Idealization of continuity[1]

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Local function

$\langle X, \tau \rangle$ - topological space

$$\mathrm{Cl}(A) = \{x \in X : A \cap U \neq \emptyset \text{ for each } U \in \tau(x)\}$$

$\mathcal{I}$ - an ideal on X

$\langle X, \tau, \mathcal{I} \rangle$ - ideal topological space [Kuratowski 1933]

$$A^*_{(\tau,\mathcal{I})} = \{x \in X : A \cap U \notin \mathcal{I} \text{ for each } U \in \tau(x)\}$$

$A^*_{(\tau,\mathcal{I})}$ (briefly A^*) - **local function**

Local function

For $\mathcal{I} = \{\emptyset\}$ we have that $A^*(\mathcal{I}, \tau) = \mathrm{Cl}(A)$.
For $\mathcal{I} = P(X)$ we have that $A^*(\mathcal{I}, \tau) = \emptyset$.
For $\mathcal{I} = Fin$ we have that $A^*(\mathcal{I}, \tau)$ is the set of ω-accumulation points of A.
For $\mathcal{I} = \mathcal{I}_{count}$ we have that $A^*(\mathcal{I}, \tau)$ is the set of condensation points of A.

Local function

(1) $A \subseteq B \Rightarrow A^* \subseteq B^*$;
(2) $A^* = \mathrm{Cl}(A^*) \subseteq \mathrm{Cl}(A)$;
(3) $(A^*)^* \subseteq A^*$;
(4) $(A \cup B)^* = A^* \cup B^*$
(5) If $I \in \mathcal{I}$, then $(A \cup I)^* = A^* = (A \setminus I)^*$.

Topology τ^*

Definition

$\mathrm{Cl}^*(A) = A \cup A^*$ is a Kuratowski closure operator, and therefore it generates a topology on X

$$\tau^*(\mathcal{I}) = \{A : \mathrm{Cl}^*(X \setminus A) = X \setminus A\}.$$

Set A is closed in τ^* iff $A^* \subseteq A$.

$$\psi(A) = X \setminus (X \setminus A)^*$$

$$O \in \tau^* \Leftrightarrow O \subseteq \psi(O); \quad \psi(\tau) = \{\psi(U) : U \in \tau\}.$$

$$\psi(\tau) \subseteq \langle \psi(\tau) \rangle \subseteq \tau \subseteq \tau^* = \tau^{**}$$

$\beta(\mathcal{I}, \tau) = \{V \setminus I : V \in \tau, I \in \mathcal{I}\}$ is a basis for τ^*

Topology τ^*

For $\mathcal{I} = \{\emptyset\}$ we have that $\tau^*(\mathcal{I}) = \tau$.
For $\mathcal{I} = P(X)$ we have that $\tau^*(\mathcal{I}) = P(X)$.
If $\mathcal{I} \subseteq \mathcal{J}$ then $\tau^*(\mathcal{I}) \subseteq \tau^*(\mathcal{J})$.
If $Fin \subseteq \mathcal{I}$ then $\langle X, \tau^* \rangle$ is T_1 space.
If $\mathcal{I} = Fin$, then $\tau^*_{ad}(\mathcal{I})$ is the cofinite topology on X.
If $\mathcal{I} = \mathcal{I}_{m0}$ - ideal of the sets of measure zero, then τ^*-Borel sets are precisely the Lebesgue measurable sets. (Scheinberg 1971)
For $\mathcal{I} = \mathcal{I}_{nwd}$ then $A^* = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ and $\tau^*(\mathcal{I}_{nwd}) = \tau^\alpha$.
(α-open sets, $A \subseteq \mathrm{Int}(Cl(Int(A)))$. (Njástad 1965)

Compatibility

Definition (Njástad 1966)

Let $\langle X, \tau, \mathcal{I}\rangle$ be an ideal topological space. We say τ is compatible with the ideal $\mathcal{I}$, denoted $\tau \sim \mathcal{I}$ if the following holds for every $A \subseteq X$: if for every $x \in A$ there exists a $U \in \tau(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Theorem

$\tau \sim \mathcal{I}$ implies $\beta = \tau^*$. (Njástad 1966)
$\tau \sim \mathcal{I}$ iff $A \setminus A^* \in \mathcal{I}$, for each A. (Vaidyanathaswamy, 1960)

Theorem

$\langle X, \tau\rangle$ is hereditarily Lindelöf iff $\tau \sim \mathcal{I}_{count}$;
$\tau \sim \mathcal{I}_{nwd}$; $\tau \sim \mathcal{I}_{mgr}$.

