Weak compactness cardinals for abstract logics

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Introduction

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I now discuss the main example of such a connection.

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A cardinal δ is *subtle* if for every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma < \delta \rangle$ and every closed unbounded subset C of δ, there exist $\beta < \gamma$ in C with the property that $A_\beta = A_\gamma \cap \beta$.

We let "Ord *is subtle*" denote the scheme of axioms stating that for every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma \in \mathrm{Ord}\rangle$ and every closed unbounded class C of ordinals, there exist $\beta < \gamma$ in C with the property that $A_\beta = A_\gamma \cap \beta$.

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Proposition

If Φ *is a sentence in the language of set theory with the property that* $\mathbf{ZFC} + \Phi$ *is consistent, then*

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Weakly $C^{(n)}$-shrewd cardinals

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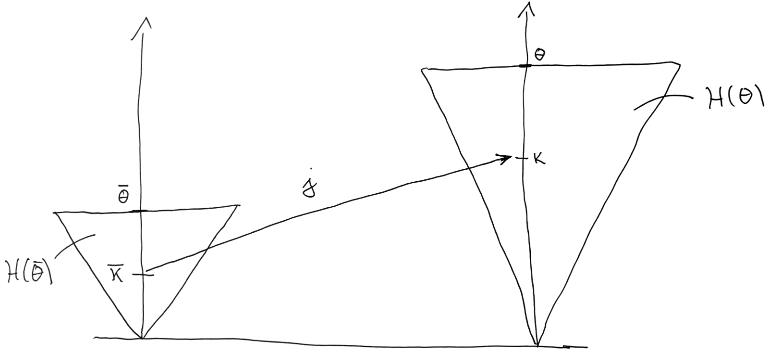

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A cardinal κ is *shrewd* if for every $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, every ordinal $\gamma > \kappa$ and every subset A of V_κ such that $\Phi(A, \kappa)$ holds in V_γ, there exist ordinals $\alpha < \beta < \kappa$ such that $\Phi(A \cap \mathrm{V}_\alpha, \alpha)$ holds in V_β.

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The following statements are equivalent for every cardinal κ:

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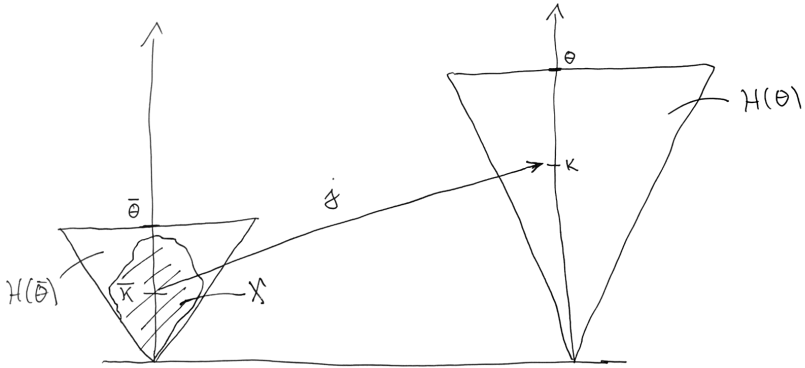

θ
H(θ)
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Definition

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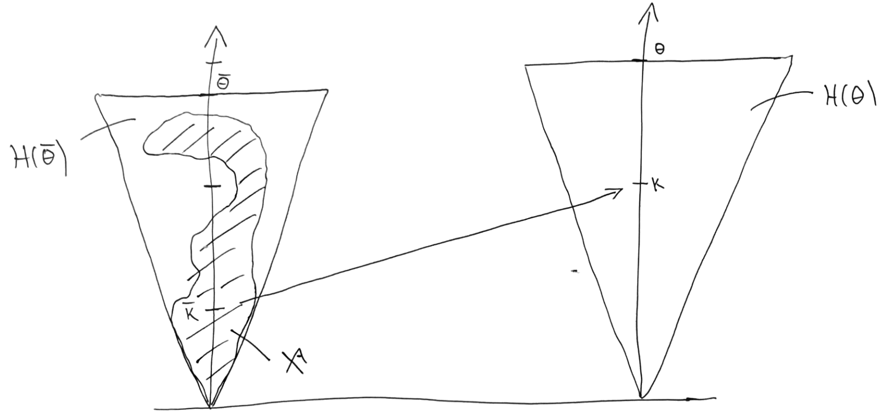

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A cardinal is strongly unfoldable if and only if it is shrewd.

