Convex Embeddability on Countable Linear Orders and Knot Theory

Joint work with Alberto Marcone, Luca Motto Ros, and Vadim Weinstein (former Kulikov)

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Classification problems and Borel reducibility

Definition

Given two classification problems (X, E) and (Y, F), we say that E **reduces** to F iff there exists a map $\varphi : X \to Y$ such that

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If X and Y are two standard Borel spaces and φ is Borel we say that E is **Borel reducible** to F, and write $E \leq_B F$.

We say that E and F are Borel bi-reducible, and write $E \sim_B F$, if $E \leq_B F$ and $F \leq_B E$.

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If X and Y are two topological spaces and φ is a Baire measurable map, we say that E is **Baire reducible** to F, and write $E \leq_{Baire} F$.

Example

Let LO be the Polish space of codes for linear orders on $\mathbb{N}$, i.e.

$$\mathsf{LO} = \{L \in 2^{\mathbb{N}\times\mathbb{N}} : L \text{ codes a linear order}\},$$

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- $\cong_{\mathsf{LO}}$ is an analytic equivalence relation: it is induced by a continuous action of the infinite symmetric group S_∞.
- $\cong_{\mathsf{LO}}$ is S_∞-complete, i.e. any other equivalence relation arising from a Borel action of the group S_∞ Borel reduces to $\cong_{\mathsf{LO}}$.

Connections between linear orders and knots

Definition

Let $\bar{B}$ be a space homeomorphic to a closed ball in $\mathbb{R}^3$. Given a map $f\colon [0,1] \to \bar{B}$, we say that the pair $(\bar{B}, \operatorname{Im} f)$ is a **proper arc** in $\bar{B}$ if f is a topological embedding and $f(x) \in \partial\bar{B} \iff x = 0$ or $x = 1$. The collection of proper arcs is denoted by $\mathcal{A}$.

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Two proper arcs $(\bar{B}, f)$ and $(\bar{B}', f')$ are **equivalent**, in symbols

$$(\bar{B}, f) \equiv_{\mathcal{A}} (\bar{B}', f'),$$

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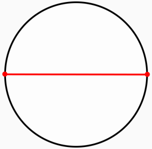

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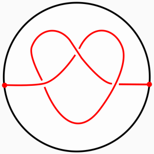

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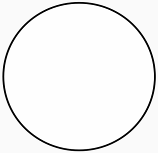

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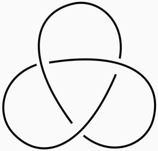

Trefoil knot

Theorem (V. Kulikov, 2017)

(a) $\cong_{\mathsf{LO}} \leq_B \equiv_{\mathcal{A}}, \equiv_{\mathcal{K}}$.

(b) There is a turbulent equivalence relation E such that $E \leq_B \equiv_{\mathcal{A}}, \equiv_{\mathcal{K}}$, hence $\equiv_{\mathcal{A}}, \equiv_{\mathcal{K}} \not\leq_B \cong_{\mathsf{LO}}$. Thus $\cong_{\mathsf{LO}} <_B \equiv_{\mathcal{A}}, \equiv_{\mathcal{K}}$.

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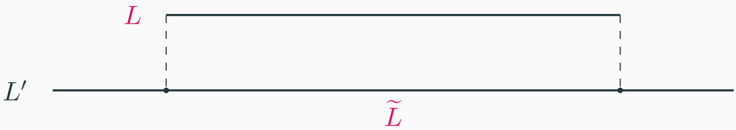
$$? \leq_B \precsim_{\mathcal{A}}$$

Convex embeddability on LO

Consider the relation of **convex embeddability** $\trianglelefteq_{\mathsf{LO}}$ between two linear orders L and L' (R. Bonnet, E. Corominas and M. Pouzet, 1973):

$$L \trianglelefteq_{\mathsf{LO}} L' \text{ if } L \text{ is isomorphic to a convex subset } \widetilde{L} \text{ of } L'.$$



We call **convex bi-embeddability**, and denote by $\bowtie_{\mathsf{LO}}$, the equivalence relation on LO induced by $\trianglelefteq_{\mathsf{LO}}$.

Clearly, for $L, L' \in LO$,

$$L \cong_{\mathsf{LO}} L' \Rightarrow L \bowtie_{\mathsf{LO}} L',$$

but the converse is not true.

Example

$\omega + \mathbb{Z}\omega \bowtie_{\mathsf{LO}} \mathbb{Z}\omega$, but $\omega + \mathbb{Z}\omega \ncong_{\mathsf{LO}} \mathbb{Z}\omega$.