$X = X^*$

Theorem (Samules 1975)

Let $\langle X, \tau, \mathcal{I} \rangle$ be an ideal topological space. Then $X = X^*$ iff $\tau \cap \mathcal{I} = \{\emptyset\}$.

Theorem (Janković, Hamlett 1990)

Let $\langle X, \tau \rangle$ be a space with an ideal $\mathcal{I}$ on X. If $X = X^*$ then $\tau_s = {\tau^*}_s$, where τ_s is the topology generated by the basis of regular open sets ($U = \mathrm{Int}(\mathrm{Cl}(U))$) in τ.

Theorem

Semiregular properties (properties shared by $\langle X, \tau \rangle$ and $\langle X, \tau_s \rangle$, like Hausdorffness, property of a space being Urysohn ($T_{2\frac{1}{2}}$), connectedness, H-closedness, ...) are shared by $\langle X, \tau \rangle$ and $\langle X, \tau^* \rangle$ if $X = X^*$.

Problem

Question

If

$$f : \langle X, \tau \rangle \to \langle Y, \sigma \rangle$$

is continuous (open, closed, homeomorphism), what are sufficient conditions for

$$f : \langle X, \tau^* \rangle \to \langle Y, \sigma^* \rangle$$

to remain continuous (open, closed, homeomorphism)?

Previous results

Theorem (Samuels 1971)

If $X = X^*$ $(\mathcal{I} \cap \tau = \{\emptyset\})$ and Y is regular then $f : \langle X, \tau \rangle \to Y$ is continuous iff $f : \langle X, \tau^* \rangle \to Y$ is continuous.

Theorem (Natkaniec 1986)

Let $f : X \to \mathbb{R}$, where X is a Polish space with topology τ, and $\mathcal{I}$ a σ-complete ideal on X such that $Fin \subset \mathcal{I}$ and $\mathcal{I} \cap \tau = \{\emptyset\}$. If $f : \langle X, \tau^* \rangle \to \langle R, \mathcal{O}_{nat} \rangle$ is a continuous function, then $f : \langle X, \tau \rangle \to \langle R, \mathcal{O}_{nat} \rangle$ is also continuous.

Previous results

Definition (Newcomb 1968, Rančin 1972)

$\langle X, \tau, \mathcal{I} \rangle$ is $\mathcal{I}$-compact iff for each open cover $\{U_\lambda : \lambda \in \Lambda\}$ exists finite subcollection $\{U_{\lambda_k} : k \leq n\}$ such that $X \setminus \bigcup\{U_{\lambda_k} : k \leq n\} \in \mathcal{I}$.

Theorem (Hamlett, Janković 1990)

Let $f : \langle X, \tau, \mathcal{I} \rangle \to \langle Y, \sigma, f[\mathcal{I}] \rangle$ be a bijection such that $\langle X, \tau \rangle$ is $\mathcal{I}$-compact and $\langle Y, \sigma \rangle$ is Hausdorff. If $f : \langle X, \tau^* \rangle \to \langle Y, \sigma \rangle$ is continuous, then $f : \langle X, \tau^* \rangle \to \langle Y, \sigma^* \rangle$ is a homeomorphism.

Previous results

Theorems (Hamlett, Rose 1990)

Let $\langle X, \tau, \mathcal{I} \rangle, \langle Y, \sigma, \mathcal{J} \rangle$ be ideal topological spaces.

If $f : \langle X, \tau \rangle \to \langle Y, \langle \psi(\sigma) \rangle \rangle$ is a continuous injection, $\mathcal{J} \sim \sigma$ and $f^{-1}[\mathcal{J}] \subset \mathcal{I}$ then $\psi(f[A]) \subseteq f[\psi(A)]$, for each $A \subseteq X$.

If $f : \langle X, \langle \psi(\tau) \rangle \rangle \to \langle Y, \sigma \rangle$ is an open bijection, $\mathcal{I} \sim \tau$ and $f[\mathcal{I}] \subset \mathcal{J}$ then $f[\psi(A)] \subseteq \psi(f[A])$, for each $A \subseteq X$.