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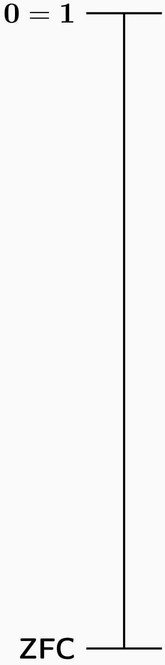
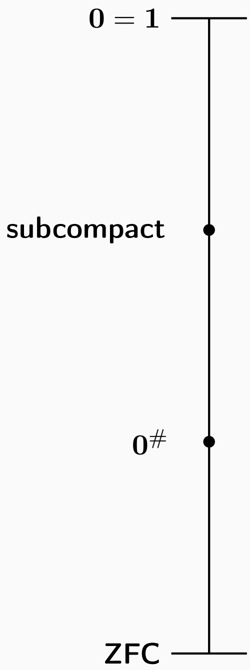
In combination with results of Bagaria and Wilson, this shows that certain patterns repeat in all parts of the large cardinal hierarchy.


0 = 1
ZFC



$0 = 1$
subcompact
$0^{\#}$
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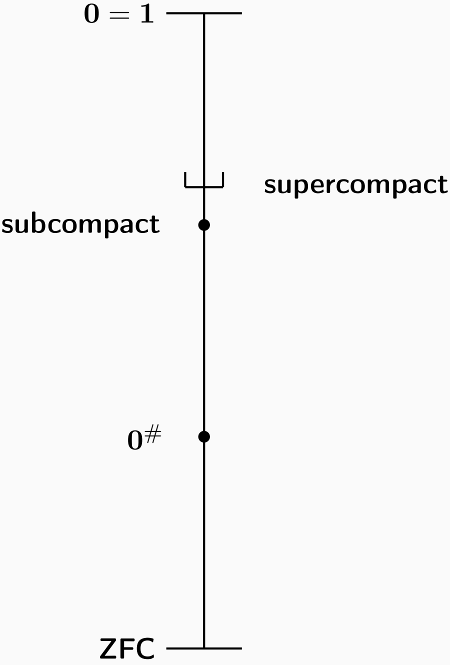
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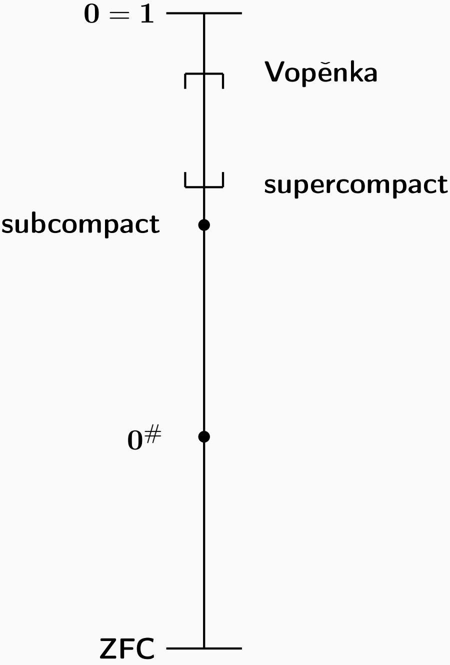

