Union theorems for trees

Stevo Todorčević

S.A.S.A., Belgrade; C.N.R.S., Paris; University of Toronto

Novi Sad, August 18-21, 2014
Joint work with K. Tyros
Outline

Part I: Classical Union Theorems

(1) Folkman Theorem
(2) Carlson-Simpson Theorem
(3) Dual Ramsey Theorem
Outline

Part I: Classical Union Theorems
(1) Folkman Theorem
(2) Carlson-Simpson Theorem
(3) Dual Ramsey Theorem

Part II: Ramsey Theory of Trees
(4) Halpern-Läuchli Theorem
(5) Dense-set version
(6) Strong-subtree version
Outline

Part I: Classical Union Theorems
(1) Folkman Theorem
(2) Carlson-Simpson Theorem
(3) Dual Ramsey Theorem

Part II: Ramsey Theory of Trees
(4) Halpern-Läuchli Theorem
(5) Dense-set version
(6) Strong-subtree version

Part III: Dual Ramsey Theory of Trees
(7) Hales-Jewett Theorem for Trees
(8) Union Theorem for Trees
(9) Union Theorem for Trees in Dimension $> 1$
(10) Conjectures
Part I: Finite Union Theorem

Theorem (Folkman, 1969)

For every pair of positive integers $k$ and $c$ there is integer $F(k,c)$ such that for every $c$-coloring of the power-set $P(X)$ of some set $X$ of cardinality $\geq F(k,c)$, there is a family $D = (D_i)_{i=1}^k$ of pairwise disjoint nonempty subsets of $X$ such that the family $U(D) = \{ \bigcup_{I \subseteq \{1,2,\ldots,k\}} D_i : \emptyset \neq I \subseteq \{1,2,\ldots,k\} \}$ of unions is monochromatic.
Theorem (Folkman, 1969)

For every pair of positive integers $k$ and $c$ there is integer $F = F(k, c)$ such that for every $c$-coloring of the power-set $\mathcal{P}(X)$ of some set $X$ of cardinality $\geq F$, there is a family $\mathbf{D} = (D_i)_{i=1}^k$ of pairwise disjoint nonempty subsets of $X$ such that the family

$$\mathcal{U}(\mathbf{D}) = \left\{ \bigcup_{i \in I} D_i : \emptyset \neq I \subseteq \{1, 2, \ldots, k\} \right\}$$

of unions is monochromatic.
Theorem (Carlson-Simpson, 1984)

For every finite Souslin measurable coloring of the power-set $\mathcal{P}(\omega)$ of $\omega$, there is a sequence $D = (D_n)_{n<\omega}$ of pairwise disjoint nonempty subsets of the natural numbers such that the set $U(D) = \{ \bigcup_{M \in D_n} M : M \text{ is a non-empty subset of } \omega \}$ is monochromatic.
Theorem (Carlson-Simpson, 1984)

For every finite Souslin measurable coloring of the power-set $\mathcal{P}(\omega)$ of $\omega$, there is a sequence $D = (D_n)_{n<\omega}$ of pairwise disjoint nonempty subsets of the natural numbers such that the set

$$U(D) = \left\{ \bigcup_{n \in M} D_n : M \text{ is a non-empty subset of } \omega \right\}$$

is monochromatic.
Dual Ramsey Theorem

Theorem (Carlson-Simpson, 1984)

For every finite Souslin-measurable coloring of the collection $U_\omega = U_\omega(\omega)$ of all infinite families of pairwise disjoint nonempty subsets of $\omega$, there is a family $D = \{D_n : n < \omega\} \in U_\omega$ such that $U_\omega \upharpoonright D = \{E_n : n < \omega\} \in U_\omega$ is monochromatic.
Dual Ramsey Theorem

Theorem (Carlson-Simpson, 1984)

For every finite Souslin-measurable coloring of the collection

\[ \mathcal{U}[\omega] = \mathcal{U}[\omega](\omega) \]

of all infinite families of pairwise disjoint nonempty subsets of \( \omega \), there is a family \( \mathbf{D} = \{D_n : n < \omega\} \in \mathcal{U}[\omega] \) such that

\[ \mathcal{U}[\omega]\upharpoonright \mathbf{D} = \{\{E_n : n < \omega\} \in \mathcal{U}[\omega] : (\forall n < \omega)E_n \in \mathcal{U}(\mathbf{D})\} \]

is monochromatic.
Part II: Halpern-Läuchli Theorem

A tree is a partially ordered set \((T, \leq_T)\) such that

\[
\text{Pred}_t(T) = \{ s \in T : s <_T t \}
\]

is finite and totally ordered.
Part II: Halpern-Läuchli Theorem

A tree is a partially ordered set \((T, \leq_T)\) such that

\[
\text{Pred}_t(T) = \{ s \in T : s <_T t \}
\]

is finite and totally ordered.

We consider only **rooted and finitely branching trees with no maximal nodes**.
For $n < \omega$, the $n$th level of $T$, is the set

$$T(n) = \{ t \in T : |\text{Pred}_t(T)| = n \}.$$
For $n < \omega$, the $n$th level of $T$, is the set

$$T(n) = \{ t \in T : |\text{Pred}_t(T)| = n \}.$$
For a subset $D$ of $T$, we define its **level set**

$$L_T(D) = \{ n \in \omega : D \cap T(n) \neq \emptyset \}$$
For a subset $D$ of $T$, we define its **level set**

\[ L_T(D) = \{ n \in \omega : D \cap T(n) \neq \emptyset \} \]

$LT(D) = \{1, 3\}$
From now on, fix an integer \( d \geq 1 \).
From now on, fix an integer $d \geq 1$.

A vector tree

$$T = (T_1, ..., T_d)$$

is a $d$-sequence of rooted and finitely branching trees with no maximal nodes.
From now on, fix an integer $d \geq 1$.

A **vector tree**

$$T = (T_1, \ldots, T_d)$$

is a $d$-sequence of rooted and finitely branching trees with no maximal nodes.
For a vector tree $T = (T_1, ..., T_d)$ we define its **level product** as

$$\otimes T = \bigcup_{n<\omega} T_1(n) \times ... \times T_d(n)$$
For a vector tree $\mathbf{T} = (T_1, \ldots, T_d)$ we define its level product as

$$\bigotimes \mathbf{T} = \bigcup_{n<\omega} T_1(n) \times \cdots \times T_d(n)$$

The $n$-th level of the level product of $\mathbf{T}$ is

$$\bigotimes \mathbf{T}(n) = T_1(n) \times \cdots \times T_d(n).$$
Let $\mathbf{T} = (T_1, \ldots, T_d)$ a vector tree.
Let \( T = (T_1, \ldots, T_d) \) a vector tree.
For \( t = (t_1, \ldots, t_d) \) and \( s = (s_1, \ldots, s_d) \) in \( \otimes T \), set
\[
    t \leq_T s \text{ iff } t_i \leq_{T_i} s_i \text{ for all } i = 1, \ldots, d.
\]
Let $\mathbf{T} = (T_1, ..., T_d)$ a vector tree.
For $\mathbf{t} = (t_1, ..., t_d)$ and $\mathbf{s} = (s_1, ..., s_d)$ in $\otimes \mathbf{T}$, set

$$\mathbf{t} \leq_T \mathbf{s} \text{ iff } t_i \leq_{T_i} s_i \text{ for all } i = 1, ..., d.$$ 

For $\mathbf{t} = (t_1, ..., t_d)$ in $\otimes \mathbf{T}$, we define

$$\text{Succ}_\mathbf{t}(\mathbf{T}) = \{ \mathbf{s} \in \otimes \mathbf{T} : \mathbf{t} \leq_T \mathbf{s} \}$$
A sequence $\mathbf{D} = (D_1, \ldots, D_d)$ is called a vector subset of $\mathbf{T}$ if
1. if $D_i$ is a subset of $T_i$ for all $i = 1, \ldots, d$ and
2. $L_{T_1}(D_1) = \ldots = L_{T_d}(D_d)$. 

