POSETS OF COPIES, EMBEDDING MONOIDS, AND INTERPRETABILITY OF RELATIONAL STRUCTURES

Miloš Kurilić

Department of Mathematics and Informatics, University of Novi Sad, Serbia

August 21, 2014

Contents

Contents

- The poset of copies of a structure

Contents

Contents

Relational structures and complete Boolean algebras

Relational structures and complete Boolean algebras

If $\mathbb{X} = \langle X, \langle \rho_i : i \in I \rangle \rangle$ is a relational structure,
by $\mathbb{P}(\mathbb{X})$ we denote the **set of domains of isomorphic substructures of** $\mathbb{X}$, that is

Relational structures and complete Boolean algebras

If $\mathbb{X} = \langle X, \langle \rho_i : i \in I \rangle \rangle$ is a relational structure,
by $\mathbb{P}(\mathbb{X})$ we denote the **set of domains of isomorphic substructures of** $\mathbb{X}$, that is

$$\mathbb{P}(\mathbb{X}) \quad = \quad \left\{ A \subset X : \left\langle A, \langle \rho_i \cap A^{n_i} : i \in I \rangle \right\rangle \cong \mathbb{X} \right\}$$

Relational structures and complete Boolean algebras

If $\mathbb{X} = \langle X, \langle \rho_i : i \in I \rangle \rangle$ is a relational structure,
by $\mathbb{P}(\mathbb{X})$ we denote the **set of domains of isomorphic substructures of** $\mathbb{X}$, that is

$$\mathbb{P}(\mathbb{X}) \quad = \quad \left\{ A \subset X : \left\langle A, \langle \rho_i \cap A^{n_i} : i \in I \rangle \right\rangle \cong \mathbb{X} \right\}$$

To $\mathbb{X}$ we adjoin

- the poset $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$

Relational structures and complete Boolean algebras

If $\mathbb{X} = \langle X, \langle \rho_i : i \in I \rangle \rangle$ is a relational structure,
by $\mathbb{P}(\mathbb{X})$ we denote the **set of domains of isomorphic substructures of** $\mathbb{X}$, that is

$$\mathbb{P}(\mathbb{X}) \quad = \quad \left\{ A \subset X : \left\langle A, \langle \rho_i \cap A^{n_i} : i \in I \rangle \right\rangle \cong \mathbb{X} \right\}$$

To $\mathbb{X}$ we adjoin

- the poset $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$
- its separative quotient $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$

Relational structures and complete Boolean algebras

If $\mathbb{X} = \langle X, \langle \rho_i : i \in I \rangle \rangle$ is a relational structure,
by $\mathbb{P}(\mathbb{X})$ we denote the **set of domains of isomorphic substructures of** $\mathbb{X}$, that is

$$\mathbb{P}(\mathbb{X}) \quad = \quad \Big\{ A \subset X : \Big\langle A, \langle \rho_i \cap A^{n_i} : i \in I \rangle \Big\rangle \cong \mathbb{X} \Big\}$$

To $\mathbb{X}$ we adjoin

- the poset $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$
- its separative quotient $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$
- its Boolean completion $\mathrm{ro}\,\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$

Relational structures and complete Boolean algebras

If $\mathbb{X} = \langle X, \langle \rho_i : i \in I \rangle \rangle$ is a relational structure,
by $\mathbb{P}(\mathbb{X})$ we denote the **set of domains of isomorphic substructures of** $\mathbb{X}$, that is

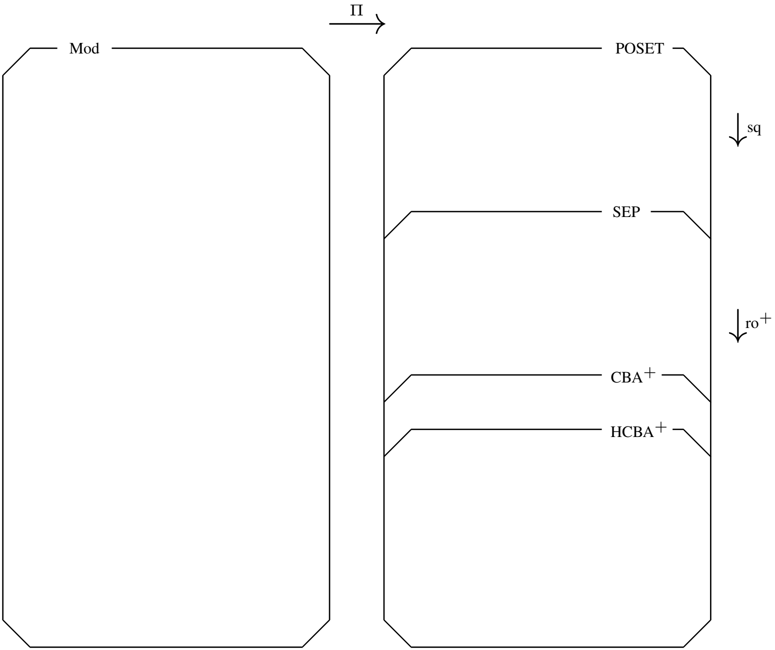
$$\mathbb{P}(\mathbb{X}) \quad = \quad \Big\{ A \subset X : \Big\langle A, \langle \rho_i \cap A^{n_i} : i \in I \rangle \Big\rangle \cong \mathbb{X} \Big\}$$