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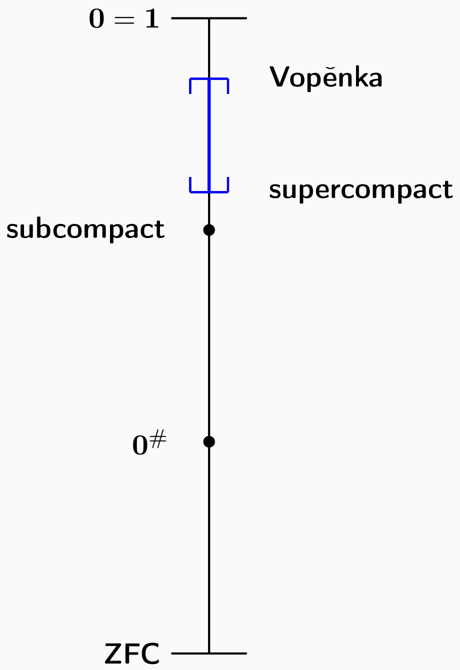
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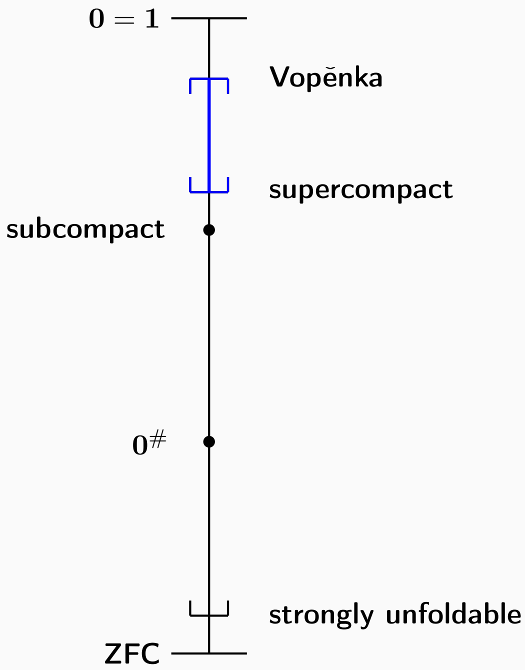

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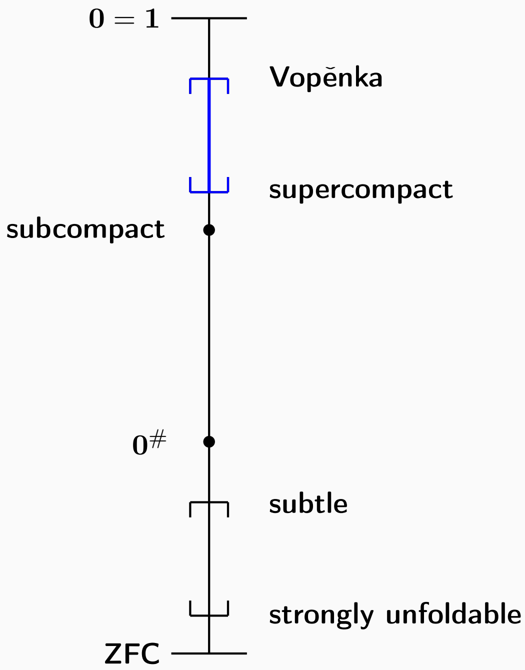

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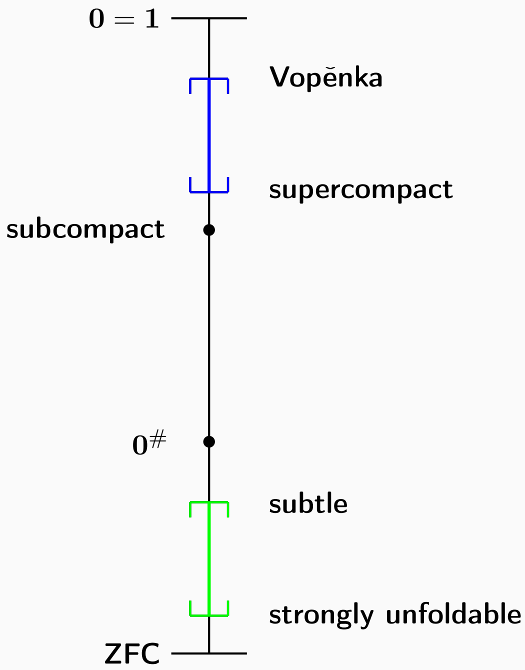
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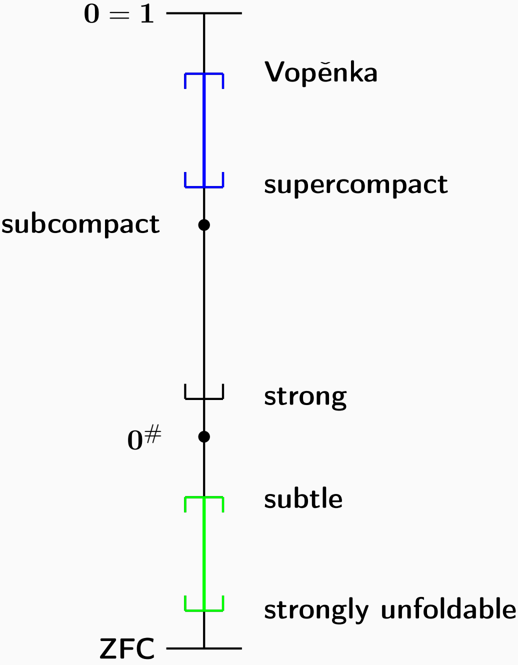