A sequence $\mathbf{D} = (D_1, \ldots, D_d)$ is called a vector subset of $\mathbf{T}$ if

1. if $D_i$ is a subset of $T_i$ for all $i = 1, \ldots, d$ and
2. $L_{T_1}(D_1) = \cdots = L_{T_d}(D_d)$.

For a vector subset $\mathbf{D}$ of $\mathbf{T}$ we define its level product

$$\otimes \mathbf{D} = \bigcup_{n < \omega} (T_1(n) \cap D_1) \times \cdots \times (T_d(n) \cap D_d).$$
A sequence $D = (D_1, ..., D_d)$ is called a **vector subset** of $T$ if

1. if $D_i$ is a subset of $T_i$ for all $i = 1, ..., d$ and
2. $L_{T_1}(D_1) = ... = L_{T_d}(D_d)$.

For a vector subset $D$ of $T$ we define its **level product**

$$\otimes D = \bigcup_{n < \omega} (T_1(n) \cap D_1) \times ... \times (T_d(n) \cap D_d).$$

For $t \in \otimes T$, a vector subset $D$ of $T$ is **t-dense**, if

$$(\forall n)(\exists m)(\forall s \in \otimes T(n) \cap \text{Succ}_T(t)(\exists s' \in \otimes T(m) \cap \otimes D) \ s \leq_T s').$$
A sequence $\mathbf{D} = (D_1, \ldots, D_d)$ is called a vector subset of $\mathbf{T}$ if

1. if $D_i$ is a subset of $T_i$ for all $i = 1, \ldots, d$ and
2. $L_{T_1}(D_1) = \ldots = L_{T_d}(D_d)$.

For a vector subset $\mathbf{D}$ of $\mathbf{T}$ we define its level product

$$\otimes \mathbf{D} = \bigcup_{n < \omega} (T_1(n) \cap D_1) \times \ldots \times (T_d(n) \cap D_d).$$

For $t \in \otimes \mathbf{T}$, a vector subset $\mathbf{D}$ of $\mathbf{T}$ is $t$-dense,

$$(\forall n)(\exists m)(\forall s \in \otimes \mathbf{T}(n) \cap \text{Succ}_T(t)(\exists s' \in \otimes \mathbf{T}(m) \cap \otimes \mathbf{D}) \ s \leq_T s').$$

$\mathbf{D}$ is called dense if it is root($\otimes \mathbf{T}$)-dense.
Theorem (Halpern–Läuchli, 1966)

Let $T$ be a vector tree. Then for every dense vector subset $D$ of $T$ and every subset $P$ of $\otimes D$, there exists a vector subset $D'$ of $D$ such that either

(i) $\otimes D'$ is a subset of $P$ and $D'$ is a dense vector subset of $T$,

(ii) $\otimes D'$ is a subset of $P^c$ and $D'$ is a $t$-dense vector subset $D'$ of $T$ for some $t$ in $\otimes T$.
Theorem (Halpern–Läuchli, 1966)

Let $T$ be a vector tree. Then for every dense vector subset $D$ of $T$ and every subset $P$ of $\otimes D$, there exists a vector subset $D'$ of $D$ such that either

(i) $\otimes D'$ is a subset of $P$ and $D'$ is a dense vector subset of $T$, or
Theorem (Halpern–Läuchli, 1966)

Let $T$ be a vector tree. Then for every dense vector subset $D$ of $T$ and every subset $P$ of $\otimes D$, there exists a vector subset $D'$ of $D$ such that either

(i) $\otimes D'$ is a subset of $P$ and $D'$ is a dense vector subset of $T$, or

(ii) $\otimes D'$ is a subset of $P^c$ and $D'$ is a $t$-dense vector subset $D'$ of $T$ for some $t$ in $\otimes T$. 
Fix a rooted and finitely branching tree \((T, \leq_T)\) of height \(\omega\) with no maximal nodes.
Fix a rooted and finitely branching tree \((T, \leq_T)\) of height \(\omega\) with no maximal nodes.

A subset \(S\) of \(T\) is called a strong subtree of \(T\) if,
Strong Subtree

Fix a rooted and finitely branching tree \((T, \leq_T)\) of height \(\omega\) with no maximal nodes.

A subset \(S\) of \(T\) is called a **strong subtree of** \(T\) if,

1. \(S\) has a minimum.
Fix a rooted and finitely branching tree \((T, \leq_T)\) of height \(\omega\) with no maximal nodes.

A subset \(S\) of \(T\) is called a **strong subtree of** \(T\) if,

1. \(S\) has a minimum.
2. Every level of \(S\) is subset of some level of \(T\),
Strong Subtree

Fix a rooted and finitely branching tree \((T, \leq_T)\) of height \(\omega\) with no maximal nodes.

A subset \(S\) of \(T\) is called a **strong subtree of** \(T\) if,

1. \(S\) has a minimum.
2. Every level of \(S\) is subset of some level of \(T\),
3. For every \(s\) in \(S\) and \(t'\) in \(\text{ImmSucc}_T(s)\) there is unique \(s'\) in \(\text{ImmSucc}_S(s)\) with \(t \leq_T s'\).
Fix a vector tree $\mathbf{T} = (T_1, \ldots, T_d)$. 

Theorem (Strong Subtree Version of HL) 

Let $T$ be a vector tree. Then for every finite coloring of $\otimes T$ there exists a vector strong subtree $S$ of $T$ such that $\otimes S$ is monochromatic.
Fix a vector tree $\mathbf{T} = (T_1, \ldots, T_d)$. A vector subset $\mathbf{S} = (S_1, \ldots, S_d)$ of $\mathbf{T}$ is called a **vector strong subtree of $\mathbf{T}$** whenever

1. $S_i$ is a strong subtree of $T_i$ for all $i = 1, \ldots, d$, and
2. $L_{T_1}(S_1) = \cdots = L_{T_d}(S_d)$.

**Theorem (Strong Subtree Version of HL)**

Let $\mathbf{T}$ be a vector tree. Then for every finite coloring of $\otimes_{\mathbf{T}}$ there exists a vector strong subtree $\mathbf{S}$ of $\mathbf{T}$ such that $\otimes_{\mathbf{S}}$ is monochromatic.
Fix a vector tree $\mathbf{T} = (T_1, \ldots, T_d)$. A vector subset $\mathbf{S} = (S_1, \ldots, S_d)$ of $\mathbf{T}$ is called a **vector strong subtree of $\mathbf{T}$** whenever

1. $S_i$ is a strong subtree of $T_i$ for all $i = 1, \ldots, d$,
Vector Strong subtree

Fix a vector tree $\mathbf{T} = (T_1, ..., T_d)$. A vector subset $\mathbf{S} = (S_1, ..., S_d)$ of $\mathbf{T}$ is called a

**vector strong subtree of $\mathbf{T}$** whenever

1. $S_i$ is a strong subtree of $T_i$ for all $i = 1, ... d$,
2. $L_{T_1}(S_1) = ... = L_{T_d}(S_d)$. 

