A plethora of big Ramsey degrees

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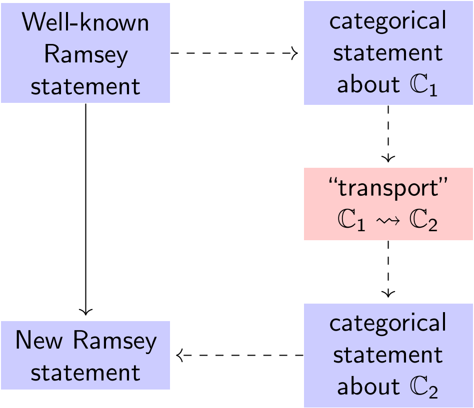
SETTOP 2022
23 Aug 2022

Benefits of categorification

- Duality Principle facilitates reasoning about dual Ramsey phenomena $\to$ "automatic dualization";
- "transport principles" enable piggyback proof strategies:



"Automatic" dualization

Duality Principle of Category Theory

If φ holds for all cat's then φ^{op} holds for all cat's.

Example.

If $\mathbb{C}$ is a locally small directed category with the Ramsey property then $\mathbb{C}$ has amalgamation.

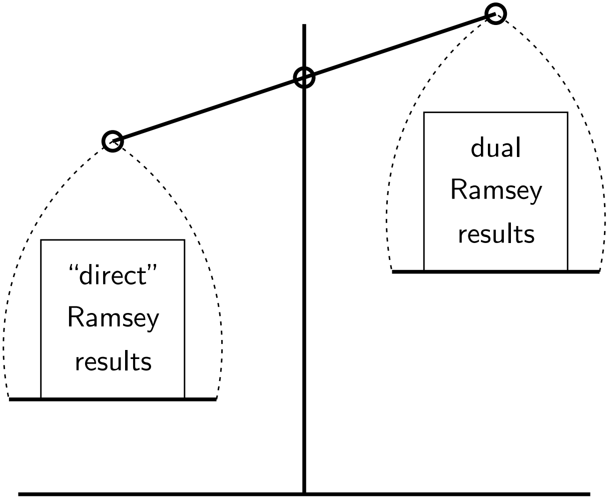
$$\partial \Big\updownarrow$$

If $\mathbb{C}$ is a locally small dually directed category with the dual Ramsey property then $\mathbb{C}$ has projective amalgamation.

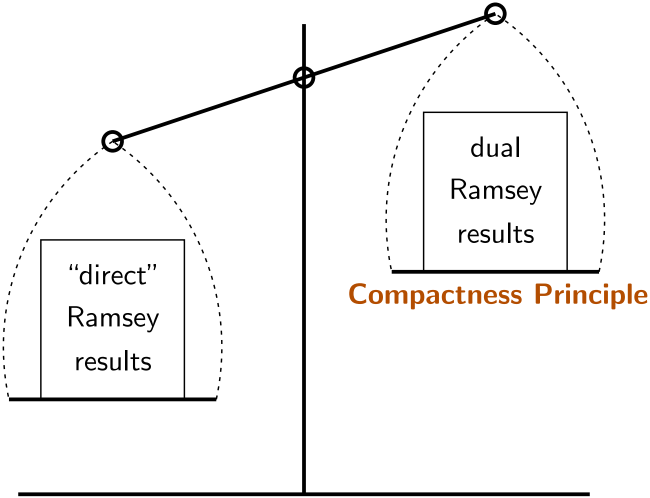
"Automatic" dualization

Why bother?

“Automatic” dualization



“Automatic” dualization



Compactnes Principle

Theorem. Infinite Ramsey Theorem $\Rightarrow$ Finite Ramsey Theorem.

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Theorem (General Compactness Principle – "direct").
Let $\mathbb{D}$ be a full subcategory of $\mathbb{C}$ such that $\hom(A, B)$ is finite for all $A, B \in \mathrm{Ob}(\mathbb{D})$ and let S be a universal and weakly locally finite object for $\mathbb{D}$. Then for every $A \in \mathrm{Ob}(\mathbb{D})$:

$$t_{\mathbb{D}}(A) \leqslant T_{\mathbb{C}}(A, S).$$

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Useful statement in the dual case???