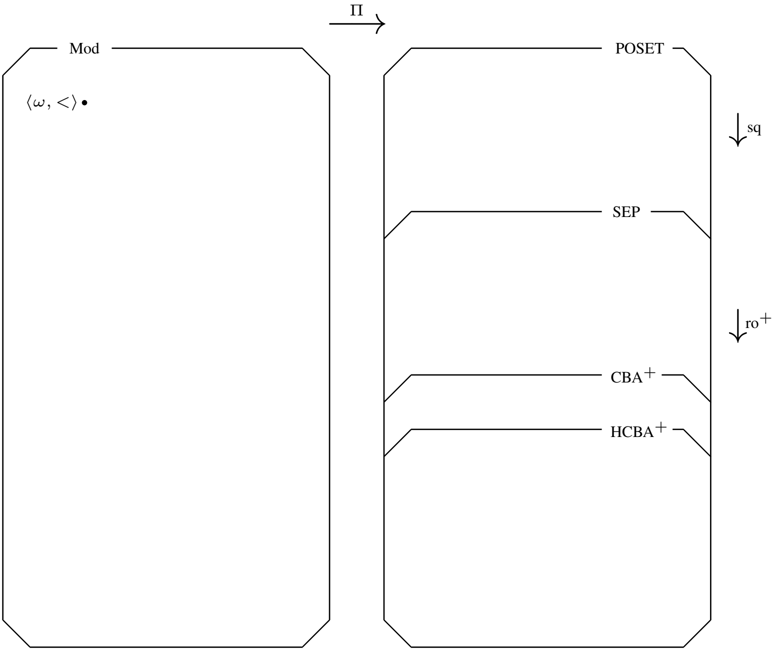
To $\mathbb{X}$ we adjoin

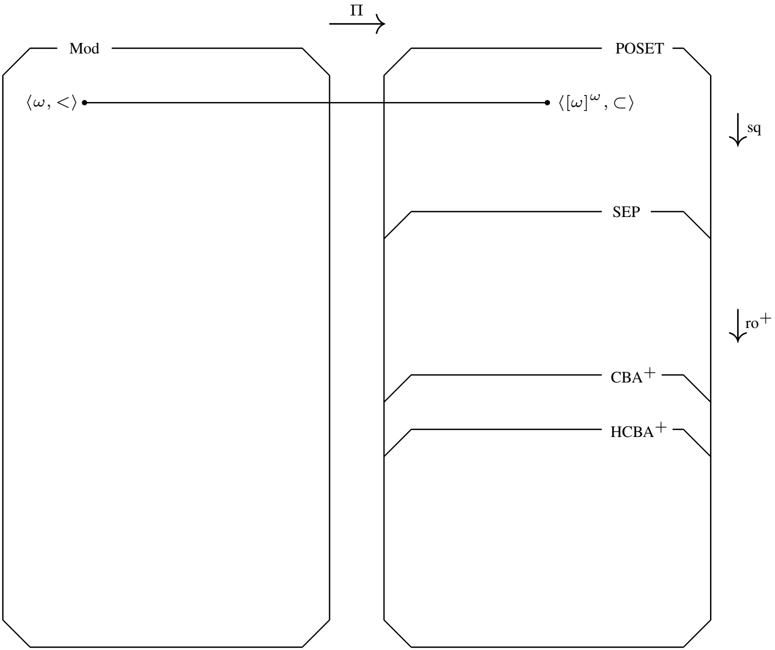
- the poset $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$
- its separative quotient $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$
- its Boolean completion $\mathrm{ro}\,\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$

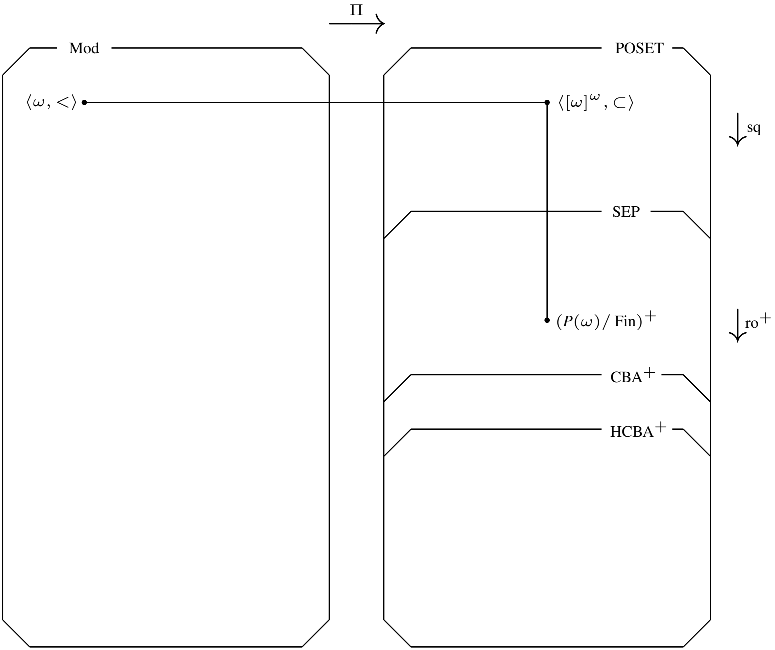
Theorem ([9])

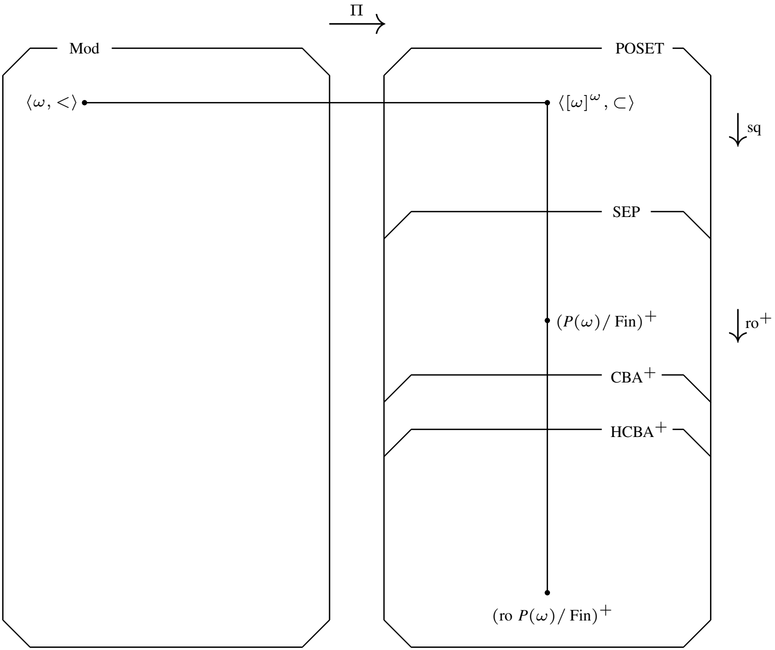
$\mathrm{ro}\,\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is a homogeneous c. B. a.