**Theorem (Strong Subtree Version of HL)**

Let $\mathbf{T}$ be a vector tree. Then for every finite coloring of $\otimes \mathbf{T}$ there exists a vector strong subtree $\mathbf{S}$ of $\mathbf{T}$ such that $\otimes \mathbf{S}$ is monochromatic.
Fix a vector tree $\mathbf{T} = (T_1, ..., T_d)$. A vector subset $\mathbf{S} = (S_1, ..., S_d)$ of $\mathbf{T}$ is called a **vector strong subtree of $\mathbf{T}$** whenever

1. $S_i$ is a strong subtree of $T_i$ for all $i = 1, ..., d$,
2. $L_{T_1}(S_1) = ... = L_{T_d}(S_d)$.

**Theorem (Strong Subtree Version of HL)**

Let $\mathbf{T}$ be a vector tree. Then for every finite coloring of $\otimes \mathbf{T}$ there exists a vector strong subtree $\mathbf{S}$ of $\mathbf{T}$ such that $\otimes \mathbf{S}$ is monochromatic.
Part III: Union Theorem for Trees
Part III: Union Theorem for Trees

Let $T$ be a vector tree.
Part III: Union Theorem for Trees

Let $\mathbf{T}$ be a vector tree. We define

$$U(\mathbf{T}) = \{U \subseteq \otimes \mathbf{T} : U \text{ has a minimum}\}.$$
Part III: Union Theorem for Trees

Let $\mathbf{T}$ be a vector tree. We define

$$\mathcal{U}(\mathbf{T}) = \{ U \subseteq \otimes \mathbf{T} : U \text{ has a minimum} \}.$$ 

We let $\mathcal{U}(\mathbf{T})$ take its topology from $\{0, 1\}^{\otimes \mathbf{T}}$. 
Part III: Union Theorem for Trees

Let $\mathbf{T}$ be a vector tree. We define

$$\mathcal{U}(\mathbf{T}) = \{ U \subseteq \otimes \mathbf{T} : U \text{ has a minimum} \}.$$  

We let $\mathcal{U}(\mathbf{T})$ take its topology from $\{0, 1\} \otimes \mathbf{T}$. Let $\mathbf{D}$ be a vector subset of $\mathbf{T}$. 
Part III: Union Theorem for Trees

Let $T$ be a vector tree. We define

$$U(T) = \{ U \subseteq \otimes T : U \text{ has a minimum} \}.$$ 

We let $U(T)$ take its topology from $\{0, 1\}^{\otimes T}$. Let $D$ be a vector subset of $T$. A **$D$-subspace** of $U(T)$ is a family

$$U = (U_t)_{t \in \otimes D}$$

such that
Part III: Union Theorem for Trees

Let $T$ be a vector tree. We define

$$U(T) = \{ U \subseteq \otimes T : U \text{ has a minimum} \}.$$  

We let $U(T)$ take its topology from $\{0, 1\} \otimes T$.

Let $D$ be a vector subset of $T$.

A **D-subspace** of $U(T)$ is a family

$$U = (U_t)_{t \in \otimes D}$$

such that

1. $U_t \in U(T)$ for all $t \in \otimes D$, 

Part III: Union Theorem for Trees

Let $T$ be a vector tree. We define

$$U(T) = \{ U \subseteq \otimes T : U \text{ has a minimum} \}.$$  

We let $U(T)$ take its topology from $\{0, 1\}^{\otimes T}$. Let $D$ be a vector subset of $T$. A $D$-subspace of $U(T)$ is a family

$$U = (U_t)_{t \in \otimes D}$$

such that

1. $U_t \in U(T)$ for all $t \in \otimes D$,
2. $U_s \cap U_t = \emptyset$ for $s \neq t$,  


Part III: Union Theorem for Trees

Let $T$ be a vector tree. We define

$$\mathcal{U}(T) = \{U \subseteq \otimes T : U \text{ has a minimum}\}.$$  

We let $\mathcal{U}(T)$ take its topology from $\{0, 1\}^T$. 
Let $D$ be a vector subset of $T$. 
A **D-subspace** of $\mathcal{U}(T)$ is a family

$$U = (U_t)_{t \in \otimes D}$$  

such that

1. $U_t \in \mathcal{U}(T)$ for all $t \in \otimes D$,  
2. $U_s \cap U_t = \emptyset$ for $s \neq t$,  
3. $\min U_t = t$ for all $t \in \otimes D$. 
For a subspace $U = (U_t)_{t \in \otimes D(U)}$ we define its span by

\[ [U] = \left\{ \bigcup_{t \in \Gamma} U_t : \Gamma \subseteq \otimes D(U) \right\} \cap U(T). \]
For a subspace $\mathbf{U} = (U_t)_{t \in \otimes D(\mathbf{U})}$ we define its span by

$$[\mathbf{U}] = \left\{ \bigcup_{t \in \Gamma} U_t : \Gamma \subseteq \otimes D(\mathbf{U}) \right\} \cap \mathcal{U}(\mathbf{T}).$$

If $\mathbf{U}$ and $\mathbf{U}'$ are two subspaces of $\mathcal{U}(\mathbf{T})$, we say that

$\mathbf{U}'$ is a subspace of $\mathbf{U}$, and write $\mathbf{U}' \leq \mathbf{U}$, if
For a subspace $\mathbf{U} = (U_t)_{t \in \otimes D(\mathbf{U})}$ we define its span by

$$[\mathbf{U}] = \left\{ \bigcup_{t \in \Gamma} U_t : \Gamma \subseteq \otimes D(\mathbf{U}) \right\} \cap \mathcal{U}(\mathbf{T}).$$

If $\mathbf{U}$ and $\mathbf{U}'$ are two subspaces of $\mathcal{U}(\mathbf{T})$, we say that

$\mathbf{U}'$ is a subspace of $\mathbf{U}$, and write $\mathbf{U}' \leq \mathbf{U}$, if

$[\mathbf{U}'] \subseteq [\mathbf{U}].$
For a subspace $U = (U_t)_{t \in \otimes D(U)}$ we define its span by

$$[U] = \left\{ \bigcup_{t \in \Gamma} U_t : \Gamma \subseteq \otimes D(U) \right\} \cap \mathcal{U}(T).$$

If $U$ and $U'$ are two subspaces of $\mathcal{U}(T)$, we say that $U'$ is a subspace of $U$, and write $U' \leq U$, if $[U'] \subseteq [U]$.

