Continuity of coordinate functionals related to ideal (filter) Schauder basis

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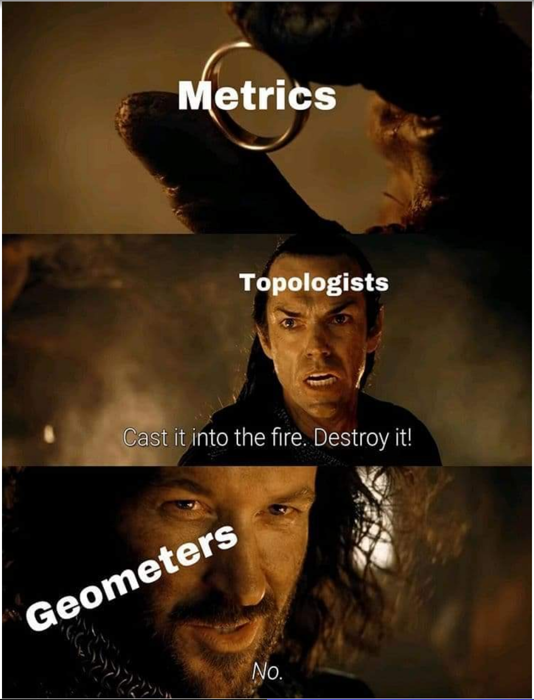
joint work with Noé de Rancourt and Tomasz Kania


Metrics
Topologists
Cast it into the fire. Destroy it!
Geometers
No.

Ideal convergence

Definition

We say that a sequence (x_n) is $\mathcal{I}$-convergent to x if for every $\varepsilon > 0$ we have $\{n \in \mathbb{N} : d(x, x_n) > \varepsilon\} \in \mathcal{I}$.

Observation

For the ideal *Fin* $\mathcal{I}$-convergence is just the classical one.

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Given an ideal $\mathcal{I}$ on ω, we say that a sequence (e_n) is $\mathcal{I}$-basis if for every $x \in X$ there exists a unique sequence $(\alpha_n) \in \mathbb{K}^\omega$ such that $x = \sum_{n,\mathcal{I}} \alpha_n e_n$. We denote the coordinate functionals by $e_n^\star$ and we set $P_n := \sum_{i=1}^n e_i^\star e_i$.

Question (Kadets)

Are $e_n^\star$ continuous for the $\mathcal{I}$ basis? At least for nice filters, e.g. $\mathcal{I}_{st}$?

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Proof for $\mathcal{I} = Fin$

We consider the space $S := \{(\alpha_n) \in \mathbb{K}^\omega : \sum_{n=0}^{\infty} \alpha_n e_n \text{ is convergent}\}$ equipped with the norm $|||(\alpha_n)||| = \sup_{n\in\omega} \|\sum_{i=0}^{n} \alpha_i e_i\|$, and map $T \colon S \to X$ given by $T((\alpha_n)) = \sum_{n=0}^{\infty} \alpha_n e_n$. Clearly T is a bijection. It is also continuous. Now it remains to prove that S is a Banach space, and use the open mapping principle. Byproduct: the norms of projections have common bound.

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Problems with classical proof

We consider the space ℓ_2 and we let $x_n = \sum_{i=1}^{n} e_i$, where (e_n) stands for standard basis. Sequence (x_n) is a $\mathcal{I}_{st}$ basis, but projections P_n related to it are not uniformly bounded.

The standard proof will not work.

Partial answer (Kochanek 2012)

If $\mathcal{I}$ is an ideal generated by less than $\mathfrak{p}$ sets, then the coordinate projections associated with $\mathcal{I}$-basis are continuous.

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Outdated proof - large cardinals

Theorem

We assume enough Large Cardinals (eg. infinitely many Woodin's with a measureable above) to get that

- every subset of $\mathbb{R}$ that is in $L(\mathbb{R})$ has the Baire property,
- in $L(\mathbb{R})$ every linear map between Fréchet spaces (in particular, Banach spaces) is continuous (Garnir, Wright)
- every projective formula is absolute between V and $L(\mathbb{R})$

Outdated proof - the space SB

Let $\mathcal{F}(C(\Delta))$ denote the hyperspace comprising all non-empty closed subsets of $C(\Delta)$.

Following Godefroy and Saint-Raymond, we shall call a Polish topology τ on $\mathcal{F}(C(\Delta))$ *admissible*, whenever

- $E^+(U) \in \tau$ for every open set $U \subseteq C(\Delta)$,
- there is a subbase $\mathcal{B}$ of τ such that every set $U \in \mathcal{B}$ may be written as a union of countably many sets of the form $E^+(U) \setminus E^+(V)$, where U and V are open in $C(\Delta)$.

It turns out that the set SB comprising all closed linear subspaces of $C(\Delta)$ is $\mathcal{G}_\delta$ in $\mathcal{F}(C(2^\omega))$ and, as such, the relative topology on SB is Polish. Recently some other approaches to the universal space for separable Banach spaces was made (see eg paper by Cúth, Doležal, Doucha and Kurka).

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Outdated Main Theorem

Theorem (Kania, S.)