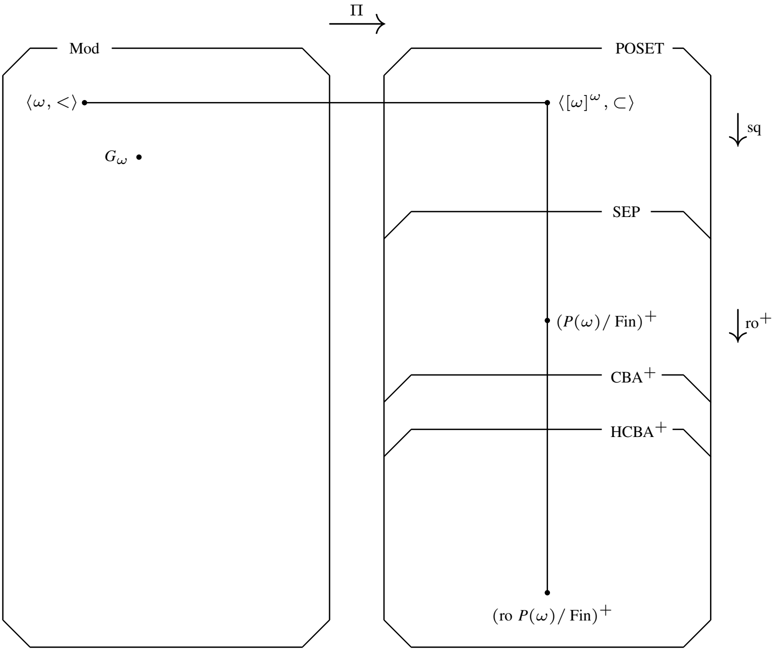

Π
Mod
POSET
sq
SEP
ro+
CBA+
HCBA+

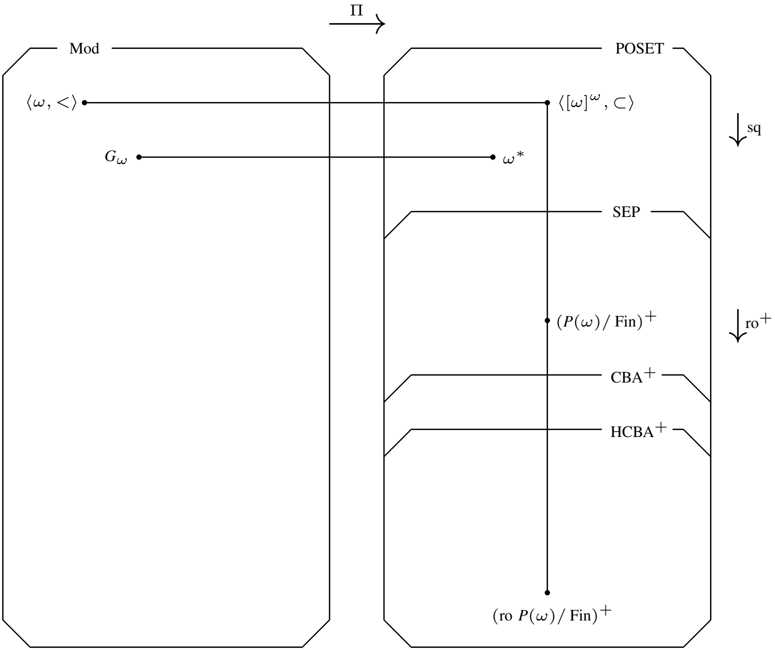

Π
Mod
$\langle \omega, < \rangle$
POSET
sq
SEP
ro$^+$
CBA$^+$
HCBA$^+$

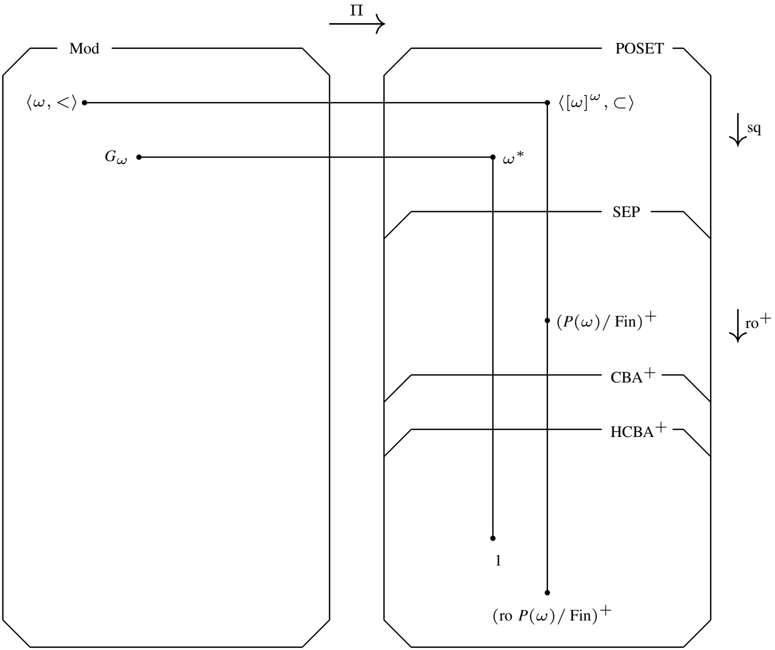

Π
Mod
POSET
⟨ω, <⟩
⟨[ω]^ω, ⊂⟩
sq
SEP
ro+
CBA+
HCBA+

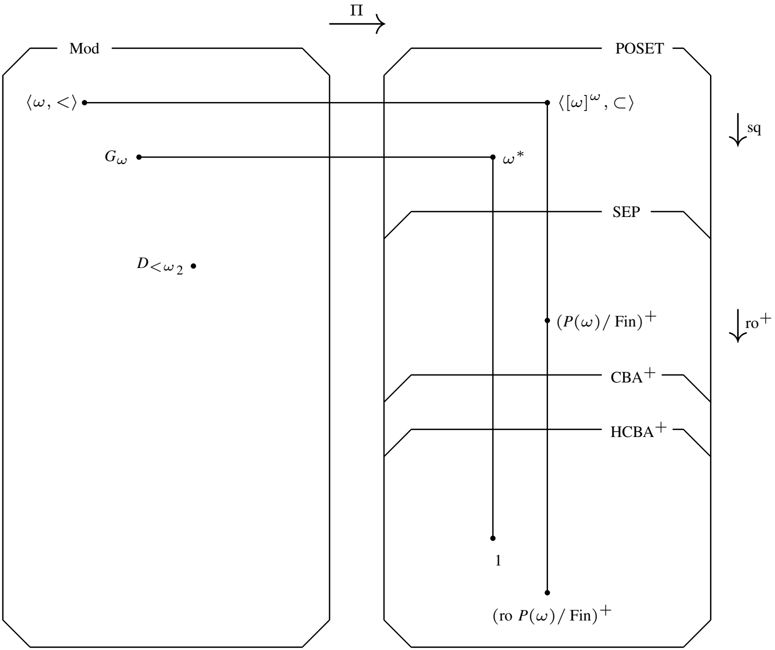

Π
Mod
POSET
⟨ω, <⟩
⟨[ω]^ω, ⊂⟩
sq
SEP
(P(ω) / Fin)+
ro+
CBA+
HCBA+

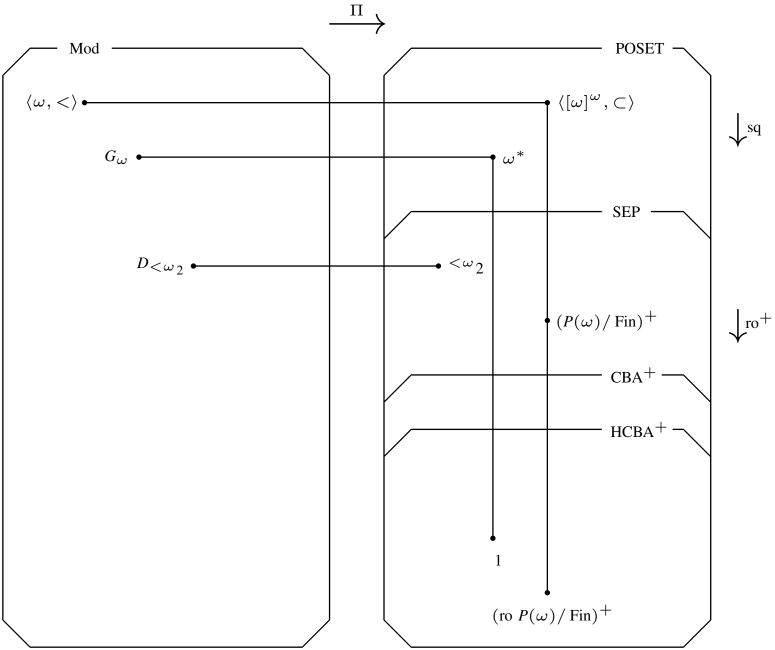

Π
Mod
POSET
⟨ω, <⟩
⟨[ω]^ω, ⊂⟩
sq
SEP
(P(ω) / Fin)+
ro+