**Remark**

$U' \leq U$ implies that $D(U')$ is a vector subset of $D(U)$. 
Union Theorem for $\mathbf{T}$
Union Theorem for $T$

Theorem

Let $T$ be a vector tree and $P$ a Souslin measurable subset of $U(T)$. Then there exists a subspace $U'$ of $U(T)$ with $U' \leq U$ such that either (i) $U'$ is a subset of $P$ and $D(U')$ is a dense vector subset of $T$, or (ii) $U'$ is a subset of $P^c$ and $D(U')$ is a $t$-dense vector subset of $T$ for some $t$ in $\otimes T$. 
Union Theorem for $T$

Theorem

Let $T$ be a vector tree and $P$ a Souslin measurable subset of $\mathcal{U}(T)$. Also let $D$ be a dense level vector subset of $T$ and $U$ a $D$-subspace of $\mathcal{U}(T)$. Then there exists a subspace $U'$ of $\mathcal{U}(T)$ with $U' \leq U$ such that either

(i) $U'$ is a subset of $P$ and $D(U')$ is a dense vector subset of $T$,

(ii) $U'$ is a subset of $P^c$ and $D(U')$ is a $t$-dense vector subset of $T$ for some $t$ in $\mathcal{T}$. 

Union Theorem for \( T \)

**Theorem**

Let \( T \) be a vector tree and \( \mathcal{P} \) a Souslin measurable subset of \( \mathcal{U}(T) \). Also let \( D \) be a dense level vector subset of \( T \) and \( U \) a \( D \)-subspace of \( \mathcal{U}(T) \). Then there exists a subspace \( U' \) of \( \mathcal{U}(T) \) with \( U' \leq U \) such that either

(i) \( U' \) is a subset of \( \mathcal{P} \) and \( D(U') \) is a dense vector subset of \( T \),

(ii) \( U' \) is a subset of \( \mathcal{P}^c \) and \( D(U') \) is a \( t \)-dense vector subset of \( T \) for some \( t \) in \( \mathcal{O}(T) \).
Union Theorem for $T$

Theorem

Let $T$ be a vector tree and $\mathcal{P}$ a Souslin measurable subset of $\mathcal{U}(T)$. Also let $D$ be a dense level vector subset of $T$ and $U$ a $D$-subspace of $\mathcal{U}(T)$. Then there exists a subspace $U'$ of $\mathcal{U}(T)$ with $U' \leq U$ such that either

(i) $[U']$ is a subset of $\mathcal{P}$ and $D(U')$ is a dense vector subset of $T$, or
Theorem

Let $T$ be a vector tree and $P$ a Souslin measurable subset of $U(T)$. Also let $D$ be a dense level vector subset of $T$ and $U$ a $D$-subspace of $U(T)$. Then there exists a subspace $U'$ of $U(T)$ with $U' \leq U$ such that either

(i) $[U']$ is a subset of $P$ and $D(U')$ is a dense vector subset of $T$,

or

(ii) $[U']$ is a subset of $P^c$ and $D(U')$ is a $t$-dense vector subset of $T$ for some $t$ in $\otimes T$. 
Consequences

Let $T$ be a vector tree and let $S$ be a vector strong subtree of $T$. Let $U$ be an $S$-subspace of $U(T)$. Then for every finite Souslin measurable coloring of $U(T)$ there exist a vector strong subtree $S'$ of $S$ and an $S'$-subspace $U'$ of $U(T)$ with $U' \leq U$ such that $[U']$ is monochromatic.

Corollary (Carlson-Simpson, 1984) For every finite Souslin measurable coloring of $P(\omega)$ there is a sequence $D = (D_n)_{n<\omega}$ of pairwise disjoint subsets of $\omega$ such that the set $U(D) = \{ \bigcup_{m \in M} D_m : M \text{ is a non-empty subset of } \omega \}$ is monochromatic.
Corollary

Let $T$ be a vector tree and let $S$ be a vector strong subtree of $T$. Then for every finite Souslin measurable coloring of $U(T)$ there exist a vector strong subtree $S'$ of $S$ and an $S'$-subspace $U'$ of $U(T)$ with $U' \leq U$ such that $[U']$ is monochromatic.

Corollary (Carlson-Simpson, 1984)

For every finite Souslin measurable coloring of $P(\omega)$ there is a sequence $D = (D_n)_{n < \omega}$ of pairwise disjoint subsets of $\omega$ such that the set $U(D) = \{ \bigcup_{M \in D_n} M : M$ is a non-empty subset of $\omega \}$ is monochromatic.
Consequences

Corollary

Let $\mathbf{T}$ be a vector tree and let $\mathbf{S}$ be a vector strong subtree of $\mathbf{T}$. Let $\mathbf{U}$ be an $\mathbf{S}$-subspace of $\mathcal{U}(\mathbf{T})$. Then for every finite Souslin measurable coloring of $\mathcal{U}(\mathbf{T})$ there exist a vector strong subtree $\mathbf{S}'$ of $\mathbf{S}$ and an $\mathbf{S}'$-subspace $\mathbf{U}'$ of $\mathcal{U}(\mathbf{T})$ with $\mathbf{U}' \leq \mathbf{U}$ such that $\left[\mathbf{U}'\right]$ is monochromatic.

Corollary (Carlson-Simpson, 1984)

For every finite Souslin measurable coloring of $\mathcal{P}(\omega)$ there is a sequence $\mathbf{D} = (\mathbf{D}_n)_{n<\omega}$ of pairwise disjoint subsets of $\omega$ such that the set $\mathcal{U}(\mathbf{D}) = \{ \bigcup_{n \in M} \mathbf{D}_n : M \text{ is a non-empty subset of } \omega \}$ is monochromatic.
Consequences

Corollary

Let $T$ be a vector tree and let $S$ be a vector strong subtree of $T$. Let $U$ be an $S$-subspace of $\mathcal{U}(T)$. Then for every finite Souslin measurable coloring of $\mathcal{U}(T)$ there exist a vector strong subtree $S'$ of $S$ and an $S'$-subspace $U'$ of $\mathcal{U}(T)$ with $U' \leq U$ such that $[U']$ is monochromatic.
Consequences

Corollary

Let $T$ be a vector tree and let $S$ be a vector strong subtree of $T$. Let $U$ be an $S$-subspace of $U(T)$. Then for every finite Souslin measurable coloring of $U(T)$ there exist a vector strong subtree $S'$ of $S$ and an $S'$-subspace $U'$ of $U(T)$ with $U' \leq U$ such that $[U']$ is monochromatic.

Corollary (Carlson-Simpson, 1984)

For every finite Souslin measurable coloring of $\mathcal{P}(\omega)$ there is a sequence $D = (D_n)_{n<\omega}$ of pairwise disjoint subsets of $\omega$ such that the set

$$U(D) = \left\{ \bigcup_{n \in M} D_n : M \text{ is a non-empty subset of } \omega \right\}$$

is monochromatic.
Hales-Jewett Theorem

Theorem (Hales-Jewett, 1963)

Let $\Lambda$ be a finite alphabet and let $v \notin \Lambda$ be a variable. Then for every integer $c \geq 1$ there is a number $HJ(\Lambda, c)$ such that for every integer $N \geq HJ(\Lambda, c)$ and every $c$-coloring of the set of $\Lambda$-words of length $N$, i.e., the cube $\Lambda^N$ there is a variable word $x(v)$ of length $N$, an element of $(\Lambda \cup \{v\})^N \setminus \Lambda^N$ such that the set of all substitutions

$$\{x[\lambda] : \lambda \in \Lambda\}$$

is monochromatic.
**Hales-Jewett Theorem**

**Theorem (Hales-Jewett, 1963)**

Let $\Lambda$ be a finite alphabet and let $v \notin \Lambda$ be a variable. Then for every integer $c \geq 1$ there is a number $HJ(\Lambda, c)$ such that for every integer $N \geq HJ(\Lambda, c)$ and every $c$-coloring of the set of $\Lambda$-words of length $N$, i.e., the cube $\Lambda^N$ there is a **variable word** $x(v)$ of length $N$, an element of $(\Lambda \cup \{v\})^N \setminus \Lambda^N$ such that the set of all substitutions

$$\{x[\lambda] : \lambda \in \Lambda\}$$

is monochromatic.

