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Tutorial on Suslin trees and related topics

This tutorial for the Young Set Theory Workshop will cover Suslin trees and related topics such as Aronszajn trees, special trees, tree embeddings, stationary antichains, and club isomorphisms, as well as consistency results related to the Suslin hypothesis. The intended audience of these talks is an advanced graduate student or postdoc who is familiar with Aronszajn trees, Suslin's hypothesis, and forcing, but who may not be an expert in these areas.

The tutorial will divide roughly into three parts. The first part will be a chronological survey of the above topics. We start with a brief history of Suslin's problem, starting with Suslin's work, followed by Kurepa's contributions to the theory of trees, and ending with the independence of the Suslin hypothesis proven by Jech, Tennenbaum, and Solovay after the invention of forcing. We then discuss Aronszajn trees and their embeddings into the reals and the rationals, stationary antichains, club isomorphisms, and rigid and homogeneous Aronszajn trees, focusing on work of Baumgartner, Jensen, Todorčević, Abraham, and Shelah from the 1970s and '80s.

In the second part of the tutorial we will start to focus specifically on Suslin trees. After describing Suslin trees as particular c.c.c. forcing notions, we give a detailed survey of coherent Suslin trees and free Suslin trees as well as some applications of these ideas. In the third part of the tutorial, we describe several iterated forcing constructions of models in which there exist a small number of Suslin trees and every non-Suslin Aronszajn tree is special. In this part we will provided detailed proofs, including a complete proof of a special case of the above consistency result and a discussion of more complicated variations. In particular, we will describe a model in which there exist a small number of Suslin trees, but any two normal Aronszajn trees, neither of which contains a Suslin subtree, are club isomorphic.

[1] U. Abraham. Construction of a rigid Aronszajn tree. *Proc. Amer. Math. Soc.*, 77(1):136–137, 1979.

[2] U. Abraham and S. Shelah. Martin's axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic. *Israel J. Math.*, 38(1-2):161–176, 1981.

[3] U. Abraham and S. Shelah. Isomorphism types of Aronszajn trees. *Israel J. Math.*, 50(1-2):75–113, 1985.

[4] U. Abraham and S. Shelah. A Δ^2_2-well order of the reals and incompactness of $L(Q^{MM})$. *Ann. Pure Appl. Logic*, 59(1):1–32, 1993.

[5] J. Baumgartner. Decompositions and embeddings of trees. *Notices Amer. Math. Soc.*, 17(6):967, 1970. Abstract.

[6] J. Baumgartner. *Results and Independence Proofs in Combinatorial Set Theory*. PhD thesis, University of California Berkeley, 1970.

[7] J. Baumgartner, J. Malitz, and W. Reinhardt. Embedding trees in the rationals. *Proc. Nat. Acad. Sci.*, 67(4):1748–1753, 1970.

[8] P. Cohen. *Set Theory and the Continuum Hypothesis*. W. A. Benjamin, Inc., New York-Amsterdam, 1966. vi+154 pp.

[9] K. Devlin. Note on a theorem of J. Baumgartner. *Fund. Math.*, 76(3):255–260, 1972.

[10] K. Devlin. *Constructibility*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1984. xi+425 pp.

[11] K. Devlin and H. Johnsbråten. *The Souslin problem*, volume 405 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1974.

[12] K. Devlin and S. Shelah. Souslin properties and tree topologies. *Proc. London Math. Soc.*, 39(2):237–252, 1979.

[13] B. Dushnik and E. W. Miller. Partially ordered sets. *Amer. J. Math*, 63(3):600–610, 1941.

[14] G. Fuchs and J. Hamkins. Degrees of rigidity for Souslin trees. *J. Symbolic Logic*, 74(2):423–454, 2009.

[15] H. Gaifman and E. P. Specker. Isomorphism types of trees. *Proc. Amer. Math. Soc.*, 15:1–7, 1964.

[16] K. Gödel. *The Consistency of the Continuum Hypothesis*, volume 3 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J., 1940. 66 pp.

[17] G. Gruenhage and P. Nyikos. Normality in X^2 for compact X. *Trans. Amer. Math. Soc.*, 340(2):563–586, 1993.

[18] T. Jech. Nonprovability of Souslin's hypothesis. *Comment. Math. Univ. Carolinae*, 8:291–305, 1967.

[19] T. Jech. Trees. *J. Symbolic Logic*, 36(1):1–14, 1971.

[20] T. Jech. Automorphisms of ω_1-trees. *Trans. Amer. Math. Soc.*, 173:57–70, 1972.

[21] T. Jech. Simple complete Boolean algebras. *Israel J. Math*, 18:1–10, 1974.

[22] T. Jech. *Set Theory. The Third Millennium Edition, revised and expanded*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. xiv+769.

[23] R. B. Jensen. Souslin's hypothesis is incompatible with $V = L$. *Notices Amer. Math. Soc.*, 15:935, 1968. Abstract.

