Partial orders of isomorphic substructures of ultrahomogeneous relational structures

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Problem

Formulation

Describe maximal chains in partial orders of the form $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, for an ultrahomogeneous relational structure $\mathbb{X}$, where $\mathbb{P}(\mathbb{X}) = \{A \subset \mathbb{X} : A \cong \mathbb{X}\}$.

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Theorem (Our starting point, Kurilić 2010)

Linear order is isomorphic to a maximal chain in $\langle \mathbb{P}(\mathbb{Q}, <) \cup \{\emptyset\}, \subset \rangle$ if and only if it is complete, $\mathbb{R}$-embeddable with minimum non-isolated.

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Linear order is isomorphic to a maximal chain in $\langle P(\kappa), \subset \rangle$ if and only if it is isomorphic to $\mathrm{Init}(L)$ for some linear order L of cardinality κ.

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Open problem on maximal chains (1907)

Does every maximal chain in $\mathbb{R}^\omega$ contain (ω_1, ω_1^*) gap?

Notation

$\mathcal{L}_{\mathbb{X}}$ - the class of order types of maximal chains in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$, for a relational structure $\mathbb{X}$;

$\mathcal{C}_{\mathbb{R}}$ - the class of order types of complete $\mathbb{R}$-embeddable linear orders with minimum non-isolated.
By complete we mean Dedekind-complete with minimum and maximum.
Also, it holds that $\mathcal{C}_{\mathbb{R}} =$ the class of order types of compact sets of reals K, such that $\min K \in K'$;

$\mathcal{B}_{\mathbb{R}}$ - the class of order types of boolean $\mathbb{R}$-embeddable linear orders with minimum non-isolated.
By boolean we mean complete with dense jumps.
Also, it holds that $\mathcal{B}_{\mathbb{R}} =$ the class of order types of nowhere dense compact sets of reals K, such that $\min K \in K'$.

Positive families

Definition

Let X be a countable set. We call $\mathcal{P} \subset P(X)$ *a positive family* on a set X if and only if:

- $\emptyset \notin \mathcal{P}$;
- $A \in \mathcal{P} \wedge B \in [A]^{<\omega} \Rightarrow A \setminus B \in \mathcal{P}$;
- $A \in \mathcal{P} \wedge A \subset B \subset X \Rightarrow B \in \mathcal{P}$;
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Lemma

If there exists a positive family $\mathcal{P} \subset \mathbb{P}(\mathbb{X})$ then $\mathcal{B}_{\mathbb{R}} \subset \mathcal{L}_{\mathbb{X}}$.

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Theorem

Let $\mathbb{X}$ be a countable ultrahomogeneous relational structure which satisfies $\mathbb{P}(\mathbb{X}) \neq \{X\}$. Then $\mathcal{L}_{\mathbb{X}} \subset \mathcal{C}_{\mathbb{R}}$.

How to find maximal chains

Theorem

Let $\mathbb{X}$ be a countable relational structure and $\mathbb{Q}$ the set of rationals.

(A) If there exists a partition $\{J_n : n \in \omega\}$ of $\mathbb{Q}$ and a structure with the domain $\mathbb{Q}$ of the same signature as $\mathbb{X}$ such that

- **(i)** J_0 is a dense subset of $\langle \mathbb{Q}, < \rangle$;
- **(ii)** J_n, $n \in \mathbb{N}$, are coinitial subsets of $\langle \mathbb{Q} < \rangle$;
- **(iii)** $J_0 \cap (-\infty, x) \subset A \subset \mathbb{Q} \cap (-\infty, x)$ implies $A \cong \mathbb{X}$, for all $x \in \mathbb{R} \cup \{\infty\}$;
- **(iv)** $J_0 \cap (-\infty, q] \subset C \subset \mathbb{Q} \cap (-\infty, q]$ implies $C \ncong \mathbb{X}$, for each $q \in J_0$;

then for each uncountable $\mathbb{R}$-embeddable complete linear order L with minimum non-isolated, such that all initial segments of $L \setminus \{\min L\}$ are uncountable there is a maximal chain in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$ isomorphic to L.

(B) If, in addition,

- **(v)** for each countable complete linear order L with minimum non-isolated there is a maximal chain in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$ isomorphic to L,

then $\mathcal{L}_{\mathbb{X}} = \mathcal{C}_{\mathbb{R}}$.

Finding positive families

Lemma

The family $\mathcal{P} = \{A \subset \mathbb{G}_{Rado} : \mathbb{G}_{Rado} \setminus \mathbb{K}_\omega \subset^* A\}$ is a positive family on $\mathbb{G}_{Rado}$ such that $\mathcal{P} \subset \mathbb{P}(\mathbb{G}_{Rado})$.

Lemma

The family $\mathcal{P} = \{B \subset \mathbb{D} : \mathbb{D} \setminus \mathbb{A}_\omega \subset^* B\}$ is a positive family on $\mathbb{D}$ such that $\mathcal{P} \subset \mathbb{P}(\mathbb{D})$.

Finding positive families

Let $\mathbb{X}$ be any ultrahomogeneous relational structure (binary) whose age satisfies strong (disjoint) amalgamation property. Define $\langle \mathbb{P}, \leq \rangle$ to be the partial order of all pairs $p = \langle X_p, \rho_p \rangle$ such that:

- $X_p \in [\mathbb{Q}]^{<\omega}$;
- $p \in \operatorname{Age} \mathbb{X}$.
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$$\mathcal{D}_{B,K,m} = \{p \in \mathbb{P} : K \subset X_p \ \wedge \ (p \restriction K \ntriangleleft B \ \vee \\ \exists q \in (m_K, m_K + \tfrac{1}{m})_{\mathbb{Q}} \ \ p \restriction (K \cup \{q\}) \cong B)\}$$

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is dense in $\mathbb{P}$ for all $K \in [\mathbb{Q}]^{<\omega}$, $B \in \operatorname{Age}_{|K|+1} \mathbb{X}$ and $m \in \mathbb{N}$.

Hence, there is a filter in $\mathbb{P}$ intersecting all these dense sets.

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Theorem

If the family $\mathcal{P}$ is given by

$$\mathcal{P} = \left\{ \mathbb{Q} \setminus \bigcup_{n \in \mathbb{Z}} F_n : F_n \in \Big[[n, n+1) \Big]^{<\omega} \right\},$$

then for each $A \in \mathcal{P}$ we have $\langle A, \rho \rangle \cong \mathbb{X}$. In particular, we have that $\mathcal{P} \subset \mathbb{P}(\mathbb{Q}, \rho)$ is a positive family.

Corrolary

Let L be an $\mathbb{R}$-embeddable boolean linear order with minimum non-isolated. Then there is a maximal chain $\mathcal{L} \subset \langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$ isomorphic to L.

Corrolary

Let $\mathbb{X}$ be a countable ultrahomogeneous relational structure. Then the following conditions are equivalent:

- there is a positive family $\mathcal{P}$ on $\mathbb{X}$ which satisfies $\mathcal{P} \subset \mathbb{P}(\mathbb{X})$;
- $\mathrm{Age}\,\mathbb{X}$ satisfies the strong amalgamation property.

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