**Theorem (Carlson-Simpson, 1984)**

Let $\Lambda$ be a finite alphabet and let $v \notin \Lambda$ be a variable. Then for every finite coloring of the semigroup $W_\Lambda$ of all $\Lambda$-words, there is an infinite sequence $(x_n(v))$ of variable words such that the set

$$\{x_0[\lambda_0] \circ \cdots \circ x_n[\lambda_n] : n < \omega, \lambda_0, ..., \lambda_n \in \Lambda\}$$

is monochromatic.
Hales-Jewett Theorem for Trees

We fix a vector tree $T$. Fix a finite alphabet $\Lambda$. For $m < n < \omega$, set $W(\Lambda, T, m, n) = \Lambda \otimes T \upharpoonright [m, n)$, where $\otimes T \upharpoonright [m, n) = \bigcup_{j = m}^{n - 1} \otimes T(j)$. We also set $W(\Lambda, T) = \bigcup_{m \leq n} W(\Lambda, T, m, n)$. 
Hales-Jewett Theorem for Trees

We fix a vector tree $T$. 

\[ W(\Lambda, T, m, n) = \Lambda \otimes T|_{[m, n]} \]
We also set $W(\Lambda, T) = \bigcup_{m \leq n} W(\Lambda, T, m, n)$. 
Hales-Jewett Theorem for Trees

We fix a vector tree \( T \).
Fix a finite alphabet \( \Lambda \).
Hales-Jewett Theorem for Trees

We fix a vector tree $T$.

Fix a finite alphabet $\Lambda$.

For $m < n < \omega$, set

$$W(\Lambda, T, m, n) = \Lambda \otimes T_{|[m,n]}$$

where $\otimes T_{|[m,n]} = \bigcup_{j=m}^{n-1} \otimes T(j)$. 
We fix a vector tree $T$.
Fix a finite alphabet $\Lambda$.
For $m < n < \omega$, set

$$W(\Lambda, T, m, n) = \Lambda \otimes T \upharpoonright [m, n),$$

where $\otimes T \upharpoonright [m, n) = \bigcup_{j=m}^{n-1} \otimes T(j)$. We also set

$$W(\Lambda, T) = \bigcup_{m \leq n} W(\Lambda, T, m, n).$$
Let \((v_s)_{s \in \bigotimes T}\) be a collection of distinct \textbf{variables}, set of symbols disjoint from \(\Lambda\).
Let \((v_s)_{s \in \otimes T}\) be a collection of distinct variables, set of symbols disjoint from \(\Lambda\).

Fix a vector level subset \(D\) of \(T\). Let

\[
W_v(\Lambda, T, D, m, n)
\]

to be the set of all functions

\[
f : \otimes T \upharpoonright [m, n) \to \Lambda \cup \{v_s : s \in \otimes D\}
\]

such that
Let $(v_s)_{s \in \otimes T}$ be a collection of distinct variables, set of symbols disjoint from $\Lambda$.

Fix a vector level subset $D$ of $T$. Let

$$W_v(\Lambda, T, D, m, n)$$

to be the set of all functions

$$f : \otimes T \upharpoonright [m, n) \rightarrow \Lambda \cup \{v_s : s \in \otimes D\}$$

such that

- The set $f^{-1}(\{u_s\})$ is nonempty and admits $s$ as a minimum in $\otimes T$, for all $s \in \otimes D$. 
Let \((v_s)_{s \in \otimes T}\) be a collection of distinct variables, set of symbols disjoint from \(\Lambda\).

Fix a vector level subset \(D\) of \(T\). Let

\[ W_v(\Lambda, T, D, m, n) \]

to be the set of all functions

\[ f : \otimes T \upharpoonright [m, n) \to \Lambda \cup \{ v_s : s \in \otimes D \} \]

such that

- The set \(f^{-1}(\{u_s\})\) is nonempty and admits \(s\) as a minimum in \(\otimes T\), for all \(s \in \otimes D\).

- For every \(s\) and \(s'\) in \(\otimes D\), we have
  \[ L_{\otimes T}(f^{-1}(\{u_s\})) = L_{\otimes T}(f^{-1}(\{u_{s'}\})). \]
For $f \in W_v(\Lambda, T, D, m, n)$, set

$$\text{ws}(f) = D, \text{bot}(f) = m \text{ and } \text{top}(f) = n.$$
For $f \in W_v(\Lambda, T, D, m, n)$, set

$$ws(f) = D, \ bot(f) = m \text{ and } top(f) = n.$$ 

Moreover, we set

$$W_v(\Lambda, T) = \bigcup \{ W_v(\Lambda, T, D, m, n) : m \leq n \text{ and } D \text{ is a vector level subset of } T \text{ with } L_T(D) \subset [m, n) \}.$$ 

The elements of $W_v(\Lambda, T)$ are viewed as **variable words over the alphabet** $\Lambda$. 

For variable words $f$ in $W_v(\Lambda, T)$ we take **substitutions**:
For every family $a = (a_s)_{s \in \otimes ws(f)} \subseteq \Lambda$, let 
$f(a) \in W(\Lambda, T)$ be the result of substituting for every $s$ in $\otimes ws(f)$ each occurrence of $\nu_s$ by $a_s$. 
For variable words $f$ in $W_v(\Lambda, T)$ we take substitutions: For every family $a = (a_s)_{s \in \otimes \text{ws}(f)} \subseteq \Lambda$, let $f(a) \in W(\Lambda, T)$ be the result of substituting for every $s$ in $\otimes \text{ws}(f)$ each occurrence of $v_s$ by $a_s$. Moreover, we set

$$[f]_{\Lambda} = \{ f(a) : a = (a_s)_{s \in \otimes \text{ws}(f)} \subseteq \Lambda \},$$

the constant span of $f$. 
An infinite sequence $X = (f_n)_{n<\omega}$ in $W_v(\Lambda, T)$ is a **subspace**, if:

1. $\text{bot}(f_0) = 0$,
2. $\text{bot}(f_{n+1}) = \text{top}(f_n)$ for all $n < \omega$,
3. Setting $D_i = \bigcup_{n < \omega} \text{ws}_i(f_n)$ for all $i = 1, \ldots, d$, where $\text{ws}_i(f_n) = (\text{ws}_1(f_n), \ldots, \text{ws}_d(f_n))$, we have that $(D_1, \ldots, D_d)$ forms a dense vector subset of $T$.

For a subspace $X = (f_n)_{n<\omega}$ we define $[X]_\Lambda = \{n \cup \{q : n < \omega \text{ and } g_q \in [f_q]_\Lambda \text{ for all } q = 0, \ldots, n\} \}$.