CBA+
HCBA+
(ro P(ω) / Fin)+

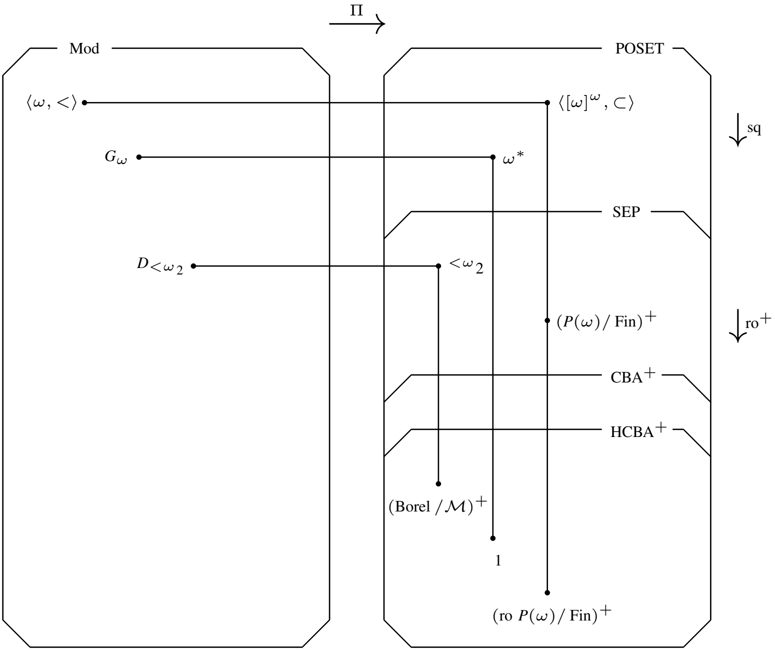

Π
Mod
POSET
$\langle \omega, < \rangle$
$\langle [\omega]^\omega, \subset \rangle$
sq
G_ω
SEP
$(P(\omega)/\operatorname{Fin})^+$
ro^+
CBA^+
HCBA^+
$(\text{ro } P(\omega)/\operatorname{Fin})^+$

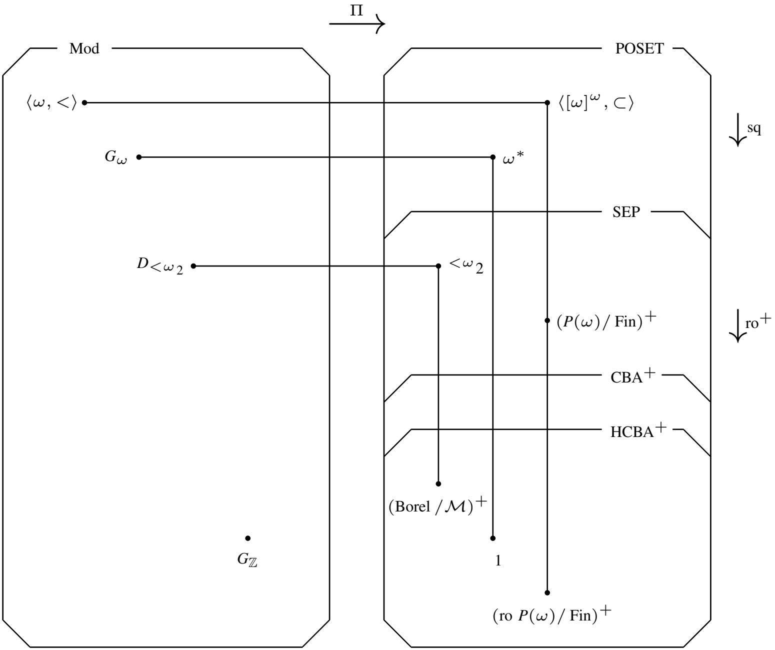

Π
Mod
POSET
$\langle \omega, < \rangle$
$\langle [\omega]^\omega, \subset \rangle$
sq
G_ω
ω^*
SEP
$(P(\omega)/\mathrm{Fin})^+$
ro^+
CBA^+
HCBA^+
$(\mathrm{ro}\ P(\omega)/\mathrm{Fin})^+$


