Hereditarily indecomposable continua as Fraïssé limits

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Abstract Convergence Schemes And Their Complexities

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- In a metric space, $x \approx_\varepsilon y$ means $d(x, y) < \varepsilon$. For maps $f, g \colon X \to Y$, $f \approx_\varepsilon g$ means $\sup_{x \in X} d(f(x), g(x)) < \varepsilon$.

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- Let $\mathcal{I}$ denote the category of all continuous surjections on $\mathbb{I}$, let $\sigma\mathcal{I}$ denote the category of all arc-like continua and continuous surjections.

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Definition

A continuous map $f\colon \mathbb{I} \to \mathbb{I}$ is ε-crooked if for every $x \leq y \in \mathbb{I}$ there are $x \leq y' \leq x' \leq y$ such that $f(x) \approx_\varepsilon f(x')$ and $f(y) \approx_\varepsilon f(y')$.

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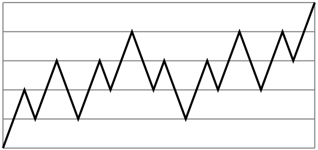

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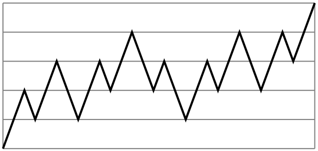

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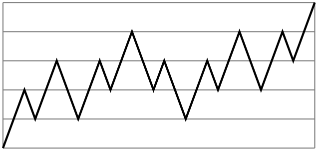

- There is a general notion of ε-crooked map between metric compacta, based on ideas of Krasinkiewicz–Minc (1976) and Maćkowiak (1985), that simplifies to the definition above for $\mathbb{I}$.
- A space X is crooked iff id_X is crooked, where crooked means ε-crooked for every $\varepsilon > 0$.

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So to obtain a hereditarily indecomposable continuum, it is enough to build a crooked sequence.

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- They characterized $\mathbb{P}$ as the unique arc-like continuum such that for every continuous surjections $f, g\colon \mathbb{P} \to Y$ onto an arc-like continuum Y and $\varepsilon > 0$, there is a homeomorphism $h\colon \mathbb{P} \to \mathbb{P}$ such that $f \approx_\varepsilon g \circ h$.

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- It follows that $\mathbb{P}$ maps onto every arc-like continuum as well as that every continuous surjection $\mathbb{P} \to \mathbb{P}$ is arbitrarily close to a homeomorphism.
- The characterization condition above looks like an approximate version of projective homogeneity.

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Let $\mathcal{K} \subseteq \mathcal{L}$ be MU-categories (categories where the hom-sets are metric spaces, subject to some coherence axioms; generalizes metric-enriched category; imagine $\langle \mathcal{I}, \sigma\mathcal{I} \rangle$ as $\langle \mathcal{K}, \mathcal{L} \rangle$).

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- projective in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-map $g \colon Y \to X$, $\mathcal{L}$-map $f \colon U \to Y$, and $\varepsilon > 0$ there is an $\mathcal{L}$-map $h \colon U \to X$ such that $f \approx_\varepsilon g \circ h$.

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The pair $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion if it satisfies certain conditions (L1), (L2), (F1), (F2), (C) assuring that $\mathcal{L}$ arised essentially by freely and continuously adding all limits of sequences to $\mathcal{K}$.

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Moreover, a Fraïssé sequence in $\mathcal{K}$ exists, and so the Fraïssé limit exists, if and only if $\mathcal{K}$ is directed, dominated by a countable subcategory, and has the amalgamation property (for every $f, g \in \mathcal{K}$ and $\varepsilon > 0$ there are $f', g' \in \mathcal{K}$ with $f' \circ f \approx_{\varepsilon} g' \circ g$).

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- For every full $\mathcal{P} \subseteq \mathbf{CPol_s}$, $\sigma\mathcal{P}$ is the full subcategory consisting of all $\mathcal{P}$-like continua, $\langle \mathcal{P}, \sigma\mathcal{P} \rangle$ is a free completion, and $\mathcal{P}$ is a Fraïssé category, and so the Fraïssé limit exists, if and only if $\mathcal{P}$ has the amalgamation property.

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- By a result of Russo (1979) there is no cofinal object in $\sigma\mathcal{P}$ unless $\mathcal{P} \subseteq \{*, \mathbb{I}, \mathbb{S}\}$.
- It turns out $\sigma\mathcal{P}$ has a Fraïssé limit if and only if $\mathcal{P} \subseteq \{*, \mathbb{I}\}$ (and the limit is $\mathbb{P}$ or $*$), and it has a cofinal object if and only if $\mathcal{P} \subseteq \{*, \mathbb{I}, \mathbb{S}\}$ (and the cofinal object is the universal pseudo-solenoid $\mathbb{P}_{\Pi}$ if $\mathbb{S} \in \mathcal{P}$).

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Theorem (somewhat folklore)

For every $\mathcal{I}$-map g and every $\varepsilon > 0$ there is $\delta > 0$ such that for every δ-crooked $f \in \mathcal{I}$ there is $h \in \mathcal{I}$ with $f \approx_\varepsilon g \circ h$.

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- $\mathcal{I}$ has AP (mountain-climbing theorem), and so there is a Fraïssé limit of $\langle \mathcal{I}, \sigma\mathcal{I} \rangle$.
- Since there are arbitrarily crooked $\mathcal{I}$-maps, and a Fraïssé sequence absorbs them, every Fraïssé sequence is a crooked sequence.
- Hence, the Fraïssé limit is a hereditarily indecomposable arc-like continuum, and so $\mathbb{P}$ by Bing's theorem.

Theorem (somewhat folklore)

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- So on the other hand, every crooked $\mathcal{I}$-sequence is Fraïssé, every hereditarily indecomposable arc-like continuum is a Fraïssé limit, and Bing's theorem follows by uniqueness of Fraïssé limits.

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Together, we obtain:

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- Hence, every $\langle \mathcal{S}_P, \sigma\mathcal{S}_P \rangle$ has a Fraïssé limit $\mathbb{P}_P$.
- But what is $\mathbb{P}_P$ and what is $\sigma\mathcal{S}_P$ (it is not full in $\sigma\mathcal{S}$)?

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- These include the pseudo-arc (type 0), the pseudo-circle (type 1), the universal pseudo-solenoid (type Π^∞), and more generally P-adic pseudo-solenoids (type P^∞ for $P \subseteq \Pi$).

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- These include the pseudo-arc (type 0), the pseudo-circle (type 1), the universal pseudo-solenoid (type Π^∞), and more generally P-adic pseudo-solenoids (type P^∞ for $P \subseteq \Pi$).
- A circle-like continuum X is an $\sigma\mathcal{S}_P$-object iff $T(X) \leq P^\infty$. A continuous surjection $f\colon X \to Y$ between $\sigma\mathcal{S}_P$-objects is a $\sigma\mathcal{S}_P$-map iff $T(f)$ is a multiplication by $t \leq P^\infty$.

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- Hence, the uniqueness of the P-adic pseudo-solenoid follows from the uniqueness of the Fraïssé limit.

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[arXiv:2208.06886]

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