Under LC the coordinate functionals of $\mathcal{I}$ basis are continuous for any projective filter $\mathcal{I}$ on $\mathbb{N}$.

Main proof

$$\forall_{X \in \mathrm{SB}} \forall_{(x_k)_{k=1}^{\infty} \in X^{\mathbb{N}}} \Big[\neg \Big(\forall_{y \in X} \exists!_{(a_k)_{k=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}} \sum_{k,\mathcal{F}} a_k x_k = y \Big) \vee$$

$$\vee \Big(\exists_{(M_k)_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}} \forall_{y \in X} \exists_{(a_k)_{k=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}} \sum_{k,\mathcal{F}} a_k x_k = y \wedge |a_k| \le \|y\| \cdot M_k \Big) \Big].$$

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Lemma

Let X be a separable Banach space and let $\mathcal{I}$ be a projective filter on $\mathbb{N}$ of class $\prod_n^1$. Suppose that $(z_k)_{k=1}^{\infty}$ is a sequence in X. Then, the following formula is $\prod_n^1$:

$$\varphi((a_k)_{k=1}^{\infty}, z) \equiv \sum_{j,\mathcal{I}} a_k z_k = z.$$

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Modern results

Main Theorem

Let $\mathcal{I}$ be analytic filter on $\mathbb{N}$. Then for every $\mathcal{I}$-basis of a Banach space the corresponding coordinate functionals are continuous.

Proof

$$e_n^\star(x) \in U \Leftrightarrow \exists_{(\alpha_i)\in\mathbb{K}^\mathbb{N}} \sum_{i,\mathcal{I}} \alpha_i e_i = x \wedge \alpha_n \in U$$

$$\Leftrightarrow \exists_{(\alpha_i)\in\mathbb{K}^\mathbb{N}} \forall_{l\in\mathbb{N}} \exists_{A\in\mathcal{I}} \forall_{m\notin A} \left\| \sum_{i=1}^{m} \alpha_i e_i - x \right\| \leq \frac{1}{l} \wedge \alpha_n \in U$$

$$e_n^\star(x) \in U \Leftrightarrow \forall_{b\in\mathbb{K}} (b \in U) \vee e_n^\star(x) \neq b$$

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Theorem

Let $\mathcal{I}$ by an ideal on ω (not necessarily projective). Let (e_n) be an $\mathcal{I}$-basis with continuous coordinate functionals. Then there exists an analytic ideal $\mathcal{I}' \subset \mathcal{I}$ on ω such that (x_n) is also an $\mathcal{I}'$-basis.

Proof

$$\mathcal{A} := \left\{ A \subset \omega \colon \exists_{x \in X} \exists_{\varepsilon > 0} \, A \subset \left\{ n \in \mathbb{N} \colon \| \sum_{i=1}^{n} e_i^\star(x) e_i - x \| > \varepsilon \right\} \right\}$$

$$\mathcal{B}_n = \left\{ (x, \varepsilon, A) \in X \times \mathbb{R}^+ \times 2^\omega \colon n \in A \vee \left\| \sum_{i=1}^{n} e_i^\star(x) e_i - x \right\| \leq \varepsilon \right\}$$

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Main Theorem

Let $\mathcal{I}$ be analytic filter on $\mathbb{N}$. Then for every $\mathcal{I}$-basis of a Banach space the corresponding coordinate functionals are continuous.

Theorem

Assume that all Δ^1_n-sets are Baire-measureable. Let $\mathcal{F}$ be Σ^1_n-ideal on ω. Then for every $\mathcal{I}$-basis the corresponding coordinate functionals are continuous.

Theorem

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Questions

Q1

Does there exists a Banach space X, an ideal $\mathcal{I}$ on $\mathbb{N}$ and an $\mathcal{I}$-basis (e_n) for X such that not all coordinate projections $(e_n^\star)$ are continuous?

Q2

Let $\mathcal{I}$ be a filter and let (e_n) be an $\mathcal{I}$-basis for a Banach space X with continuous coordinate functionals. Does there exists a Borel filter $\mathcal{I}'$ such that (e_n) is also an $\mathcal{I}'$-basis? Should it be the case, what is the smallest complexity of such $\mathcal{I}'$? In particular, is the $\mathcal{I}_{st}$-basis of ℓ_2 provided earlier also an $\mathcal{I}'$-basis for some F_σ ideal $\mathcal{I}'$?

Q3

Can one find an example of Banach space X, sequence $(e_n) \subset X$ and pair of ideal $\mathcal{I}$, $\mathcal{I}'$ such that (e_n) is both $\mathcal{I}$-base and $\mathcal{I}'$-base, but the coordinate functionals differs in those situations?

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Thank you for your attention! Хвала на пажњи!
Gratiam vobis ago pro animis attentis!
Σας ευχαριστώ για την προσοχή σας!
Dziękuję za uwagę! Děkuji za pozornost!
Köszönöm a figyelmet! Grazie per l'attenzione!
Danke für Ihre Aufmerksamkeit!
ध्यान देने के लिए धन्यवाद! با تشکر از توجه شما
Gracias por su atención!
Hvala za vašo pozornost!
ขอขอบคุณสำหรับความสนใจของคุณ!