Mathias–Příkrý forcing and generic ultrafilters

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Ultrafilters on $\omega$

**Definition**
An ultrafilter $\mathcal{F}$ on $\omega$ is a *P-point* if for each $\mathcal{C} \in [\mathcal{F}]^\omega$ there is a pseudo-intersection $P \in \mathcal{F}$ such that $P \subset^* F$ for each $F \in \mathcal{C}$.

**Definition**
An ultrafilter $\mathcal{F}$ on $\omega$ is *selective* if for each $\{A_i : i \in \omega\}$, a partition of $\omega$ disjoint with $\mathcal{F}$ there is a selector $S \in \mathcal{F}$ such that $|S \cap A_i| = 1$ for each $i \in \omega$. 

**Theorem (Zapletal)**
An ultrafilter $\mathcal{F}$ is a P-point iff for each analytic ideal $I \subset \mathcal{F}^*$ there is an $\mathcal{F}$-ideal $C$ such that $I \subseteq C \subseteq \mathcal{F}^*$. 
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Let $\mathcal{F}$ be an ultrafilter in $\omega$. The following properties are equivalent:

1. $\mathcal{F}$ is selective,
2. For each $c : [\omega]^2 \rightarrow 2$ there exists a $c$-homogeneous set $F \in \mathcal{F}$, for each tall analytic ideal $I \neq \emptyset$. 

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**Theorem (Zapletal)**
An ultrafilter $\mathcal{F}$ is a P-point iff for each analytic ideal $\mathcal{I} \subseteq \mathcal{F}^*$ there is an $F_\sigma$ ideal $\mathcal{C}$ such that $\mathcal{I} \subseteq \mathcal{C} \subseteq \mathcal{F}^*$. 
Generic filters

Theorem

The generic filter on the poset \((\mathcal{P}(\omega) \setminus \text{Fin}, \subset^*)\) is a selective ultrafilter.
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**Theorem (Todorcevic)**

\((\text{LC})^1\) An ultrafilter is selective if and only if it is a generic filter on \(\mathcal{P}(\omega) \setminus \text{Fin}\) over \(L(\mathbb{R})\).

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\(^1\text{(LC)}\) denotes the assumption that there exist sufficiently large cardinals in \(V\). In this talk infinitely many Woodin and a measurable above.
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Let $\mathcal{I}$ be an $F_\sigma$ ideal on $\omega$. Denote by $\mathbb{Q}_\mathcal{I}$ the forcing $(\mathcal{P}(\omega) \setminus \mathcal{I}, \subset^*)$.

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Theorem (Zapletal, Ch.)
\((\text{LC})\) \(\mathcal{F}\) is a \(\mathbb{Q}_\mathcal{I}\)-generic filter over \(L(\mathbb{R})\) iff
\begin{enumerate}
  \item \(\mathcal{F}\) is a \(P\)-point disjoint with \(\mathcal{I}\), and
  \item for each closed set \(C \subseteq \mathcal{P}(\omega)\) disjoint with \(\mathcal{F}\) there is \(e \in \mathcal{F}^*\) such that \(C \subseteq \langle \mathcal{I}, \{e\} \rangle\).
\end{enumerate}

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Definition (Mathias forcing)

\[ M = \{ \langle a, F \rangle : a \in [\omega]^<\omega, F \in [\omega]^\omega \} \]

\[ \langle a, F \rangle < \langle b, H \rangle \text{ if } b \subseteq a, F \subseteq H, \text{ and } a \setminus b \subseteq H. \]
Definition (Mathias–Příkrý forcing)

Let $\mathcal{F}$ be a filter on $\omega$.

$\mathcal{M}(\mathcal{F}) = \{\langle a, F \rangle : a \in [\omega]<\omega, F \in \mathcal{F}\}$

$\langle a, F \rangle < \langle b, H \rangle$ if $b \subseteq a$, $F \subseteq H$, and $a \setminus b \subseteq H$. 

Fact

A Mathias real is a pseudo-intersection of $\mathcal{F}$ ($x \subseteq \ast F$ for each $F \in \mathcal{F}$).

Definition

$U \subseteq [\omega]<\omega$ is an $\mathcal{F}$-universal set if $[\mathcal{F}]<\omega \cap U \neq \emptyset$ for each $F \in \mathcal{F}$.

Fact

$[x]<\omega \cap U \neq \emptyset$ for each $\mathcal{F}$-universal set $U$. 

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Definition (Mathias real for $\mathcal{F}$)

$x = \bigcup \{ a : \langle a, F \rangle \} \in \mathbb{G}$, where $\mathbb{G}$ is an $\mathbb{M}(\mathcal{F})$ generic filter.