Π
Mod
POSET
⟨ω, <⟩
⟨[ω]^ω, ⊂⟩
sq
G_ω
ω*
SEP
(P(ω) / Fin)+
ro+
CBA+
HCBA+
1
(ro P(ω) / Fin)+

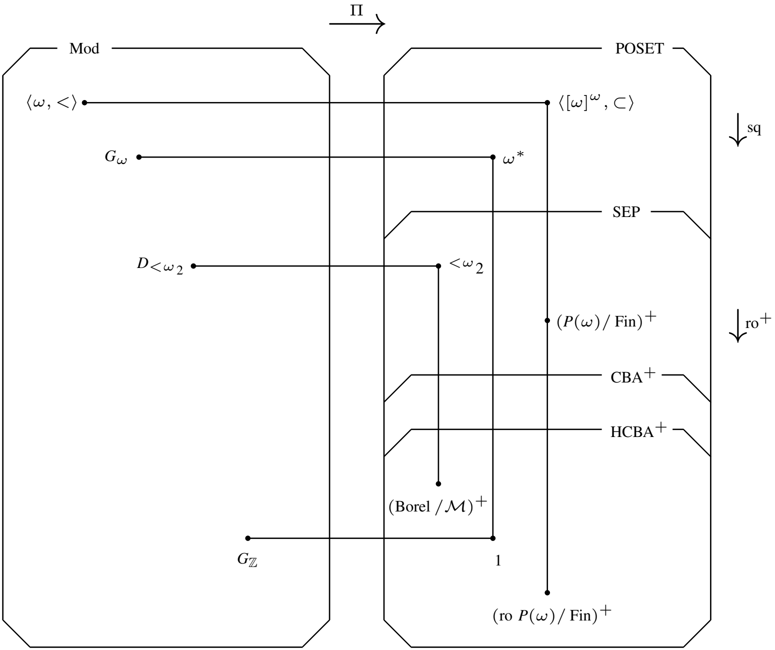

Π
Mod
POSET
$\langle \omega, < \rangle$
$\langle [\omega]^\omega, \subset \rangle$
sq
G_ω
ω^*
SEP
$D_{<\omega_2}$
$(P(\omega) / \mathrm{Fin})^+$
ro^+
CBA^+
HCBA^+
1
$(\mathrm{ro}\ P(\omega) / \mathrm{Fin})^+$


Π
Mod
POSET
⟨ω, <⟩
⟨[ω]^ω, ⊂⟩
sq
G_ω
ω*
SEP
D_{<ω_2}
<ω_2
(P(ω) / Fin)+
ro+
CBA+
HCBA+
1
(ro P(ω) / Fin)+

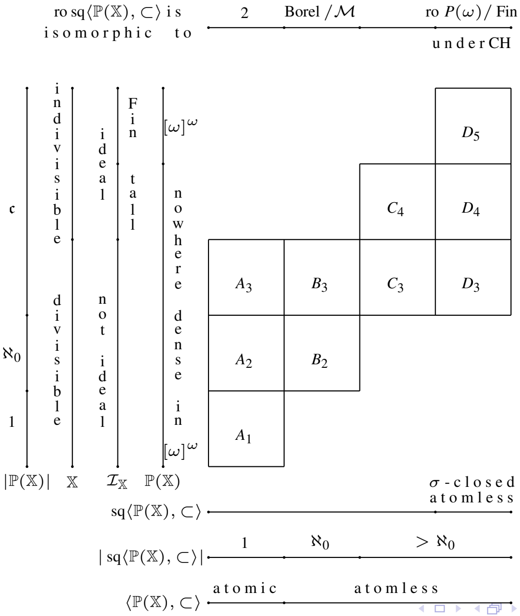

Π
Mod
POSET
$\langle\omega,<\rangle$
$\langle[\omega]^\omega,\subset\rangle$
sq
G_ω
ω^*
SEP
$D_{<\omega_2}$
$<\omega_2$
$(P(\omega)\,/\,\mathrm{Fin})^+$
ro^+
CBA^+
HCBA^+
$(\mathrm{Borel}\,/\,\mathcal{M})^+$
1
$(\mathrm{ro}\ P(\omega)\,/\,\mathrm{Fin})^+$



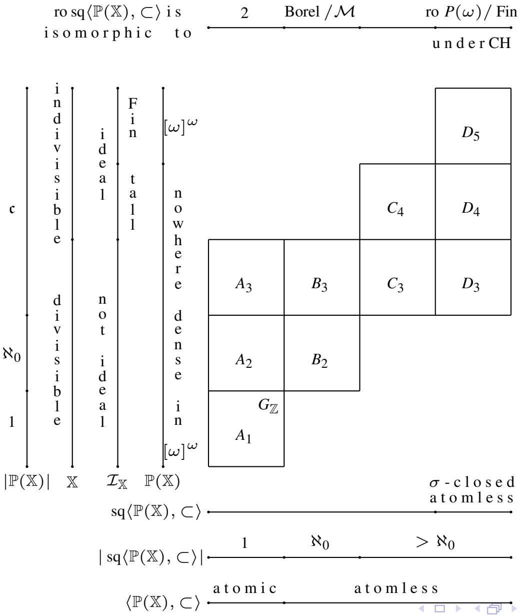
$\xrightarrow{\Pi}$
Mod
POSET
$\langle \omega, < \rangle$
$\langle [\omega]^\omega, \subset \rangle$
$\downarrow$ sq
G_ω
ω^*
SEP
$D_{<\omega_2}$
$<\omega_2$
$(P(\omega)/\,\mathrm{Fin})^+$
$\downarrow$ ro^+
CBA^+
HCBA^+
$(\mathrm{Borel}/\mathcal{M})^+$
$G_{\mathbb{Z}}$
1
$(\mathrm{ro}\ P(\omega)/\,\mathrm{Fin})^+$


