Every ccc pseudocompact crowded space is resolvable

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- *All spaces under discussion are considered to be Tychonoff.*

Definition

A space X is *resolvable* if there is a dense subset $A \subseteq X$ such that $X \setminus A$ is dense as well.

- A resolvable space has no isolated points.
- A space without isolated points is called *crowded*.
- The concept of resolvability is due to Hewitt (1943).
- Hewitt showed that every crowded metrizable space and every crowded locally compact space is resolvable.

Definition

A crowded space X is *irresolvable* if it is not resolvable.

Theorem (Hewitt (1943))

There is a crowded irresolvable space.

Proof.

Let τ denote the usual topology on $\mathbb{Q}$. Hence the collection $\mathcal{P}$ of crowded Tychonoff topologies on $\mathbb{Q}$ is not empty. The collection $\mathcal{P}$ is partially ordered by inclusion. If $\mathcal{T}$ is a chain in $\mathcal{P}$, then $\bigcup \mathcal{T}$ is the basis for a crowded Tychonoff topology on $\mathbb{Q}$. Hence there is a maximal element ρ in $\mathcal{P}$ and we claim that $(\mathbb{Q}, \rho)$ is irresolvable. Striving for a contradiction, assume that A and B are complementary dense subsets of $\mathbb{Q}$. We claim that A is crowded. If not, then there exists a nonempty open subset U in $\mathbb{Q}$ such that $U \cap A$ is a singleton. Since ρ is crowded, U is infinite. Hence $U \setminus A$ is a nonempty open subset of $\mathbb{Q}$ missing the dense set A, which is a contradiction. Hence A is crowded and so is B. Hence the topological sum topology on $A \cup B$ can be regarded as a proper strengthening of ρ in $\mathcal{P}$, which contradicts the maximality of ρ. $\square$

Theorem (Kunen, Szymański and Tall (1986))

Under $V = L$, every crowded Baire space is resolvable.

Theorem (Kunen, Szymański and Tall (1986))

If ZFC is consistent with the existence of a measurable cardinal, then ZFC is consistent with the existence of an irresolvable (zero-dimensional) Baire space.

Definition

A group G is called *strongly resolvable* if every crowded group topology on G is resolvable.

- Ol'shanski (1980) showed that there is a countably infinite group G with no group topology except the discrete topology. Hence such groups, and all finite groups, are strongly resolvable.

Theorem (Comfort and vM (1994))

Let G be an Abelian group. If G contains no isomorphic copy of $\bigoplus_\omega\{0,1\}$, then G is strongly resolvable.

Theorem (Comfort and vM (1994))

Under Martin's Axiom, if G be an Abelian containing an isomorphic copy of $\bigoplus_\omega\{0,1\}$, then G is not strongly resolvable.

- The proof of this last result depends heavily on a construction of crowded extremally disconnected topological groups by Malyhin (1975).

Problem (Arhangelskii, 1967)

Is there in ZFC an example of a crowded extremally disconnected topological group?

Definition

Let $\kappa \geq 2$ be a cardinal. A crowded space X is called *κ-resolvable* if there is a family of κ-many pairwise disjoint dense subsets of X.

Theorem (Illanes (1996))

If X is n-resolvable for every n, then X is ω-resolvable.

Definition

1. A space X ps *pseudocompact* if every realvalued continuous function $f\colon X \to \mathbb{R}$ is bounded.
2. A space X is *countably compact* if every countably infinite subset of X has an accumulation point.

- A countably compact space is pseudocompact.
- A *normal* space is pseudocompact if and only if it is countably compact.

Theorem (Comfort and García Ferreira (1996))

Every crowded countably compact space is ω-resolvable.

Theorem (Pytkeev (2002), Juhász, Soukup and Szentmiklóssy (2007))

Every countably compact crowded space is ω_1-resolvable.

Questions

1. *(Comfort and García Ferreira (1996)) Is every pseudocompact crowded space resolvable?*
2. *(Juhász, Soukup and Szentmiklóssy (2007)) Is every countably compact crowded space $\mathfrak{c}$-resolvable?*

- A space X is *ccc* is every family consisiting of pairwise disjoint nonempty open subsets of X is countable.

Theorem (vM 2013)

Every crowded ccc pseudocompact space is $\mathfrak{c}$*-resolvable.*

- Hence this answers the above questions in the class of ccc spaces.

Lemma

Let X be crowded ccc space, and let W be a nonempty open subset of X. Then there is a countably infinite family $\mathcal{U}$ of open F_σ-subsets of X such that

1. *for every $U \in \mathcal{U}$, $\overline{U} \subseteq W$,*
2. *if $U, V \in \mathcal{U}$ are distinct, then $\overline{U} \cap \overline{V} = \emptyset$,*
3. *$\bigcup \mathcal{U}$ is dense in W.*

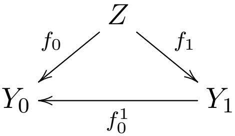
- We let Z denote any compact crowded ccc space. To Z we will associate a certain inverse sequence of compact metrizable spaces. We do not know what the 'length' is of that sequence, but we do know that it is at most ω_1.
- We assume that Z is a subspace of $\mathbb{I}^\kappa$, for some infinite cardinal κ.