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Theorem [I. - Motto Ros]

$\trianglelefteq_{\mathsf{LO}} \leq_B \precsim_{\mathcal{A}}$. Thus also $\bowtie_{\mathsf{LO}} \leq_B \approx_{\mathcal{A}}$, where $\approx_{\mathcal{A}}$ is the analytic equivalence relation associated to $\precsim_{\mathcal{A}}$.

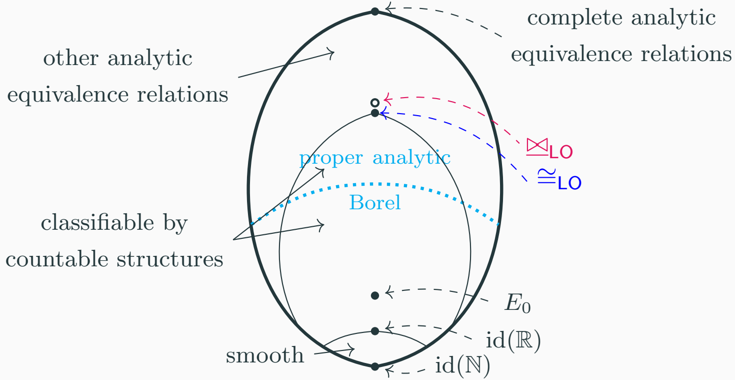
Complexity with respect to Borel Reducibility

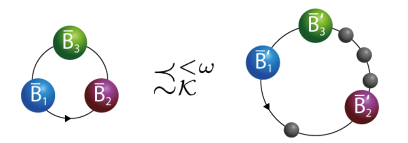
Theorem [I. - Marcone - Motto Ros - Weinstein]

(a) $\cong_{\mathsf{LO}} \leq_B \bowtie_{\mathsf{LO}}$.

(b) $\bowtie_{\mathsf{LO}} \leq_{Baire} \cong_{\mathsf{LO}}$.

(c) If X is a turbulent Polish G-space, then the equivalence relation induced by the group G on X is not Borel reducible to $\bowtie_{\mathsf{LO}}$.





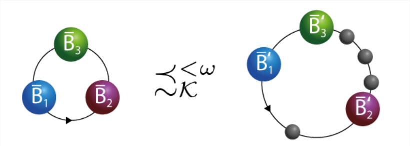
Definition

Let $K, K' \in \mathcal{K}$. Then K is a **(finite) piecewise component** of K', in symbols

$$K \precsim_{\mathcal{K}}^{<\omega} K',$$

if and only if there is an orientation of K' and a finite number of closed balls $\bar{B}'_1, ..., \bar{B}'_n$ such that

A notion of component for Knots



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if and only if there is an orientation of K' and a finite number of closed balls $\bar{B}'_1, ..., \bar{B}'_n$ such that

(a) the $(\bar{B}'_i, K' \cap \bar{B}'_i)$ are (almost) pairwise disjoint sub-arcs of K', oriented according to the chosen orientation of K', of which K is an "ordered" **(finite) tame sum**;

(b) if an endpoint of some $(\bar{B}'_i, K' \cap \bar{B}'_i)$ is singular, then it is not isolated.

Countable Circular Orders

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Definition (Cěch, 1969)

A ternary relation $C \subset X^3$ on a set X is said to be a **circular order** if the following conditions are satisfied:

- Cyclicity: $(x, y, z) \in C \Rightarrow (y, z, x) \in C$;
- Asymmetry: $(x, y, z) \in C \Rightarrow (y, x, z) \notin C$;
- Transitivity: $(x, y, z), (x, z, w) \in C \Rightarrow (x, y, w) \in C$;
- Totality: if $x, y, z \in X$ are distinct, then $(x, y, z) \in C$ or $(x, z, y) \in C$.

Denote by CO the Polish space of codes for circular orders on $\mathbb{N}$, i.e.

$$\mathsf{CO} = \{C \in 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} : C \text{ codes a circular order}\}.$$

The isomorphism relation on CO

Definition

Let $C, C' \in \mathsf{CO}$. We say that C and C' are **circularly isomorphic**, and write $C \cong_{\mathsf{CO}} C'$, if there exists a bijective function between them which preserves the circular order.

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Every $L \in \mathsf{LO}$ defines a standard circular order $C_L \in \mathsf{CO}$ as follows:

$C_L(n, m, k) \iff (n <_L m <_L k) \vee (m <_L k <_L n) \vee (k <_L n <_L m)$.

Clearly, for $L, L' \in \mathsf{LO}$,

$$L \cong_{\mathsf{LO}} L' \Rightarrow C_L \cong_{\mathsf{CO}} C_{L'}.$$

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$\omega + 1 \ncong_{\mathsf{LO}} \omega$, but $C_{\omega+1} \cong_{\mathsf{CO}} C_\omega$.