Π
Mod
POSET
⟨ω, <⟩
⟨[ω]^ω, ⊂⟩
sq
G_ω
ω*
SEP
D_{<ω2}
<ω2
(P(ω) / Fin)+
ro+
CBA+
HCBA+
(Borel / M)+
G_ℤ
1
(ro P(ω) / Fin)+

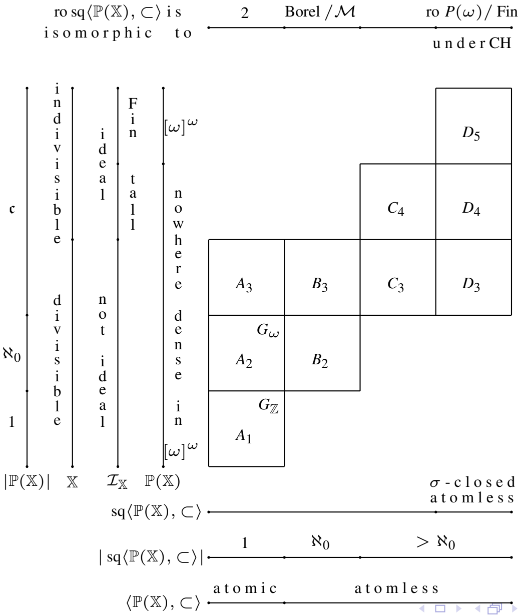
Countable binary structures [3]



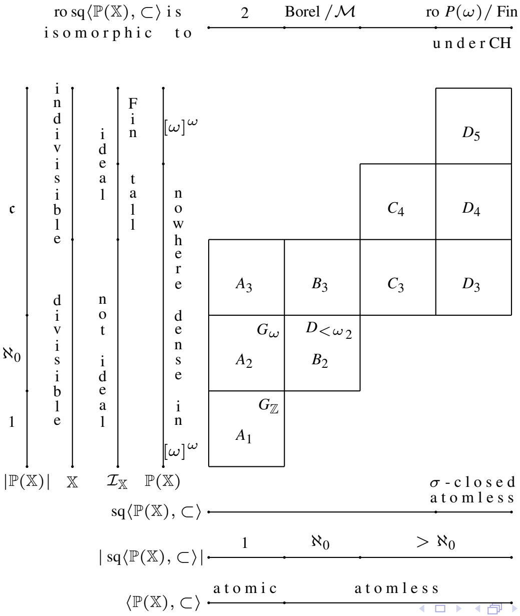
Countable binary structures [3]



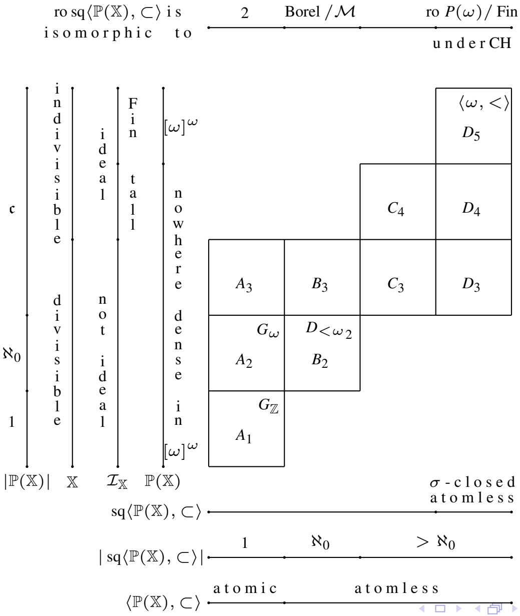
Countable binary structures [3]



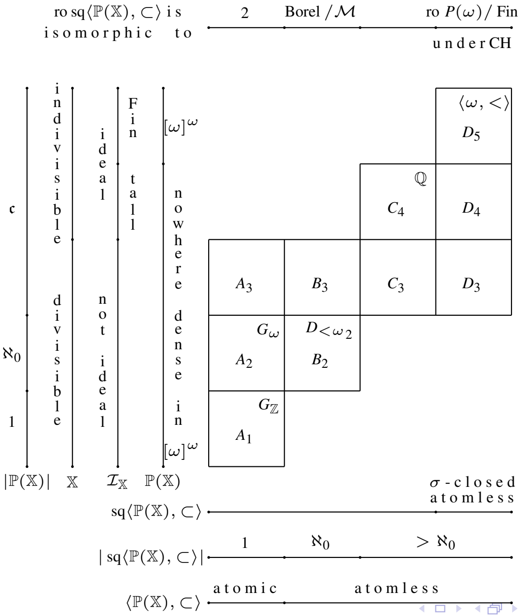
Countable binary structures [3]



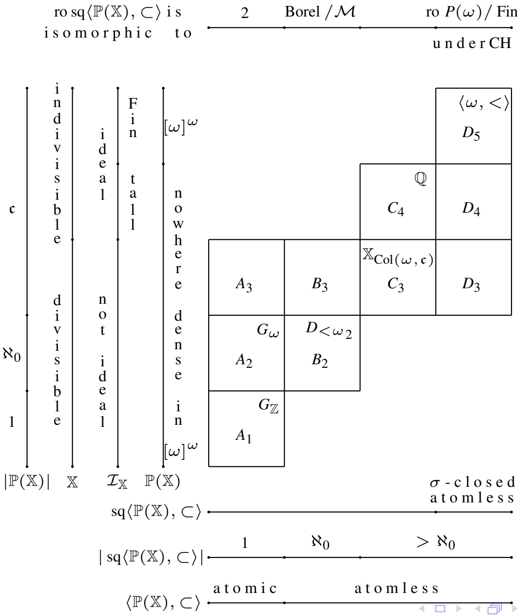
Countable binary structures [3]



Countable binary structures [3]



Countable binary structures [3]



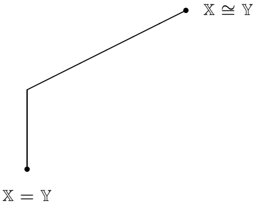
The hierarchy of similarities between relational structures

