moconja, T. 2017.)

If in a colored order we name:

1. all definable unary predicates;
2. all definable convex equivalence relations E;
3. all relations $x \leqslant S_E^n(y)$ (where $S_E^n(y)$ is the n-th successor E-class of $[y]_E$, E is definable, convex and $n \in \omega$);
4. $S_E^{-n}(x) < y$;

then the obtained structure eliminates quantifiers.

Colored trees

Theorem (Simon 2011.)

Let $\mathcal{M} = (M, \wedge, ...)$ be a colored tree

(a) If $\bar{a} = (a_1, ..., a_n)$ a subtree, then $tp_{\bar{x}}(\bar{a})$ is determined by:

- $\wedge$-atomic diagram of $\bar{a}$
- all $tp_{x_i x_j}(a_i a_j)$ where a_i and a_j are $\lhd$-consecutive in $\bar{a}$.

(b) The theory $T = Th(\mathcal{M})$ is ternary: every formula is T-equivalent to a Boolean combination of formulas with at most three free variables.

Theorem (Simon 2011.)

Colored trees are dp-minimal: there do NOT exist a colored tree $(M, \dots)$, $m, n \in \mathbb{N}$, sequences $(\bar{a}_i \mid i \in \omega)$ of n-tuples and $(\bar{b}_j \mid j \in \omega)$ of m-tuples of elements of M, formulas $\pi_1(x, \bar{y})$ and $\pi_2(x, \bar{z})$, and elements $c_{ij} \in M$ $(i, j < \omega)$ such that for all $i, j, k \in \omega$:

$$M \models \pi_1(c_{ij}, \bar{a}_k) \text{ iff } i = k, \text{ and}$$

$$M \models \pi_2(c_{ij}, \bar{b}_k) \text{ iff } j = k.$$

Colored ordered trees

Theorem

Let $\mathcal{M} = (M, \wedge, <, \ldots)$ be a cor-tree.

(a) If $\bar{a} = (a_1, \ldots, a_n)$ a subtree, then $tp_{\bar{x}}(\bar{a})$ is determined by:

- $\wedge$-atomic diagram of $\bar{a}$;
- the $<$-ordering of $\bar{a}$;
- all $tp_{x_i x_j}(a_i a_j)$ where a_i and a_j are $<$-consecutive in $\bar{a}$.

(b) The theory $T = Th(\mathcal{M})$ is ternary.

(c) $Th(\mathcal{M})$ is dp-minimal.

(d) If E a definable equivalencce relation on $\mathcal{M}$, then there is a definable partition $M = D_1 \cup \ldots \cup D_n$ such that E is $\lhd$-convex on each D_i.

Theorem (Moconja, T. 2020.)

T–countable, binary, stationarily ordered theory.

- $I(\aleph_0, T) = 2^{\aleph_0}$ provided that at least one of the following conditions holds:
  - (a) T is not small;
  - (b) there is a non-convex type $p \in S_1(T)$;
  - (c) there is a non-simple type $p \in S_1(T)$;
  - (d) there are infinitely many $\not\perp^w$-classes of non-isolated types in $S_1(T)$;
  - (e) there is a non-isolated forking extension of some $p \in S_1(T)$ over an 1-element domain.

- $I(\aleph_0, T) = \aleph_0$ iff none of the above holds and there are infinitely many $\not\perp^f$-classes of non-isolated types in $S_1(T)$;

- If none of the above holds, then:

$$I(\aleph_0, T) = \Pi_{i \in w_T \setminus u}(|\alpha_i^{\mathcal{F}}| + 2) \cdot \Pi_{j \in u}(|\alpha_j^{\mathcal{F}}|^2 + 3|\alpha_j^{\mathcal{F}}| + 2)$$

THANK YOU !!!