THE CONDENSATION ORDER ON $\mathrm{Rel}(X)$

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Plan of the presentation

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- Preliminaries

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- The condensation equivalence on $\mathrm{Rel}(X)$

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- The complexity of the equivalence classes in $\mathrm{Rel}(\omega)$
- The size of the equivalence classes in $\mathrm{Rel}(\omega)$
- A partition of the quotient $\mathrm{Rel}(X)/\sim_c$
- Suborders $D_\rho = \{[\rho \cup \Delta_A]_{\sim_c} : A \subset X\}$ for irreflexive ρ, and the properties of $\mathrm{Aut}\langle X, \rho\rangle$

Preliminaries

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- an isomorphism iff $\rho_f = \sigma$

And also

- $[\rho]_{\cong} = \{\rho_f : f \in \mathrm{Bij}(X)\}$

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- $\rho \preccurlyeq_c \sigma$ iff there exists a bijective homomorphism $f : \langle X, \rho \rangle \to \langle X, \sigma \rangle$

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- $[\rho]_{\sim_c} = \mathrm{Conv}_{\langle \mathrm{Rel}(X), \subset \rangle}([\rho]_{\cong})$
- if ρ is finite, then $[\rho]_{\sim_c} = [\rho]_{\cong}$

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- the relation $\leq$ on $\mathrm{Rel}(X)/\sim_c$ is a partial order (**the condensation order**)

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Without loss of generality we can speak only about the structures $\langle \mathrm{Rel}(\kappa), \subset \rangle$, $\langle \mathrm{Rel}(\kappa), \preccurlyeq_c \rangle$ and $\langle \mathrm{Rel}(\kappa)/\sim_c, \leq \rangle$ where $\kappa > 0$ is a cardinal

$\langle \mathrm{Rel}(2), \subset \rangle$ is a Boolean lattice

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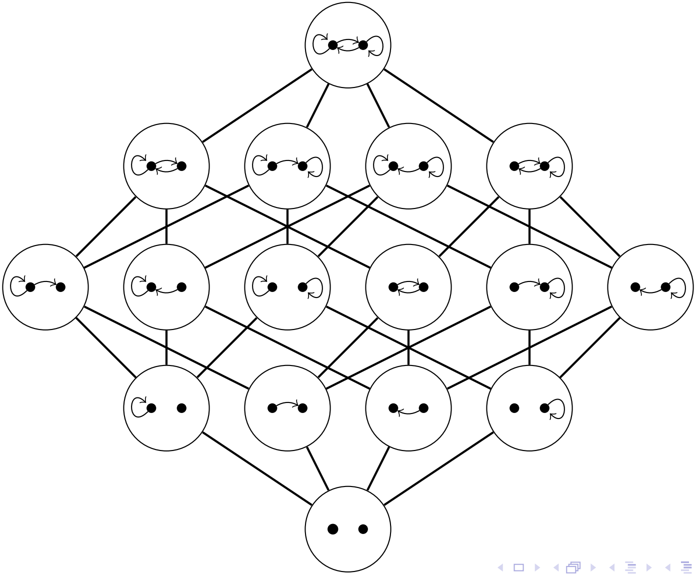

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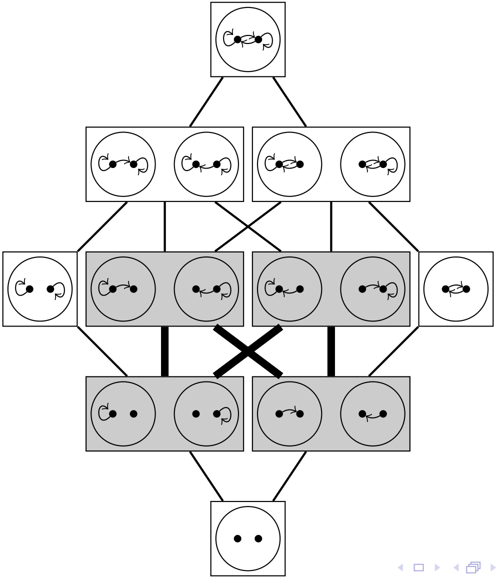

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We have

$$\rho \text{ is strongly reversible} \Rightarrow \rho \text{ is reversible} \Rightarrow \rho \text{ is weakly reversible}$$

Strongly reversible relations

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Some reversible relations are the following:

- (Strict) linear orders

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- Finite unions of tournaments (oriented complete graphs)

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Some reversible relations are the following:

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- Finite relations
- Finite unions of complete oriented graphs
- Finite unions of tournaments (oriented complete graphs)
- Equivalence relations corresponding to finite partitions

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- a weakly reversible relation $\rho \in \mathrm{Rel}(\omega)$ which is not reversible

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- a relation $\sigma \in \mathrm{Rel}(\omega)$ which is not weakly reversible

The complexity of the equivalence classes

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We shall identify $\mathrm{Rel}(\omega) = P(\omega \times \omega)$ with the Cantor cube $2^{\omega\times\omega} \cong 2^{\omega}$ by identifying each set $A \subset \omega$ with its characteristic function χ_A

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The following theorem is a known result.