The hierarchy of similarities between relational structures

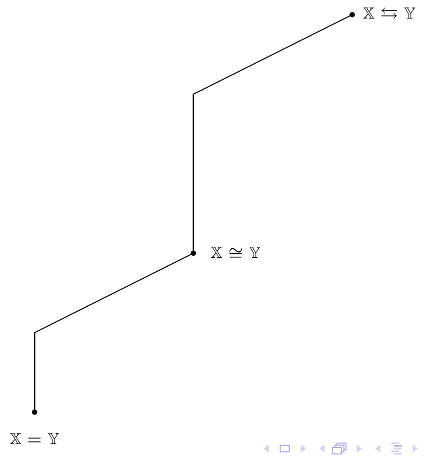
•

$\mathbb{X} = \mathbb{Y}$

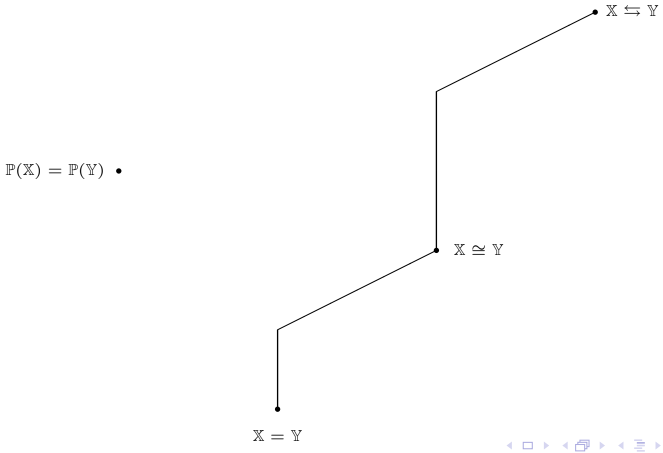
The hierarchy of similarities between relational structures



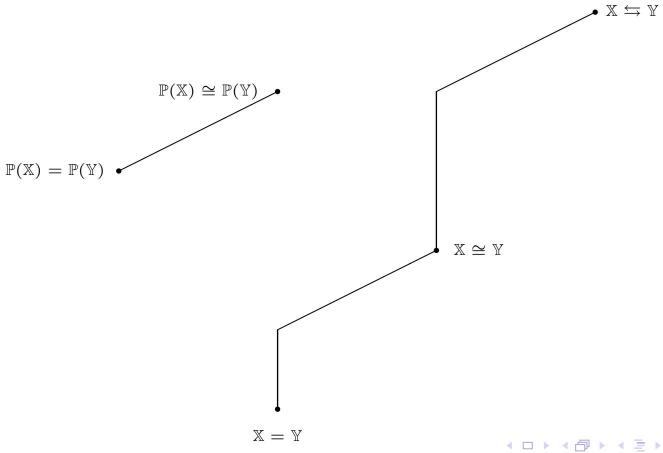
The hierarchy of similarities between relational structures



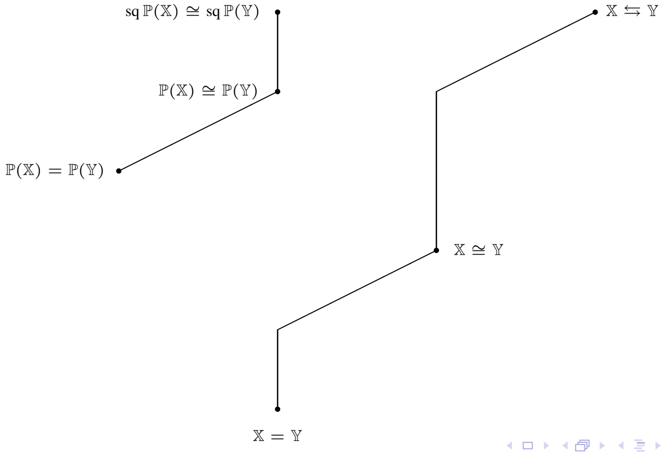
The hierarchy of similarities between relational structures



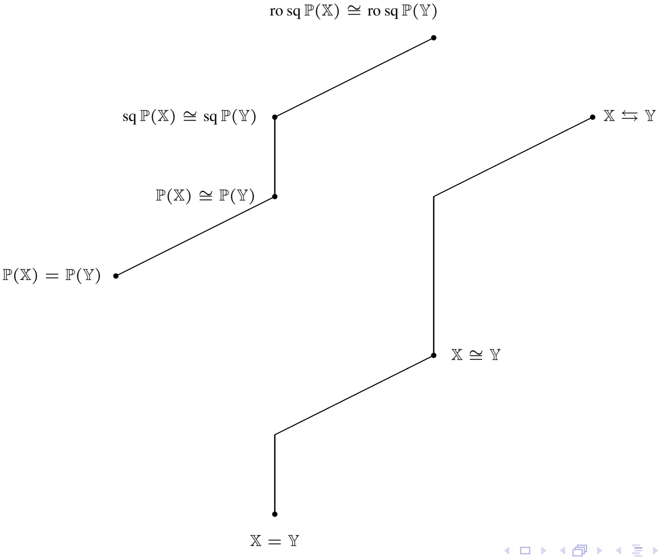
The hierarchy of similarities between relational structures



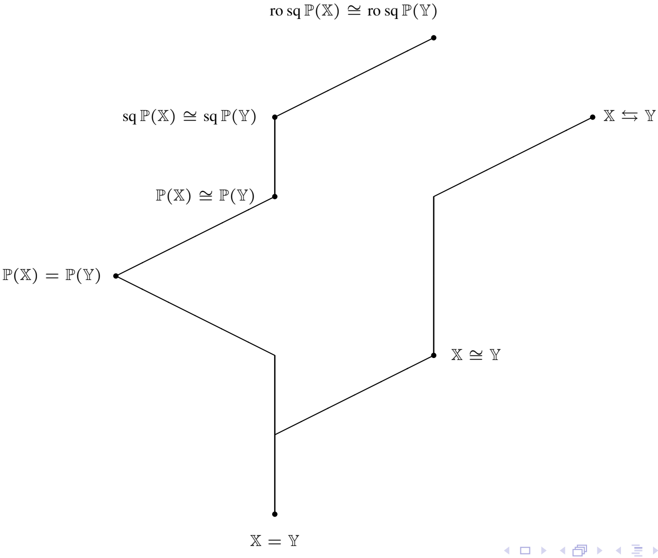
The hierarchy of similarities between relational structures