For two subspaces $X$ and $Y$, we write $X \leq Y$ if $[X]_\Lambda \subseteq [Y]_\Lambda$. 
Subspaces

An infinite sequence $X = (f_n)_{n<\omega}$ in $W_v(\Lambda, T)$ is a subspace, if:

1. $\text{bot}(f_0) = 0$,
Subspaces

An infinite sequence $X = (f_n)_{n<\omega}$ in $W_v(\Lambda, T)$ is a **subspace**, if:

1. $\text{bot}(f_0) = 0$,
2. $\text{bot}(f_{n+1}) = \text{top}(f_n)$ for all $n < \omega$, 
Subspaces

An infinite sequence \( X = (f_n)_{n<\omega} \) in \( W_v(\Lambda, T) \) is a **subspace**, if:

1. \( \text{bot}(f_0) = 0 \),
2. \( \text{bot}(f_{n+1}) = \text{top}(f_n) \) for all \( n < \omega \),
3. Setting \( D_i = \bigcup_{n<\omega} ws_i(f) \) for all \( i = 1, \ldots, d \), where \( ws(f) = (ws_1(f), \ldots, ws_d(f)) \), we have that \( (D_1, \ldots, D_d) \) forms a dense vector subset of \( T \).
Subspaces

An infinite sequence $X = (f_n)_{n<\omega}$ in $W_v(\Lambda, T)$ is a **subspace**, if:

1. $\bot(f_0) = 0$,
2. $\bot(f_{n+1}) = \top(f_n)$ for all $n < \omega$,
3. Setting $D_i = \bigcup_{n<\omega} ws_i(f)$ for all $i = 1, ..., d$, where $ws(f) = (ws_1(f), ..., ws_d(f))$, we have that $(D_1, ..., D_d)$ forms a dense vector subset of $T$.

For a subspace $X = (f_n)_{n<\omega}$ we define

$$[X]_\Lambda = \left\{ \bigcup_{q=0}^{n} g_q : n < \omega \text{ and } g_q \in [f_q]_\Lambda \text{ for all } q = 0, ..., n \right\}.$$
Subspaces

An infinite sequence $X = (f_n)_{n<\omega}$ in $W_v(\Lambda, T)$ is a subspace, if:

1. $\text{bot}(f_0) = 0$,
2. $\text{bot}(f_{n+1}) = \text{top}(f_n)$ for all $n < \omega$,
3. Setting $D_i = \bigcup_{n<\omega} ws_i(f)$ for all $i = 1, \ldots, d$, where $ws(f) = (ws_1(f), \ldots, ws_d(f))$, we have that $(D_1, \ldots, D_d)$ forms a dense vector subset of $T$.

For a subspace $X = (f_n)_{n<\omega}$ we define

$$\left[ X \right]_\Lambda = \left\{ \bigcup_{q=0}^n g_q : n < \omega \text{ and } g_q \in \left[ f_q \right]_\Lambda \text{ for all } q = 0, \ldots, n \right\}.$$

For two subspaces $X$ and $Y$, we write $X \leq Y$ if $\left[ X \right]_\Lambda \subseteq \left[ Y \right]_\Lambda$. 
An infinite Hales-Jewett theorem for trees

Theorem
Let $\Lambda$ be a finite alphabet and $T$ a vector tree. Then for every finite coloring of the set of the constant words $W(\Lambda, T)$ over $\Lambda$ and every subspace $X$ of $W(\Lambda, T)$ there exists a subspace $X'$ of $W(\Lambda, T)$ with $X' \leq X$ such that the set $[X']_\Lambda$ is monochromatic.
Theorem
Let $\Lambda$ be a finite alphabet and $T$ a vector tree. Then for every finite coloring of the set of the constant words $W(\Lambda, T)$ over $\Lambda$ and every subspace $X$ of $W(\Lambda, T)$ there exists a subspace $X'$ of $W(\Lambda, T)$ with $X' \leq X$ such that the set $[X']_\Lambda$ is monochromatic.

Remark
This will be used as a pigeonhole principle for its infinite-dimensional version.
A Ramsey space of sequences of words

Let $W^\infty(\Lambda, T)$, be the set of all sequences $(g_n)_{n<\omega}$ in $W(\Lambda, T)$ such that:

1. $\bot(g_0) = 0$ and
2. $\bot(g_{n+1}) = \top g_n$ for all $n < \omega$.

For a subspace $X$, we set $[X]_{\infty}^\Lambda = \{ (g_n)_{n<\omega} \in W^\infty(\Lambda, T) : (\forall n < \omega) \bigcup q = 0 \} \cap \Lambda$.

Theorem
Let $\Lambda$ be a finite alphabet and $T$ a vector tree. Then for every finite Souslin measurable coloring of the set $W^\infty(\Lambda, T)$ and every subspace $X$ of $W(\Lambda, T)$ there exists a subspace $X'$ of $W(\Lambda, T)$ with $X' \leq X$ such that the set $[X']_{\infty}^\Lambda$ is monochromatic.
A Ramsey space of sequences of words

Let $W^\infty(\Lambda, T)$, be the set of all sequences $(g_n)_{n<\omega}$ in $W(\Lambda, T)$ such that:

1. $\text{bot}(g_0) = 0$ and
A Ramsey space of sequences of words

Let $W^\infty(\Lambda, T)$, be the set of all sequences $(g_n)_{n<\omega}$ in $W(\Lambda, T)$ such that:

1. $\text{bot}(g_0) = 0$ and
2. $\text{bot}(g_{n+1}) = \text{top} g_n$ for all $n < \omega$. 

For a subspace $X$, we set $[X]_\Lambda^\infty = \{ (g_n)_{n<\omega} \in W^\infty(\Lambda, T) : (\forall n<\omega) \bigcup_{q} g_q \in [X]_\Lambda \}$. 

Theorem
Let $\Lambda$ be a finite alphabet and $T$ a vector tree. Then for every finite Souslin measurable coloring of the set $W^\infty(\Lambda, T)$ and every subspace $X$ of $W(\Lambda, T)$ there exists a subspace $X'$ of $W(\Lambda, T)$ with $X' \leq X$ such that the set $[X']_\Lambda^\infty$ is monochromatic.
A Ramsey space of sequences of words

Let $W^\infty(\Lambda, T)$, be the set of all sequences $(g_n)_{n<\omega}$ in $W(\Lambda, T)$ such that:

1. $\text{bot}(g_0) = 0$ and
2. $\text{bot}(g_{n+1}) = \text{top}g_n$ for all $n < \omega$.

For a subspace $X$, we set

$$[X]_{\Lambda}^\infty = \{(g_n)_{n<\omega} \in W^\infty(\Lambda, T) : (\forall n < \omega) \bigcup_{q=0}^{n} g_q \in [X]_{\Lambda} \}.$$
A Ramsey space of sequences of words

Let $W^\infty(\Lambda, T)$, be the set of all sequences $(g_n)_{n<\omega}$ in $W(\Lambda, T)$ such that:

1. $\text{bot}(g_0) = 0$ and
2. $\text{bot}(g_{n+1}) = \text{top}g_n$ for all $n < \omega$.