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For each $\rho \in \mathrm{Rel}(\omega)$ the isomorphism class $[\rho]_{\cong}$ is an analytic set

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We have the similar result.

Theorem

For each $\rho \in \mathrm{Rel}(\omega)$ the condensation class $[\rho]_{\sim_c}$ is an analytic set

The size of the equivalence classes

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Theorem

$|\,\mathrm{Rel}(\omega)/\cong\,| = |\,\mathrm{Rel}(\omega)/\sim_c\,| = \mathfrak{c}$

Nice partition of $\mathrm{Rel}(X)/\sim_c$

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Convex properties (and thus also condensation properties):

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Theorem

- $\{q_{\sim_c}[\mathrm{Refl}_X], q_{\sim_c}[\mathrm{Irrefl}_X], q_{\sim_c}[\neg\,\mathrm{Refl}_X \cap \neg\,\mathrm{Irrefl}_X]\}$ is a partition of the poset $\langle \mathrm{Rel}(X)/\sim_c, \leq\}$ into convex sets

Nice partition of $\text{Rel}(X)/\sim_c$

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- The mapping $F : \langle q_{\sim_c}[\text{Irrefl}_X], \leq\rangle \to \langle q_{\sim_c}[\text{Refl}_X], \leq\rangle$ defined by $F([\rho]_{\sim_c}) = [\rho \cup \Delta_X]_{\sim_c}$ is an isomorphism

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Let $\rho \in \text{Irrefl}_X$ and let $G : P(X) \to D_\rho$, where $G(A) = [\rho \cup \Delta_A]_{\sim_c}$. Then G is injective iff $\langle X, \rho \rangle$ is a rigid structure. And then $\langle P(X), \subset \rangle \cong_G \langle D_\rho, \leq \rangle$

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For a finite cardinal κ, let θ_κ denote the order type of $\kappa + 1$. For infinite κ let

$$\theta_\kappa = \mathrm{type}\left(\left\langle \{\mu \in \mathrm{Card} : \mu \leq \kappa\}, \leq \right\rangle + \left\langle \{\mu \in \mathrm{Card} : \mu < \kappa\}, \leq \right\rangle^*\right)$$

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Theorem

For $\rho \in \mathrm{Irrefl}_X$ the poset $\langle D_\rho, \leq \rangle$ contains a chain of the type $\theta_{|X|}$. If ρ is strongly reversible, then $\langle D_\rho, \leq \rangle \cong \theta_{|X|}$

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For $\rho \in \mathrm{Irrefl}_X$ and $n < \min\{\omega, |X| + 1\}$ let $D^n_\rho = \{[\rho \cup \Delta_A]_{\sim_c} : A \in [X]^n\}$

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For each $\rho \in \mathrm{Irrefl}_X$, where $|X| \leq \omega$, the following conditions are equivalent:

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The sets D^n_ρ

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For $\rho \in \mathrm{Irrefl}_X$ and $n < \min\{\omega, |X| + 1\}$ let $D^n_\rho = \{[\rho \cup \Delta_A]_{\sim_c} : A \in [X]^n\}$

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If $\rho \in \mathrm{Irrefl}_X$ is a linear order and $|X| \geq \omega$, then $|D^\omega_\rho| > 1$

The sets $\mathcal{D}_{[\rho]_{\sim c}}$

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We have the example of a relation $\rho \in \mathrm{Irrefl}_\omega$ such that $\mathcal{D}_{[\rho]_{\sim c}} \neq D_\rho$

References

R. Fraïssé, Theory of relations, Revised edition, With an appendix by Norbert Sauer, Studies in Logic and the Foundations of Mathematics, 145. North-Holland, Amsterdam, (2000)

W. Hodges, Model theory, Encyclopedia of Mathematics and its Applications, 42, Cambridge University Press, Cambridge, 1993.

T. Jech, Set Theory, 2nd corr. Edition, Springer, Berlin, 1997.

A. S. Kechris, Classical Descriptive Set Theory, Graduate Texts in Mathematics, 156. Springer-Verlag, New York, 1995.

A. H. Lachlan, R. E. Woodrow, Countable ultrahomogeneous undirected graphs, Trans. Amer. Math. Soc., 262,1 (1980) 51–94.

R. C. Lyndon, Properties preserved under homomorphism, Pacific J. Math. 9 (1959) 143–154.

J. G. Rosenstein, Linear orderings, Pure and Applied Mathematics, 98, Academic Press, Inc., Harcourt Brace Jovanovich Publishers, New York-London, 1982.

J. H. Schmerl, Countable homogeneous partially ordered sets, Algebra Univers. 9,3 (1979) 317-321.