Taxonomy of big Ramsey degrees

Big Ramsey degrees	embedding	structural	
"direct"	$T_{\mathbb{C}}(A, S)$	$\tilde{T}_{\mathbb{C}}(A, S)$	no topology involved
dual	$T^{\partial}_{\mathbb{C}}(A, S)$	$\tilde{T}^{\partial}_{\mathbb{C}}(A, S)$	topology is essential
	coloring morphisms	coloring subobjects	

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For small Ramsey degrees: $t^{\partial}_{\mathbb{C}}(A) = t_{\mathbb{C}^{op}}(A)$.

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With this state of affairs it is **NOT** the case that
$T^{\partial}_{\mathbb{C}}(A, B) = T_{\mathbb{C}^{op}}(A, B)$ $\leftarrow$ **Infrastructure needed!**

The setup: Step 1 – enriched categories

Top . . . topological spaces + continuous maps

$\mathbb{C}$ enriched over **Top** . . .

- homsets are topological spaces and
- composition of morphisms is continuous

Note.

1. Every category enriched over **Top** is locally small
2. Every category has a trivial (discrete) enrichment over **Top**

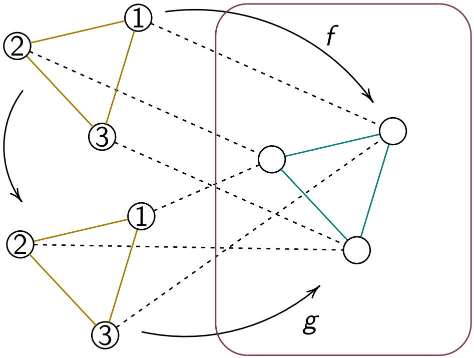
The setup: Step 2 – subobjects

"Embeddings:" hom(A, B)

"Subobjects:" $\binom{B}{A} = \text{hom}(A, B)/{\sim_A}$, where

$f \sim_A g$ iff $\exists \alpha \in \text{Aut}(A) : f = g \cdot \alpha$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$



The setup: Step 2 – subobjects

"Embedding degree:" $\chi : \mathrm{hom}(A, B) \to k$

"Structural degree:" $\chi : \mathrm{hom}(A, B)/\sim_A \to k$, where

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The setup: Step 2 – subobjects

$\mathbb{C}$ ... a locally small category

$\mathfrak{G} = (G_A)_{A \in \mathrm{Ob}(\mathbb{C})}$ ... $G_A \leqslant \mathrm{Aut}_{\mathbb{C}}(A)$

$\sim_{\mathfrak{G}}$ ... $f \sim_{\mathfrak{G}} g$ if $\exists \alpha \in G_A : f = g \cdot \alpha$ (where $f, g \in \hom_{\mathbb{C}}(A, B)$)

$\binom{B}{A}_{\mathfrak{G}} = \hom(A, B)/\sim_{\mathfrak{G}}$

NB. In the two extreme cases:

- if $\mathfrak{G} = (\{\mathrm{id}_A\})_{A \in \mathrm{Ob}(\mathbb{C})}$ then $\binom{B}{A}_{\mathfrak{G}}$ "=" $\hom(A, B)$;
- if $\mathfrak{G} = (\mathrm{Aut}(A))_{A \in \mathrm{Ob}(\mathbb{C})}$ then $\binom{B}{A}_{\mathfrak{G}} = \binom{B}{A}$.