[24] R. B. Jensen. Automorphism properties of Souslin continua. *Notices Amer. Math. Soc.*, 16:576, 1969. Abstract.

[25] R. B. Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308, 1972.

[26] M. Katětov. Complete normality of Cartestian products. *Fund. Math.*, 35:271–274, 1948.

[27] J. Krueger. Suslin tree preservation and club isomorphisms. Preprint.

[28] J. Krueger. Entangledness in Suslin lines and trees. *Topology Appl.*, 275, 2020.

[29] K. Kunen. *Set Theory*, volume 34 of *Studies in Logic*. College Publications, London, 2011. viii+401.

[30] D. Kurepa. Ensembles ordonnés et ramijiés. *Publ. Math. Univ. Belgrade*, 4:1–138, 1935.

[31] D. Kurepa. Ensembles linéaires et une classe de tableaux ramifiés (Tableaux ramifiés de M. Aronszajn). *Publ. Math. Univ. Belgrade*, 6/7:129–160, 1937/8.

[32] D. Kurepa. Transformations monotones des ensembles partiellement ordonnés. *Rev. Ci.*, 42:827–846, 1940.

[33] D. Kurepa. À propos d'une généralisation de la notion d'esembles bien ordonnés. *Acta Math.*, 75:139–150, 1943.

[34] D. Kurepa. *Selected Papers of Duro Kurepa. (French) Edited and with commentaries by Aleksandar Ivić, Zlatko Mamuzić, Žarko Mijajlović and Stevo Todorčević.* Matematički Institut SANU, Belgrade, 1996.

[35] P. Larson. An $\mathbb{S}_{\mathsf{max}}$ variation for one Souslin tree. *J. Symbolic Logic*, 64(1):81–98, 1999.

[36] P. Larson and S. Todorčević. Katětov's problem. *Trans. Amer. Math. Soc.*, 354:1783–1791, 2002.

[37] R. Laver. Random reals and Souslin trees. *Proc. Amer. Math. Soc.*, 100(3):531–534, 1987.

[38] K. McAloon. Les algèbres be Boole rigides et minimales. *C. R. Acad. Sci. Paris Sèr A*, 272:89–91, 1971.

[39] E. Miller. A note on Souslin's problem. *Amer. J. Math*, 65(4):673–678, 1943.

[40] T. Miyamoto. ω_1-Souslin trees under countable support iterations. *Fund. Math.*, 142(3):257–261, 1993.

[41] J. Moore. A five element basis for the uncountable linear orders. *Ann. of Math. (2)*, 163(2):669–688, 2006.

[42] P. Nyikos. A compact non-metrizable space P such that P^2 is completely normal. *Topology Proc.*, 2:359–364, 1977.

[43] G. Scharfenberger-Fabian. Optimal matrices of partitions and an application to Souslin trees. *Fund. Math.*, 210(2):111–131, 2010.

[44] C. Schlindwein. Suslin's hypothesis does not imply stationary antichains. *Ann. Pure Appl. Logic*, 64:153–167, 1993.

[45] S. Shelah. Can you take Solovay's inaccessible away? *Israel J. Math.*, 48(1):1–47, 1984.

[46] S. Shelah. *Proper and Improper Forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

[47] S. Shelah and J. Zapletal. Canonical models for $\aleph_1$-combinatorics. *Ann. Pure Appl. Logic*, 98(1-3):217–259, 1999.

[48] W. Sierpinski. Sur un probleme de la théorie générale des ensembles equivalent un problème de Souslin. *Fund. Math.*, 35:165–174, 1948.

[49] R. M. Solovay and S. Tennenbaum. Iterated Cohen extensions and Souslin's problem. *Ann. of Math. (2)*, 94:201–245, 1971.

[50] M. Suslin. Problème 3. *Fund. Math.*, 1:223, 1920.

[51] S. Tennenbaum. Souslin's problem. *Proc. Nat. Acad. Sci. USA*, 59:60–63, 1968.

[52] S. Todorčević. Forcing with a coherent Souslin tree. Preprint.

[53] S. Todorčević. Rigid Aronszajn trees. *Publ. Inst. Math.*, 27(41):259–265, 1980.

[54] S. Todorčević. Trees and linearly ordered sets. In *Handbook of Set-Theoretic Topology*, pages 235–293. North-Holland, Amsterdam, 1984.

[55] S. Todorčević. Partitioning pairs of countable ordinals. *Acta Math.*, 159(3-4):261–294, 1987.

[56] S. Todorčević. Lipschitz maps on trees. *J. Int. Math. Jussieu*, 6(3):527–556, 2007.

[57] S. Todorčević. *Walks on Ordinals and Their Characteristics*. Birkhäuser Verlag, Basel, 2007.

[58] W. H. Woodin. Supercompact cardinals, sets of reals, and weakly homogeneous trees. *Proc. Natl. Acad. Sci. USA*, 85:6587–6591, 1988.

[59] W. H. Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal*. De Gruyter Series in Logic and its Applications. Walter de Gruyter and Co., Berlin, 1999.