For a subspace $X$, we set

$$[X]^\infty_\Lambda = \{(g_n)_{n<\omega} \in W^\infty(\Lambda, T) : (\forall n < \omega) \bigcup_{q=0}^{n} g_q \in [X]_\Lambda \}.$$

**Theorem**

Let $\Lambda$ be a finite alphabet and $T$ a vector tree. Then for every finite Souslin measurable coloring of the set $W^\infty(\Lambda, T)$ and every subspace $X$ of $W(\Lambda, T)$ there exists a subspace $X'$ of $W(\Lambda, T)$ with $X' \leq X$ such that the set $[X']^\infty_\Lambda$ is monochromatic.
Higher Dimensions

Theorem (Graham-Rothschild) For every triple of positive integers $k, l, c$ there is integer $GR(k, l, c)$ such that for every set $X$ of cardinality $\geq GR$ and every $c$-coloring of the family $(P(X)_k)$ of all $k$-families of pairwise disjoint subsets of $X$ there is a family $D = (D_i)_{i=1}^l$ of pairwise disjoint nonempty subsets of $X$ such that the family $(U(D)_k) = \{\bigcup_{i \in I} D_i : \emptyset \neq I \subseteq \{1, 2, \ldots, l\}\}$ is monochromatic.
Theorem (Graham-Rothschild)

For every triple of positive integers \( k, l, \) and \( c \) there is integer \( GR = GR(k, l, c) \) such that for every set \( X \) of cardinality \( \geq GR \) and every \( c \)-coloring of the family \( \binom{\mathcal{P}(X)}{k} \) of all \( k \)-families of pairwise disjoint subsets of \( X \) there is a family \( D = (D_i)_{i=1}^l \) of pairwise disjoint nonempty subsets of \( X \) such that the family \( \binom{U(D)}{k} \) of \( k \)-families of pairwise disjoint subsets of \( U(D) = \{ \bigcup_{i \in I} D_i : \emptyset \neq I \subseteq \{1, 2, \ldots, l\} \} \) is monochromatic.
Finite Union Theorem for Trees in $\text{Dim} > 1$
Finite Union Theorem for Trees in Dim > 1

Fix a positive integers $b$ and $n$. 
Finite Union Theorem for Trees in Dim $> 1$

Fix a positive integers $b$ and $n$. Let $b^{<n}$ denote the uniformly $b$-branching tree of height $n$, the set of all sequences of length less than $n$ taking values from the set $b = \{0, \ldots, b - 1\}$ ordered by the relation $\sqsubseteq$ of end-extension.
Finite Union Theorem for Trees in Dim $> 1$

Fix a positive integers $b$ and $n$. Let $b^{<n}$ denote the uniformly $b$-branching tree of height $n$, the set of all sequences of length less than $n$ taking values from the set $b = \{0, \ldots, b - 1\}$ ordered by the relation $\sqsubseteq$ of end-extension.

Let $k$ be another positive integer.
Finite Union Theorem for Trees in $\text{Dim} > 1$

Fix a positive integers $b$ and $n$. Let $b^{<n}$ denote the uniformly $b$-branching tree of height $n$, the set of all sequences of length less than $n$ taking values from the set $b = \{0, ..., b - 1\}$ ordered by the relation $\sqsubseteq$ of end-extension.

Let $k$ be another positive integer. A subset subset $T$ of $b^{<n}$ is a skew subtree of height $k$ if
Finite Union Theorem for Trees in Dim $> 1$

Fix a positive integers $b$ and $n$. Let $b^{<n}$ denote the uniformly $b$-branching tree of height $n$, the set of all sequences of length less than $n$ taking values from the set $b = \{0, ..., b - 1\}$ ordered by the relation $\sqsubseteq$ of end-extension.

Let $k$ be another positive integer. A subset subset $T$ of $b^{<n}$ is a skew subtree of height $k$ if

1. $T$ has a minimum,
Finite Union Theorem for Trees in \(\text{Dim} > 1\)

Fix a positive integers \(b\) and \(n\).
Let \(b^{<n}\) denote the \textbf{uniformly} \(b\)-branching tree of height \(n\), the set of all sequences of length less than \(n\) taking values from the set \(b = \{0, \ldots, b - 1\}\) ordered by the relation \(\sqsubseteq\) of \textbf{end-extension}.

Let \(k\) be another positive integer.
A subset \(T\) of \(b^{<n}\) is a \textbf{skew subtree of height} \(k\) if

1. \(T\) has a minimum,
2. Every maximal chain in \(T\) is of size \(k\),

\[\sum_{i=0}^{n-1} (b-1)^i = b^n - 1\]
Finite Union Theorem for Trees in Dim $>1$

Fix a positive integers $b$ and $n$. Let $b^{<n}$ denote the uniformly $b$-branching tree of height $n$, the set of all sequences of length less than $n$ taking values from the set $b = \{0, \ldots, b - 1\}$ ordered by the relation $\sqsubseteq$ of end-extension.

Let $k$ be another positive integer. A subset subset $T$ of $b^{<n}$ is a skew subtree of height $k$ if

1. $T$ has a minimum,
2. Every maximal chain in $T$ is of size $k$,
3. For every non maximal $t$ in $T$ and every $s \in \text{ImmSucc}_{b^{<n}}(t)$ there exists unique $s' \in \text{ImmSucc}_T(t)$ satisfying $s \sqsubseteq s'$.
Finite Union Theorem for Trees in Dim > 1

Fix a positive integers \( b \) and \( n \). Let \( b^{<n} \) denote the \textbf{uniformly} \( b \)-branching tree of height \( n \), the set of all sequences of length less than \( n \) taking values from the set \( b = \{0, \ldots, b - 1\} \) ordered by the relation \( \sqsubseteq \) of end-extension.

Let \( k \) be another positive integer.
A subset subset \( T \) of \( b^{<n} \) is a \textbf{skew subtree of height} \( k \) if

1. \( T \) has a minimum,
2. Every maximal chain in \( T \) is of size \( k \),
3. For every non maximal \( t \) in \( T \) and every \( s \in \text{ImmSucc}_{b^{<n}}(t) \) there exists unique \( s' \in \text{ImmSucc}_T(t) \) satisfying \( s \sqsubseteq s' \).
4. For every \( \ell < k \) and \( s, t \in T(\ell) \), we have that \( s <_{\text{lex}} t \) iff \( |s| < |t| \).
Skew subtree of height 3

Skew subtree of height 3
Subspaces of finite dimension > 1

Let

\[ U(b^{<n}) = \{ U \subseteq b^{<n} : U \text{ has a minimum} \}. \]
Subspaces of finite dimension $> 1$

Let

$$\mathcal{U}(b^{<n}) = \{ U \subseteq b^{<n} : U \text{ has a minimum} \}.$$ 

A \textbf{k-dimensional subspace of } $\mathcal{U}(b^{<n})$ is a family of of the form

$$(U_t)_{t \in T} \subseteq \mathcal{U}(b^{<n})$$

such that
Subspaces of finite dimension $> 1$

Let

$$\mathcal{U}(b^{<n}) = \{ U \subseteq b^{<n} : U \text{ has a minimum} \}.$$ 

A $k$-dimensional subspace of $\mathcal{U}(b^{<n})$ is a family of of of the form

$$(U_t)_{t \in T} \subseteq \mathcal{U}(b^{<n})$$

such that

1. $T$ is a skew subtree of $b^{<n}$ height $k$, 

Subspaces of finite dimension $> 1$

Let

$\mathcal{U}(b^{<n}) = \{ U \subseteq b^{<n} : U \text{ has a minimum} \}$.