The setup: Putting it all together

$(\mathbb{C}, \mathfrak{G})$ . . . $\mathbb{C}$ enriched over **Top**

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$C \xrightarrow{\mathfrak{G}}_\flat (B)^A_{k,t}$ ... $\forall$ Borel coloring $\chi : \binom{C}{A}_{\mathfrak{G}} \to k$

$\exists\ w \in \hom(B, C)$ s.t. $|\chi(w \cdot \binom{B}{A}_{\mathfrak{G}})| \leqslant t$

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$T^{\mathfrak{G}}_{\mathbb{C}}(A, S)$ ... $S \xrightarrow{\mathfrak{G}}_\flat (S)^A_{k,t}$

Fact. $(T^{\mathfrak{G}}_{\mathbb{C}})^{\partial}(A, S) = T^{\mathfrak{G}}_{\mathbb{C}^{op}}(A, S)$

Putting it all together

The extreme cases:

$\mathfrak{G} \rightarrow$ **enrichment**$\downarrow$	$(\{\mathrm{id}_A\})_{A \in \mathrm{Ob}(\mathbb{C})}$	$(\mathrm{Aut}(A))_{A \in \mathrm{Ob}(\mathbb{C})}$
discrete	$T_{\mathbb{C}}(A, S)$	$\tilde{T}_{\mathbb{C}}(A, S)$
loc comp 2nd ctble Hausdorff	$T^{\partial}_{\mathbb{C}}(A, S)$	$\tilde{T}^{\partial}_{\mathbb{C}}(A, S)$

Putting it all together

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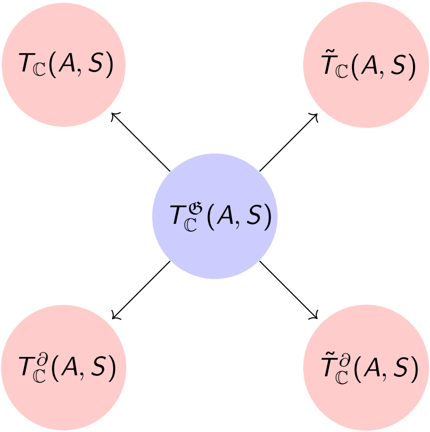
Plethora of big Ramsey degrees $T^{\mathfrak{G}}_{\mathbb{C}}$ "between" $T_{\mathbb{C}}$ and $\tilde{T}_{\mathbb{C}}$!

Putting it all together



Fundamental relationships

Theorem. [M 2022+]
Let $\mathbb{C}$ be a category enriched over **Top** whose morphisms are mono, and let $A, S \in \mathrm{Ob}(\mathbb{C})$. Assume that

- the enrichment is discrete, or
- $\hom(A, A)$ and $\hom(A, S)$ are locally compact second countable Hausdorff and $\mathrm{Aut}(A)$ is a topological group closed in $\hom(A, A)$.

Then $T(A, S) \geqslant |\mathrm{Aut}(A)|$.

In particular, if $\mathrm{Aut}(A)$ is infinite then $T(A, S) = \infty$.

Question. What happens with $T^{\mathfrak{G}}_{\mathbb{C}}(A, S)$ if $[\mathrm{Aut}(A) : G_A] < \infty$?

Fundamental relationships

Theorem. [Zucker 2019, M 2022+]
Let $\mathbb{C}$ be a category enriched over **Top** whose morphisms are mono, and let $\mathfrak{G}$ and $\mathfrak{H}$ be two choices of finite automorphism groups. Assume that

- the enrichment is discrete, or
- $\hom(A, S)$ is locally compact second countable Hausdorff and both G_A and H_A are discrete groups.

Then $|G_A| \cdot T^{\mathfrak{G}}(A, S) = |H_A| \cdot T^{\mathfrak{H}}(A, S)$.

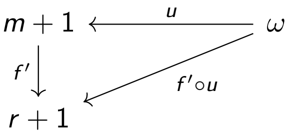
In particular, $T(A, S) = |G_A| \cdot T^{\mathfrak{G}}(A, S)$.

Towards the General Compactness Principle

- T. J. Carlson, S. G. Simpson: *A dual form of Ramsey's theorem.* Adv. Math. 53 (1984), 265–290.

