Products of CW complexes

Andrew Brooke-Taylor

UNIVERSITY OF LEEDS

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For $n \in \mathbb{N}$, denote by

- D^n the closed ball of radius 1 about the origin in $\mathbb{R}^n$ (the *n-disc*),
- $\mathring{D}^n$ its interior, and
- S^{n-1} its boundary (the $(n-1)$-*sphere*).

CW complexes

Definition

A Hausdorff space X is a *CW complex* if there exists a set of continuous functions $\varphi_\alpha : D^n \to X$ (*characteristic maps*), for α in an arbitrary index set and $n \in \mathbb{N}$ a function of α, such that:

1. $\varphi_\alpha \restriction \mathring{D}^n$ is a homeomorphism to its image, and X is the disjoint union as α varies of these homeomorphic images $\varphi_\alpha[\mathring{D}^n]$ ("cells").

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We often denote $\varphi_\alpha[\overset{\circ}{D}{}^n]$ by e^n_α or just e_α.

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X is not metrizable, as x_0 does not have a countable neighbourhood base.

Proof

Identify each edge with the unit interval, with x_0 at 0. For every $f : \mathbb{N} \to \mathbb{N}$, consider the open neighbourhood $U(x_0; f)$ of x_0 whose intersection with $e^1_{X,n}$ is the interval $[\,0, 1/(f(n)+1)\,)$.

These form a neighbourhood base, but for any countably many f_i, there is a g that is not dominated by any of them, so $U(x_0; g)$ does not contain any of the $U(x_0; f_i)$. □

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Convention

In this talk, $X \times Y$ is always taken to have the product topology, so "$X \times Y$ is a CW complex" means "the product topology on $X \times Y$ is the same as the weak topology".

Example (Dowker, 1952)

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Consider the subset of $X \times Y$

$$H = \left\{ \left(\frac{1}{f(n)+1}, \frac{1}{f(n)+1} \right) \in e^1_{X,n} \times e^1_{Y,f} : n \in \mathbb{N}, f \in \mathbb{N}^{\mathbb{N}} \right\}$$

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Then $\left(\frac{1}{g(k)+1}, \frac{1}{g(k)+1} \right) \in U \times V \cap H$. So in the product topology, $(x_0, y_0) \in \bar{H}$.

More preliminaries: subcomplexes

A *subcomplex* A of a CW complex X is what you would expect.

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A *subcomplex* A of a CW complex X is a subspace which is a union of cells of X, such that if $e^n_\alpha \subseteq A$ then its closure $\overline{e^n_\alpha} = \varphi^n_\alpha[D^n]$ is contained in A.

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Definition

Let κ be a cardinal. We say that a CW complex X is *locally less than κ* if for all x in X there is a subcomplex A of X with fewer than κ many cells such that x is in the interior of A. We write *locally finite* for locally less than $\aleph_0$, and *locally countable* for locally less than $\aleph_1$.

Proposition

If κ is a regular uncountable cardinal, then a CW complex W is locally less than κ if and only if every connected component of W has fewer than κ many cells.

Proof sketch.

$\Leftarrow$ is trivial. For $\Rightarrow$, given any point w, recursively fill out to get an open (hence clopen) subcomplex containing w with fewer than κ many cells, using the fact that the cells are compact to control the number of cells along the way if $\kappa < 2^{\aleph_0}$. □

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If X and Y are both (locally) countable, then $X \times Y$ is a CW complex.

Theorem (Y. Tanaka, 1982)

If neither X nor Y is locally countable, then $X \times Y$ is not a CW complex.

What was known, beyond ZFC

Theorem (Liu Y.-M., 1978)

Assuming the Continuum Hypothesis, $X \times Y$ is a CW complex if and only if either

- *one of them is locally finite, or*
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Assuming $\mathfrak{b} = \aleph_1$, $X \times Y$ is a CW complex if and only if either

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Can we do better?

Question

Can we show, without assuming any extra set-theoretic axioms, that the product $X \times Y$ of CW complexes X and Y is a CW complex if and only if either

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Answer (follows from Tanaka's work)

No.

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Can we characterise exactly when the product of two CW complexes is a CW complex, without assuming any extra set-theoretic axioms?

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Answer (B.-T.)

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Recall

- For $f, g \in \mathbb{N}^{\mathbb{N}}$, we write $f \leq^* g$ if for all but finitely many $n \in \mathbb{N}$, $f(n) \leq g(n)$.
- The bounding number $\mathfrak{b}$ is the least cardinality of a set of functions that is unbounded with respect to $\leq^*$, i.e. such that no one g is $\geq^*$ them all, i.e.,

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}} \wedge \forall g \in \mathbb{N}^{\mathbb{N}} \exists f \in \mathcal{F} \neg(f \leq^* g)\}.$$

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Let $g\colon \mathbb{N} \to \mathbb{N}^+$ be an increasing function such that $[0, \frac{1}{g(n)}) \subset e^1_{X,n} \cap U$ for every $n \in \mathbb{N}$. Take $f \in \mathcal{F}$ such that $f \not\leq^* g$.

Consider the edge $e^1_{Y,f}$ of Y:

Let $k \in \mathbb{N}$ be sufficiently large that $\frac{1}{f(k)+1} \in e^1_{Y,f} \cap V$ and $f(k) > g(k)$.

Then $\left(\frac{1}{f(k)+1}, \frac{1}{f(k)+1} \right) \in U \times V \cap H$. So in the product topology, $(x_0, y_0) \in \bar{H}$.

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A complete characterisation

Theorem (B.-T.)

Let X and Y be CW complexes. Then $X \times Y$ is a CW complex if and only if one of the following holds:

1. *X or Y is locally finite.*
2. *One of X and Y is locally countable, and the other is locally less than $\mathfrak{b}$.*

Key features of the proof

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- Natural first attempt: inductively, for each cell of Y, find a function $f : \mathbb{N} \to \mathbb{N}$ giving a good neighbourhood of x_0. There are fewer than $\mathfrak{b}$ of these, so dominate them all with a single g and use that.

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- But that doesn't work — $f \leq^* g$ isn't good enough, you really want $f \leq g$.
- Instead, through the induction, build up g on the X side as a limit of *Hechler conditions* — finite initial sequences, along with functions you promise to dominate thereafter. This works.