Let $f : X \to Y$ be a bijection and $f[\mathcal{I}] = \mathcal{J}$. Then the following conditions are equivalent

a) $f : \langle X, \tau^* \rangle \to \langle Y, \sigma^* \rangle$ is a homeomorphism;

b) $f[A^*] = (f[A])^*$, for each $A \subseteq X$;

c) $f[\psi(A)] = \psi(f[A])$, for each $A \subseteq X$.

Continuity

Theorem

Let $\langle X, \tau_X, \mathcal{I}_X \rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y \rangle$ be ideal topological spaces. If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is a continuous function and for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$. Then there hold the following equivalent conditions:

a) $\forall A \subseteq X \; f[A^*] \subseteq (f[A])^*$;

b) $\forall B \subseteq Y \; (f^{-1}[B])^* \subseteq f^{-1}[B^*]$.

which implies the following three equivalent conditions:

c) $\forall A \subseteq X \; f[\overline{A}^{\tau_X^*}] \subseteq \overline{f[A]}^{\tau_Y^*}$;

d) $\forall B \subseteq Y \; \overline{(f^{-1}[B])}^{\tau_X^*} \subseteq f^{-1}[\overline{B}^{\tau_Y^*}]$;

e) $f : \langle X, \tau_X^* \rangle \to \langle Y, \tau_Y^* \rangle$ is a continuous function.

Continuity of $f : \langle X, \tau_X^* \rangle \to \langle Y, \tau_Y^* \rangle$ does not imply conditions a) and b)

f is a bijection

Theorem

Let $\langle X, \tau_X, \mathcal{I}_X \rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y \rangle$ be ideal topological spaces. If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is a continuous bijection and for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$, then there hold the following equivalent conditions:

a) $\forall A \subseteq X \ \psi(f[A]) \subseteq f[\psi(A)]$;

b) $\forall B \subseteq Y \ f^{-1}[\psi(B)] \subseteq \psi(f^{-1}[B])$.

Example

If f is not a bijection mapping, then conditions a) and b) do not have to hold.

Open mappings

Theorem

Let $\langle X, \tau_X, \mathcal{I}_X \rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y \rangle$ be ideal topological spaces. If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is an open function and for all $I \in \mathcal{I}_X$ we have $f[I] \in \mathcal{I}_Y$, then there hold the *following* equivalent conditions:

a) $\forall A \subseteq X \;\; f[\Psi(A)] \subseteq \Psi(f[A])$;

b) $\forall B \subseteq Y \;\; \Psi(f^{-1}[B]) \subseteq f^{-1}[\Psi(B)]$.

which implies

c) $f : \langle X, \tau_X^* \rangle \to \langle Y, \tau_Y^* \rangle$ is an open function.

Example

c) is not equivalent with conditions a) and b).

Open bijections and closed injections

Theorem

Let $\langle X, \tau_X, \mathcal{I}_X \rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y \rangle$ be ideal topological spaces. If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is an open bijection or closed injection and for all $I \in \mathcal{I}_X$ we have $f[I] \in \mathcal{I}_Y$, then there hold the *following* equivalent conditions:

a) $\forall A \subseteq X \ (f[A])^* \subseteq f[A^*]$;

b) $\forall B \subseteq Y \ f^{-1}[B^*] \subseteq (f^{-1}[B])^*$.

Example

If f is open but not bijection, or closed but not injection then conditions a) and b) do not have to hold.

Homeomorphism

Finally, gathering all previous, we extended the result obtained by Hamlett and Rose in 1990, which was already mentioned in "Previous results" part.

Corollary

Let $\langle X, \tau_X, \mathcal{I}_X\rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y\rangle$ be ideal topological spaces. If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is homeomorphism and for each $I \subset X$ there holds $I \in \mathcal{I}_X$ iff $f[I] \in \mathcal{I}_Y$. Then the following equivalent conditions hold:

a) $f : \langle X, \tau_X^*\rangle \to \langle Y, \tau_Y^*\rangle$ is a homeomorphism;

b) $\forall A \subseteq X \; (f[A])^* = f[A^*]$;

c) $\forall B \subseteq Y \; f^{-1}[B^*] = (f^{-1}[B])^*$.

d) $\forall A \subseteq X \; \Psi(f[A]) = f[\Psi(A)]$;

e) $\forall B \subseteq Y \; f^{-1}[\Psi(B)] = \Psi(f^{-1}[B])$.

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