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Theorem [I. - Marcone]

- $\cong_{\mathsf{CO}} \sim_B \cong_{\mathsf{LO}}$.
- $\cong_{\mathsf{CO}} <_B \equiv_{\mathcal{K}}$.

Convex embeddability on CO

Definition (B. Kulpeshov, H. D. Macpherson, 2005)

Let $A \subseteq C$, where C is a circular order. The set A is said to be **convex** in C if for any $x, y \in A$ one of the following holds:

1. for any $z \in C$ with $C(x, z, y)$ we have $z \in A$;
2. for any $z \in C$ with $C(y, z, x)$ we have $z \in A$.

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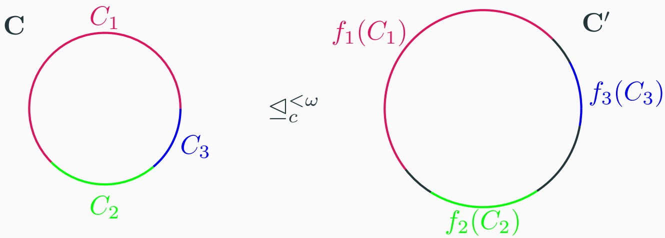
Definition

Let C and C' be circular orders. We say that C is a **convex** of C', and write $C \trianglelefteq_c C'$, if there exists a convex subset A of C' such that $C \cong_{\mathsf{CO}} A$. We denote by $(\trianglelefteq_c)_{\mathsf{CO}}$ the restriction of the convexity relation to the set CO of (codes for) countable circular linear orders.



Definition

Let $C, C' \in \mathsf{CO}$. Then $C \trianglelefteq_c^{<\omega} C'$ if and only if there exists $k \in \omega$ and (non necessarily infinite) convex subsets $C_1, \ldots, C_k$ of C such that

- $C = C_1 + ... + C_k$, and
- for every $i = 1, ..., k$ there exists $f_i : C_i \to C'$ witnessing $C_i \trianglelefteq_c C'$ such that the $f_i(C_i)$'s are pairwise disjoint in C' and

$$C'(f_i(x_i), f_j(y_j), f_h(z_h))$$

for every $x_i \in C_i, y_j \in C_j, z_h \in C_h$ and $i < j < h \leq k$.

$(\trianglelefteq_c^{<\omega})_{\mathsf{CO}}$ is an analytic quasi-order on CO. Denote by $(\bowtie_c^{<\omega})_{\mathsf{CO}}$ its induced (analytic) equivalence relation.

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Consider the equivalence relation E_1, that is defined on $\mathbb{R}^{\mathbb{N}}$ as

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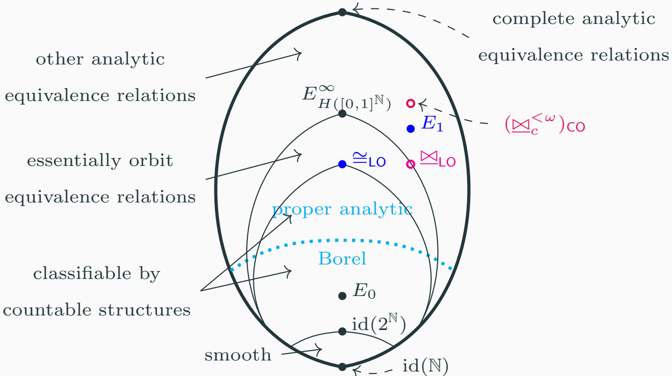
- $(\bowtie_c^{<\omega})_{\mathsf{CO}}$ is not reducible to any orbit equivalence relation;
- $\cong_{\mathsf{LO}} <_B (\bowtie_c^{<\omega})_{\mathsf{CO}}$;
- $(\bowtie_c^{<\omega})_{\mathsf{CO}}$ does not reduce to $\bowtie_{\mathsf{LO}}$.

Denote by $\approx_{\mathcal{K}}^{<\omega}$ its associated (analytic) equivalence relation and call it the **(finite) piecewise mutual component relation**.

Theorem [I. - Marcone - Motto Ros - Weinstein]

- $(\trianglelefteq_c^{<\omega})_{\mathsf{CO}} \leq_B \precsim_{\mathcal{K}}^{<\omega}$. Then, we have $(\bowtie_c^{<\omega})_{\mathsf{CO}} \leq_B \approx_{\mathcal{K}}^{<\omega}$.
- $\cong_{\mathsf{CO}} \sim_B \cong_{\mathsf{LO}} <_B \approx_{\mathcal{K}}^{<\omega}$.
- $E_1 \leq_B \approx_{\mathcal{K}}^{<\omega}$. Thus $\approx_{\mathcal{K}}^{<\omega}$ is not reducible to any orbit equivalence relation.



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Thank you for your attention!