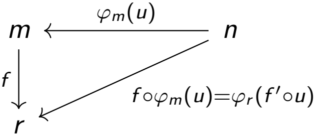
Theorem. Infinite Dual Ramsey Theorem $\Rightarrow$ Finite Dual Ramsey Theorem

∂Ramsey: YES



Countable chains
$+$ rigid surj's (enriched)

∂Ramsey: ?



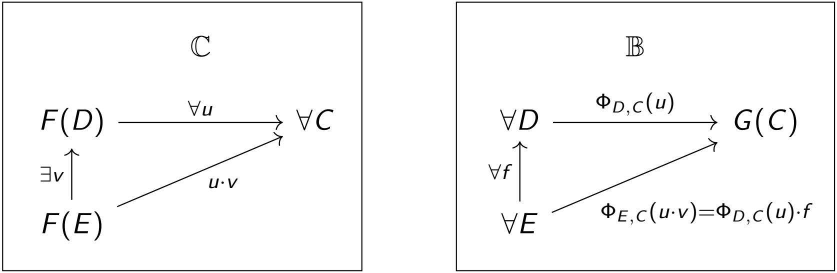
Finite chains
$+$ rigid surj's

Towards the General Compactness Principle

A Borel pre-adjunction between $\mathbb{C}$ and $\mathbb{B}$ consists of

- a pair of **maps** $F : \mathrm{Ob}(\mathbb{B}) \rightleftarrows \mathrm{Ob}(\mathbb{C}) : G$, and
- Borel maps $\Phi_{X,Y} : \hom_{\mathbb{C}}(F(X), Y) \to \hom_{\mathbb{B}}(X, G(Y))$

such that:



Towards the General Compactness Principle

$\mathbb{C}$ . . . a small category

$\mathbf{Sub}(\mathbb{C})$ . . .

- objects: all full subcategories of $\mathbb{C}$
- morphisms $\mathbb{B} \to \mathbb{D}$:
  $(f_B)_{B \in \mathrm{Ob}(\mathbb{B})}$ where $\mathrm{dom}(f_B) = B$ and $\mathrm{cod}(f_B) \in \mathrm{Ob}(\mathbb{D})$

NB. $\mathbb{C} \hookrightarrow \mathbf{Sub}(\mathbb{C})$ "canonically"

Theorem [M 2021]. $t_{\mathbb{C}}(A) = T_{\mathbf{Sub}(\mathbb{C})}(A, \mathbb{C})$.

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Theorem [M 2021]. $t_{\mathbb{C}}(A) = T_{\mathbf{Sub}(\mathbb{C})}(A, \mathbb{C})$.

Fun Fact [M 2021]. $t_{\mathbb{C}}(A) = \min_{S,\mathbb{S}} T_{\mathbb{S}}(A, S)$.

General Compactness Principle – Iteration 0

$(\mathbb{C}, \mathfrak{G})$ ... category enriched over **Top** whose morphisms are mono and with distinguished automorphism groups

$\mathbb{D}$ ... directed small full subcategory of $\mathbb{C}$

$S \in \mathrm{Ob}(\mathbb{C})$ ... universal for $\mathbb{D}$.

Theorem [M 2022+]. Assume that

- there is a Borel pre-adjunction $F : \mathbf{Sub}(\mathbb{D}) \leftrightarrows \mathbb{C} : G$ such that $G(S) = \mathbb{D}$ and $F(\mathbb{D}) \to S$;
- the enrichment of $\mathbb{D}$ is discrete or $\mathbb{D}$ has a countable skeleton;
- $\hom(F(A), S)$ is locally compact second countable Hausdorff and both G_A and $G_{F(A)}$ are finite discrete groups.

Then $t_{\mathbb{D}}^{\mathfrak{G}}(A) \leqslant T_{\mathbb{C}}^{\mathfrak{G}}(F(A), S)$ for all $A \in \mathrm{Ob}(\mathbb{D})$.

General Compactness Principle

To be continued...