A $k$-dimensional subspace of $\mathcal{U}(b^{<n})$ is a family of the form

$$(U_t)_{t \in T} \subseteq \mathcal{U}(b^{<n})$$

such that

1. $T$ is a skew subtree of $b^{<n}$ height $k$,
2. $U_s \cap U_t = \emptyset$ for $s \neq t \in T$, 
Subspaces of finite dimension > 1

Let

\[ \mathcal{U}(b^{<n}) = \{ U \subseteq b^{<n} : U \text{ has a minimum} \}. \]

A \textbf{k-dimensional subspace of} \( \mathcal{U}(b^{<n}) \) is a family of of the form

\[ (U_t)_{t \in T} \subseteq \mathcal{U}(b^{<n}) \]

such that

1. \( T \) is a skew subtree of \( b^{<n} \) height \( k \),
2. \( U_s \cap U_t = \emptyset \) for \( s \neq t \in T \),
3. \( \min U_t = t \) for all \( t \in T \).
Subspaces of finite dimension $> 1$

Let
\[ \mathcal{U}(b^{<n}) = \{ U \subseteq b^{<n} : U \text{ has a minimum} \}. \]

A \textbf{k-dimensional subspace of } \mathcal{U}(b^{<n}) \text{ is a family of of the form }

\[ (U_t)_{t \in T} \subseteq \mathcal{U}(b^{<n}) \]

such that

1. $T$ is a skew subtree of $b^{<n}$ height $k$,
2. $U_s \cap U_t = \emptyset$ for $s \neq t \in T$,
3. $\min U_t = t$ for all $t \in T$.

A subspace $(V_s)_{s \in S}$ is a \textbf{further subspace of } $(U_t)_{t \in T}$ if

\[ (\forall s \in S) \quad V_s \in \left\{ \bigcup_{t \in A} U_t : A \subseteq T \right\}. \]
Finite Union Theorem for Trees in Dimension $k$

For every positive integers $c$, $k$, $l$, $b$ with $k \leq l$ there exists a positive integer $n_0 = \text{TT}(c, k, l, b)$ such that for every integer $n \geq n_0$ and every $r$-coloring of the $k$-dimensional subspaces of $U(b < n)$, there exists a $l$-dimensional subspace $U$ such that the set of all further $k$-dimensional subspaces of $U$ is monochromatic.

Remark

The Graham-Rotschild Finite Union Theorem is the case $b = 1$ of this result.
Finite Union Theorem for Trees in Dimension $k$

Theorem

For every positive integers $c, k, l, b$ with $k \leq l$ there exists a positive integer $n_0 = TT(c, k, l, b)$ such that for every integer $n \geq n_0$ and every $r$-coloring of the $k$-dimensional subspaces of $U(b^{<n})$, there exists a $l$-dimensional subspace $U$ such that the set of all further $k$-dimensional subspaces of $U$ is monochromatic.

Remark

The Graham-Rotschild Finite Union Theorem is the case $b = 1$ of this result.
Finite Union Theorem for Trees in Dimension $k$

**Theorem**

For every positive integers $c, k, l, b$ with $k \leq l$ there exists a positive integer $n_0 = TT(c, k, l, b)$ such that for every integer $n \geq n_0$ and every $r$-coloring of the $k$-dimensional subspaces of $U(b^{<n})$, there exists a $l$-dimensional subspace $U$ such that the set of all further $k$-dimensional subspaces of $U$ is monochromatic.

**Remark**

The Graham-Rotschild Finite Union Theorem is the case $b = 1$ of this result.
Further work
Further work

An $\omega$-dimensional subspace of $\mathcal{U}(b^{<\omega})$ is a family of the form

$$(U_t)_{t \in T} \subseteq \mathcal{U}(b^{<\omega})$$

such that

1. $T$ is a skew subtree of $b^{<\omega}$ height $\omega$,
2. $U_s \cap U_t = \emptyset$ for $s \neq t \in T$,
3. $\min U_t = t$ for all $t \in T$.

As before an $\omega$-dimensional subspace $(V_s)_{s \in S}$ is a further subspace of $(U_t)_{t \in T}$ if

$$(\forall s \in S) V_s \in \{ \bigcup A \subseteq U(b^{<\omega}) : A \subseteq T \}.$$

Conjecture

For every finite Souslin-measurable coloring of the family of all $\omega$-dimensional subspaces of $\mathcal{U}(b^{<\omega})$ there is an $\omega$-dimensional subspace $(U_t)_{t \in T}$ all of whose further $\omega$-dimensional subspaces are of the same color.
Further work

An \( \omega \)-dimensional subspace of \( \mathcal{U}(b^{<\omega}) \) is a family of the form

\[
(\mathcal{U}_t)_{t \in T} \subseteq \mathcal{U}(b^{<\omega})
\]

such that

1. \( T \) is a skew subtree of \( b^{<\omega} \) height \( \omega \),
Further work

An \textit{\(\omega\)-dimensional subspace} of \(\mathcal{U}(b^{<\omega})\) is a family of the form

\[(U_t)_{t \in T} \subseteq \mathcal{U}(b^{<\omega})\]

such that

1. \(T\) is a skew subtree of \(b^{<\omega}\) height \(\omega\),
2. \(U_s \cap U_t = \emptyset\) for \(s \neq t \in T\),
Further work

An \( \omega \)-dimensional subspace of \( \mathcal{U}(b^{<\omega}) \) is a family of of the form

\[
(U_t)_{t \in T} \subseteq \mathcal{U}(b^{<\omega})
\]

such that

1. \( T \) is a skew subtree of \( b^{<\omega} \) height \( \omega \),
2. \( U_s \cap U_t = \emptyset \) for \( s \neq t \in T \),
3. \( \min U_t = t \) for all \( t \in T \).
Further work

An $\omega$-dimensional subspace of $\mathcal{U}(b^{<\omega})$ is a family of of the form

$$(U_t)_{t \in T} \subseteq \mathcal{U}(b^{<\omega})$$

such that

1. $T$ is a skew subtree of $b^{<\omega}$ height $\omega$,
2. $U_s \cap U_t = \emptyset$ for $s \neq t \in T$,
3. $\min U_t = t$ for all $t \in T$.

As before an $\omega$-dimensional subspace $(V_s)_{s \in S}$ is a further subspace of $(U_t)_{t \in T}$ if

$$(\forall s \in S) \ V_s \in \{ \bigcup_{t \in A} U_t : A \subseteq T \}.$$
Further work

An \(\omega\)-dimensional subspace of \(\mathcal{U}(b^{<\omega})\) is a family of the form

\[
(U_t)_{t \in T} \subseteq \mathcal{U}(b^{<\omega})
\]

such that

1. \(T\) is a skew subtree of \(b^{<\omega}\) height \(\omega\),
2. \(U_s \cap U_t = \emptyset\) for \(s \neq t \in T\),
3. \(\min U_t = t\) for all \(t \in T\).

As before an \(\omega\)-dimensional subspace \((V_s)_{s \in S}\) is a further subspace of \((U_t)_{t \in T}\) if

\[
(\forall s \in S) \quad V_s \in \{\bigcup U_t : A \subseteq T\}.
\]

Conjecture

For every finite Souslin-measurable coloring of the family of all \(\omega\)-dimensional subspaces of \(\mathcal{U}(b^{<\omega})\) there is an \(\omega\)-dimensional subspace \((U_t)_{t \in T}\) all of whose further \(\omega\)-dimensional subspaces are of the same color.