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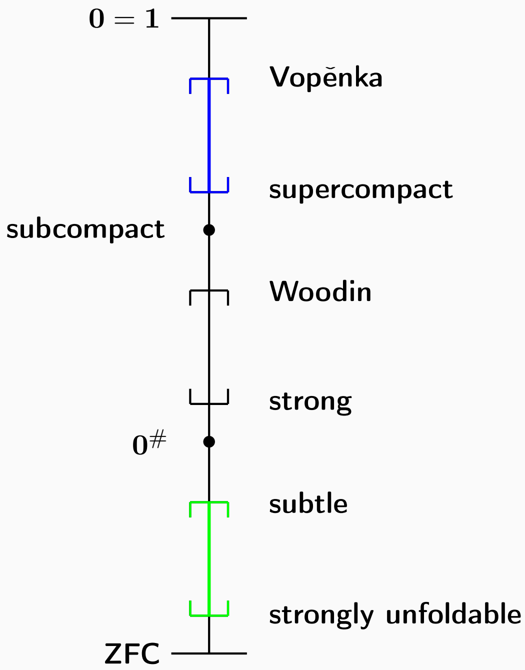
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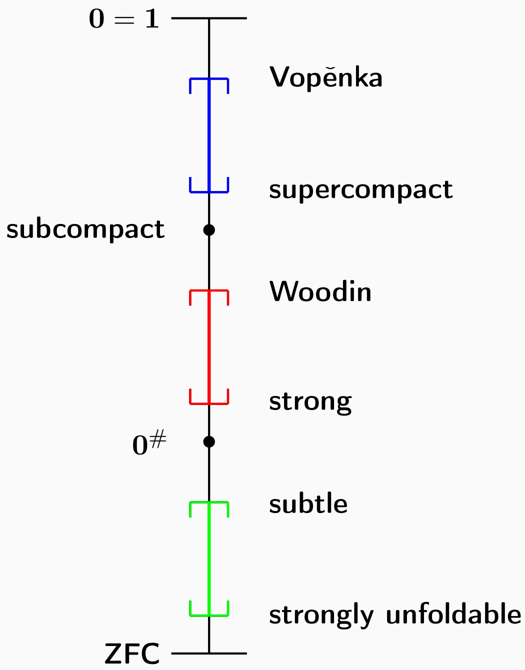

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Weak compactness cardinals for abstract logics

Definition

Given a natural number $n > 0$, a cardinal κ is *weakly $C^{(n)}$-shrewd* if for every cardinal $\kappa < \theta \in C^{(n)}$ and every $z \in \mathrm{H}(\theta)$, there exist a cardinal $\bar{\theta} \in C^{(n)}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \upharpoonright \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.

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In the remainder of this talk, I want to outline the proof of the following implication:

Lemma

Assume that for every natural number $n > 0$, there exist unboundedly many weakly $C^{(n)}$-shrewd cardinals. Then every abstract logic has unboundedly many weak compactness cardinals.

- A *language* is a tuple $\tau = \langle \mathfrak{C}_\tau, \mathfrak{F}_\tau, \mathfrak{R}_\tau, \mathfrak{a}_\tau \rangle$, where $\mathfrak{C}_\tau$, $\mathfrak{F}_\tau$ and $\mathfrak{R}_\tau$ are pairwise disjoint sets and $\mathfrak{a}_\tau : \mathfrak{F}_\tau \cup \mathfrak{R}_\tau \longrightarrow \omega \setminus \{0\}$ is a function.

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  We then call $\mathfrak{C}_\tau$ the *set of constant symbols* of τ, $\mathfrak{F}_\tau$ the *set of function symbols* of τ, $\mathfrak{R}_\tau$ the *set of relation symbols* of τ and $\mathfrak{a}_\tau$ the *arity function* of τ.