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Let $\mathcal{F}$ be a filter on $\omega$.

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$U \subset [\omega]^{<\omega}$ is an $\mathcal{F}$-universal set if $[F]^{<\omega} \cap U \neq \emptyset$ for each $F \in \mathcal{F}$.

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$[x]^{<\omega} \cap U \neq \emptyset$ for each $\mathcal{F}$-universal set $U.$
Definition (Pseudogeneric real for $\mathcal{F}$)

Let $\mathcal{F}$ be a filter on $\omega$. A set $m \subset \omega$ is a pseudogeneric real for $\mathcal{F}$ if

1. $m \subseteq^* F$ for each $F \in \mathcal{F}$,
2. $[m]^{<\omega} \cap U \neq \emptyset$ for each $\mathcal{F}$-universal set $U$. 

Theorem

If $m \subset \omega$ is a pseudogeneric real for $\mathcal{F}$ and $c \subset \omega$ is a Cohen real, then $m \cap c$ is a Mathias real for $\mathcal{F}$. 
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Theorem

(LC) $\mathcal{F}$ is a $\mathbb{Q}_I$-generic filter over $L(\mathbb{R})$ iff

1. $\mathcal{F}$ is a $P$-point disjoint with $\mathcal{I}$, and

2. for each closed set $C \subset \mathcal{P}(\omega)$ disjoint with $\mathcal{F}$ there is $e \in \mathcal{F}^*$ such that $C \subseteq \langle \mathcal{I}, \{e\} \rangle$. 

Lemma

Let $U$ be a $P$-point. Assume there is (in some extension of $V$) an elementary embedding $j: V \rightarrow M$ such that $R \cap V$ is countable in $M$. Then there is a Mathias real $g \in j(F)$ (over $V$).

Lemma

Let $\mathcal{F}$ be as in the theorem. Suppose $D \in L(\mathbb{R})$ is open dense in $\mathbb{Q}_I$. Then $M(\mathcal{F}) \forces \dot{g} \in D_{V[\dot{g}]}$. 
**Theorem**

(LC) $\mathcal{F}$ is a $Q\mathcal{I}$-generic filter over $L(\mathbb{R})$ iff

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**Lemma**

Let $\mathcal{U}$ be a $P$-point. Assume there is (in some extension of $V$) an elementary embedding $j: V \rightarrow M$ such that $\mathbb{R} \cap V$ is countable in $M$. Then there is a Mathias real $g \in j(\mathcal{F})$ (over $V$).
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(LC) \( \mathcal{F} \) is a \( \mathbb{Q}_\mathcal{I} \)-generic filter over \( L(\mathbb{R}) \) iff

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Lemma
Let \( \mathcal{F} \) be as in the theorem. Suppose \( D \in L(\mathbb{R}) \) is open dense in \( \mathbb{Q}_\mathcal{I} \). Then \( M(\mathcal{F}) \models \dot{g} \in D^{V[\dot{g}]} \).
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Let \( U \) be a P-point. Assume there is (in some extension of \( V \)) an elementary embedding \( j : V \to M \) such that \( \mathbb{R} \cap V \) is countable in \( M \). Then there is a Mathias real \( g \in j(F) \) (over \( V \)).

Lemma
Let \( F \) be as in the theorem. Suppose \( D \in L(\mathbb{R}) \) is open dense in \( \mathbb{Q}_I \). Then \( M(F) \models \dot{g} \in D^{V[\dot{g}]} \).

Suppose \( D \in L(\mathbb{R}) \) is open dense in \( \mathbb{Q}_I \). We need to show that \( D \cap F \neq \emptyset \).
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Suppose $D \in L(\mathbb{R})$ is open dense in $\mathbb{Q}_I$. We need to show that $D \cap \mathcal{F} \neq \emptyset$.

Pass to an extension $V[G]$ where $j : V \to M$ exists. There is a Mathias real $g \in j(\mathcal{F})$. 
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Lemma

Let $\mathcal{F}$ be as in the theorem. Suppose $D \in L(\mathbb{R})$ is open dense in $\mathbb{Q}_\mathcal{I}$. Then $\mathcal{M}(\mathcal{F}) \models \dot{g} \in D^V[\dot{g}]$.

Suppose $D \in L(\mathbb{R})$ is open dense in $\mathbb{Q}_\mathcal{I}$. We need to show that $D \cap \mathcal{F} \neq \emptyset$.

Pass to an extension $V[G]$ where $j: V \rightarrow M$ exists. There is a Mathias real $g \in j(\mathcal{F})$. Now $g \in D^V[G]$, i.e. $g \in j(D) \cap j(\mathcal{F})$. 