Lemma

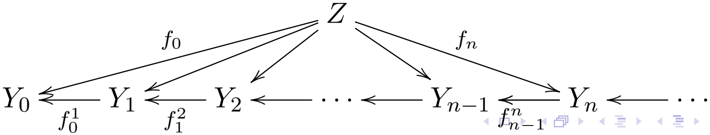
For every nonempty open F_σ-subset U of Z there are a countable $A(U) \subseteq \kappa$ and an open subset V of $\mathbb{I}^A$ such that $\pi^{-1}_{A(U)}(V) \cap Z = U$.

- Now let $\mathcal{U}_0$ be a family of open F_σ-subsets of Z given by the lemma in the previous slide with $W = Z$. For every $U \in \mathcal{U}_0$, let $A(U)$ be the countable subset of κ we get from the lemma in this slide, and put $A_0 = \bigcup_{U \in \mathcal{U}_0} A(U)$. Let $Y_0 = \pi_{A_0}(Z)$, and let $f_0 \colon Z \to Y_0$ denote the restriction of the projection π_{A_0} to Z.
- Observe that Y_0 is compact metrizable. Observe that if we enlarge A_0 to a countable subset B of κ, then $\pi_B(Z)$ and the restriction of the projection π_B to Z have the same properties.

- Now do the same thing for every $U \in \mathcal{U}_0$, thus obtaining $\mathcal{U}_1$, $A_1 \supseteq A_0$, Y_1, and a continuous surjection $f_0^1 \colon Y_1 \to Y_0$ such that the diagram

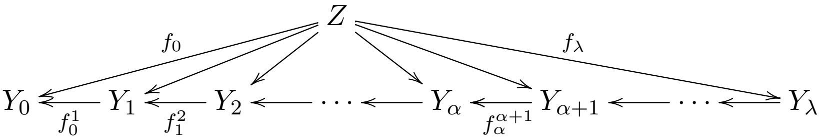


  commutes.
- We continue this process in exactly the same way for all $n < \omega$, thus obtaining an increasing sequence of countable subsets $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ of κ, corresponding spaces $Y_0, Y_1, \ldots,$ and mappings such that all subdiagrams of the following diagram commute:



- Put $A = \bigcup_{n<\omega} A_n$, and let $Y^\omega = \pi_A(Z)$. Observe that Y^ω is the inverse limit of the sequence $\{Y_n : n < \omega\}$ and hence the projections give us all mappings that are needed to 'complete' the diagram.
- The collections $\mathcal{U}_n$ for $n < \omega$ form an ω-branching tree under reverse inclusion. The intersection of each path trough the tree is a closed G_δ-subset of Z.
- By ccc, there are at most countably many of them that have nonempty interior.
- If there are no such paths, the process stops and we put $Y_\omega = Y^\omega$.
- If there are such paths, we continue the construction exactly in the same way with each of the countably many nonempty interiors.
- Continue in this way.

- By ccc, the process stops at an infinite limit ordinal $\lambda \leq \omega_1$. This brings us to the following inverse system:



  and the corresponding tree of open subsets $\mathbb{T} = \{U : (\exists\, \alpha < \lambda)(U \in \mathcal{U}_\alpha)\}$, ordered in the obvious way. Observe that $\mathbb{T}$ does not have uncountable chains nor anti-chains. Also observe that Y_α for $\alpha < \lambda$ is compact metrizable, and that Y_λ may have weight ω_1.
- Let $Z = \beta X$ for some crowded pseudocompact ccc space, and assume that $\lambda = \omega_1$.
- As before, for every $\alpha < \lambda$ and $U \in \mathcal{U}_\alpha$, let V_U in Y_α be open such that $f_\alpha^{-1}(V_U) = U$, and put $F_\alpha = Y_\alpha \setminus \bigcup_{U \in \mathcal{U}_\alpha} V_U$. Observe that $f_\alpha^{-1}(F_\alpha) = Z \setminus \bigcup \mathcal{U}_\alpha$, and that $(f_\beta^\alpha)^{-1}(F_\beta) \subseteq F_\alpha$ if $\alpha > \beta$.

Lemma

If $\lambda = \omega_1$, *then* $\{f_\alpha^{-1}(F_\alpha) : \alpha < \lambda\}$ *covers* Z.

- For every $\alpha < \omega_1$, let $\{A^\alpha_\xi : \xi < \mathfrak{c}\}$ be a partition of $F_\alpha \setminus \bigcup_{\beta<\alpha}(f^\alpha_\beta)^{-1}(F_\beta)$ in $\mathfrak{c}$ Bernstein sets in $F_\alpha \setminus \bigcup_{\beta<\alpha}(f^\alpha_\beta)^{-1}(F_\beta)$ (some members of this family may be empty). For $\xi < \mathfrak{c}$, put

$$B_\xi = \bigcup_{\alpha<\omega_1} f_\alpha^{-1}(A^\alpha_\xi) \cap X.$$

  Then $\mathcal{B} = \{B_\xi : \xi < \mathfrak{c}\}$ partitions X by the previous result.
- Every B_ξ is dense. Here the pseudocompactness of X is used.
- Every nonempty G_δ of Z intersects X. That is enough to prove that for every nonempty open subset V of X there exists $\alpha < \lambda$ such that $f_\alpha(V) \setminus \bigcup_{\beta<\alpha}(f^\alpha_\beta)^{-1}(F_\beta)$ contains a Cantor set and hence meets A^α_ξ.

Problem

How about pseudocompact spaces of cellularity ω_1?