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- Given a language τ, a τ*-structure* is a tuple

  $$M \;=\; \langle |M|, (c^M)_{c \in \mathfrak{C}_\tau}, (f^M)_{f \in \mathfrak{F}_\tau}, (R^M)_{R \in \mathfrak{R}_\tau} \rangle,$$

  where $|M|$ is a non-empty set, $c^M \in |M|$ for $c \in \mathfrak{C}_\tau$,
  $f^M : |M|^{\mathfrak{a}_\tau(f)} \longrightarrow |M|$ for $f \in \mathfrak{F}_\tau$ and $R^M \subseteq |M|^{\mathfrak{a}_\tau(R)}$ for $R \in \mathfrak{R}_\tau$.

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  We let $Str(\tau)$ denote the class of all τ-structures.

- A *morphism* between languages τ and υ is an injection

$$h : \mathfrak{C}_\tau \cup \mathfrak{F}_\tau \cup \mathfrak{R}_\tau \longrightarrow \mathfrak{C}_\upsilon \cup \mathfrak{F}_\upsilon \cup \mathfrak{R}_\upsilon$$

with $h[\mathfrak{C}_\tau] \subseteq \mathfrak{C}_\upsilon$, $h[\mathfrak{F}_\tau] \subseteq \mathfrak{F}_\upsilon$, $h[\mathfrak{R}_\tau] \subseteq \mathfrak{R}_\upsilon$, $\mathfrak{a}_\upsilon(h(f)) = \mathfrak{a}_\tau(f)$ for all $f \in \mathfrak{F}_\tau$ and $\mathfrak{a}_\upsilon(h(R)) = \mathfrak{a}_\tau(R)$ for all $R \in \mathfrak{R}_\tau$.

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  Such a morphism is a *renaming* if it is bijective.

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$$h : \mathfrak{C}_\tau \cup \mathfrak{F}_\tau \cup \mathfrak{R}_\tau \longrightarrow \mathfrak{C}_\upsilon \cup \mathfrak{F}_\upsilon \cup \mathfrak{R}_\upsilon$$

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Such a morphism is a *renaming* if it is bijective. Given a renaming h from τ to υ, we let

$$h_* : Str(\tau) \longrightarrow Str(\upsilon)$$

denote the unique bijection with the property that $|h_*(M)| = |M|$ and $h(x)^{h_*(M)} = x^M$ for all $M \in Str(\tau)$ and $x \in \mathfrak{C}_\tau \cup \mathfrak{F}_\tau \cup \mathfrak{R}_\tau$.

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- Given a language υ that extends a language τ, we have $\mathcal{L}(\tau) \subseteq \mathcal{L}(\upsilon)$ and, for all $\phi \in \mathcal{L}(\tau)$ and $M \in Str(\upsilon)$, we have $M \models_{\mathcal{L}} \phi$ if and only if $M \restriction \tau \models_{\mathcal{L}} \phi$.

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- If h is a renaming of a language τ into a language υ, then there is a unique bijection $h_* : \mathcal{L}_\tau \longrightarrow \mathcal{L}_\rho$ with the property that

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  holds for every τ-structure M and all $\phi \in \mathcal{L}(\tau)$.
- There exists a minimal cardinal $o(\mathcal{L})$ (the *occurrence number* of $\mathcal{L}$) such that for every language υ and all $\phi \in \mathcal{L}(\upsilon)$, there is a language τ with the property that υ extends τ, τ has less than $o(\mathcal{L})$-many symbols and ϕ is an element of $\mathcal{L}(\tau)$.

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- If $\tau \in \mathrm{H}(o(\mathcal{L}))$ is a language and π is a non-trivial permutation of $\mathcal{L}(\tau)$, then there exists a τ-structure $M_{\tau,\pi} \in \mathrm{H}(\mu)$ and $\phi_{\tau,\pi} \in \mathcal{L}(\tau)$ with the property that

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In the following, fix a sufficiently large natural number n and a weakly $C^{(n)}$-shrewd cardinal κ greater than $|\mathrm{H}(\mu)|$.

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By our assumption, we can now find $\bar{\theta} \in C^{(n)}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and the property that $\mathrm{ran}(j)$ contains the language τ and the theory T.

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This setup ensures that $\bar{\kappa}$ is a regular cardinal greater than μ and $\mathrm{H}(\mu)$ is a subset of X. By elementarity, the fact that $\tau \subseteq \mathrm{H}(\kappa)$ implies that $\tau \cap \mathrm{H}(\bar{\kappa}) \in X$ and $j(\tau \cap \mathrm{H}(\bar{\kappa})) = \tau$.

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Thank you for listening!