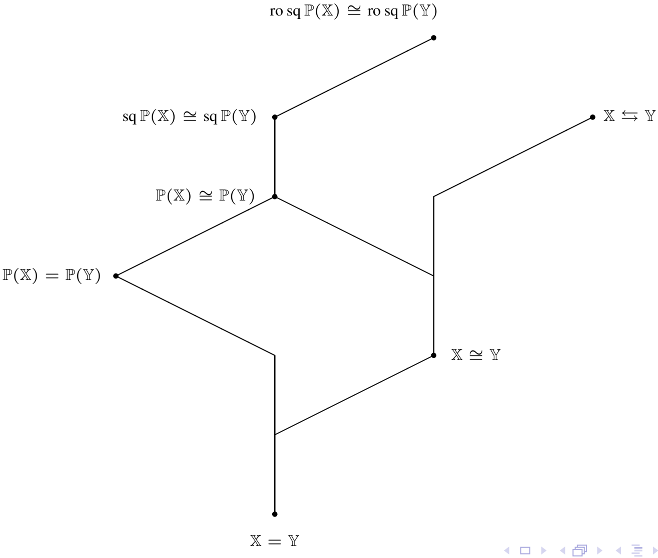
The hierarchy of similarities between relational structures



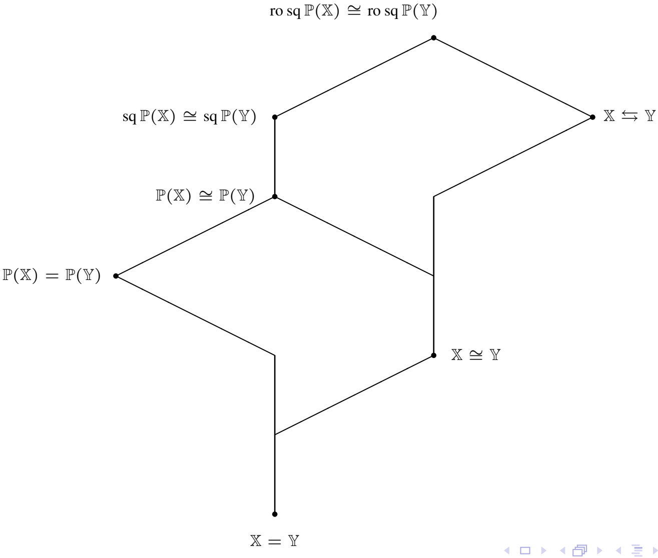
The hierarchy of similarities between relational structures



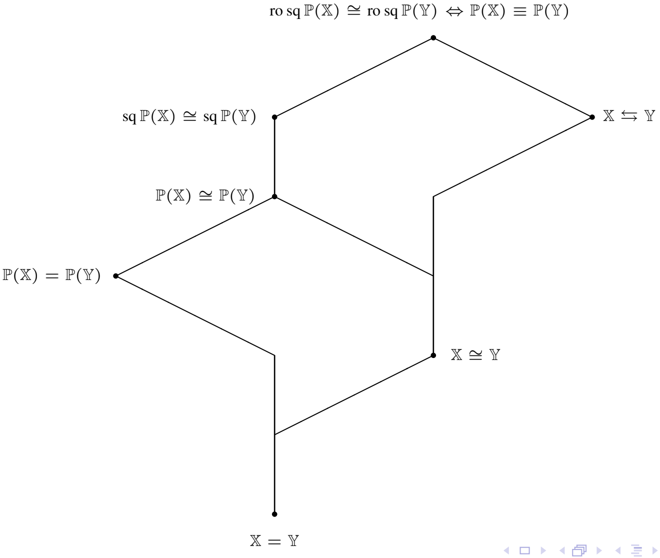
The hierarchy of similarities between relational structures



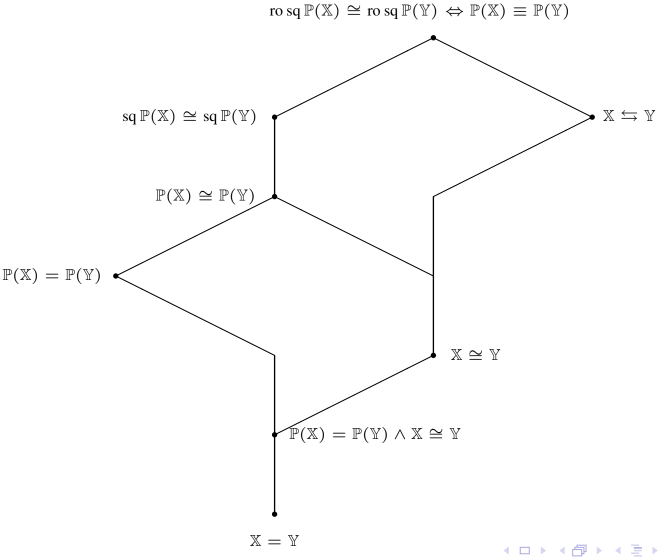
The hierarchy of similarities between relational structures



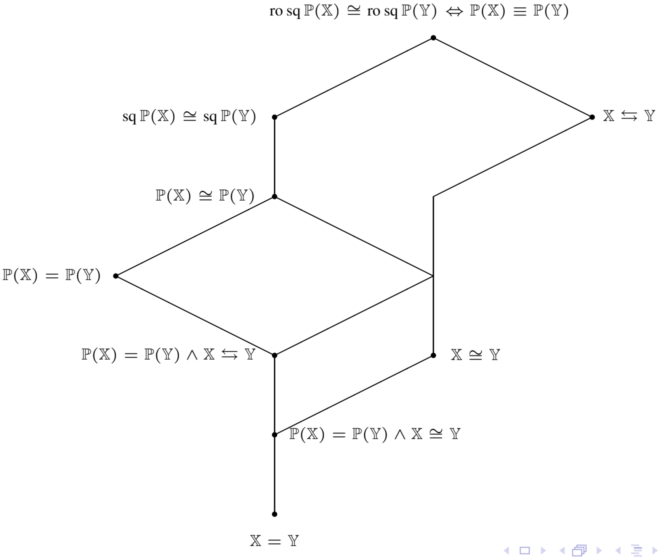
The hierarchy of similarities between relational structures



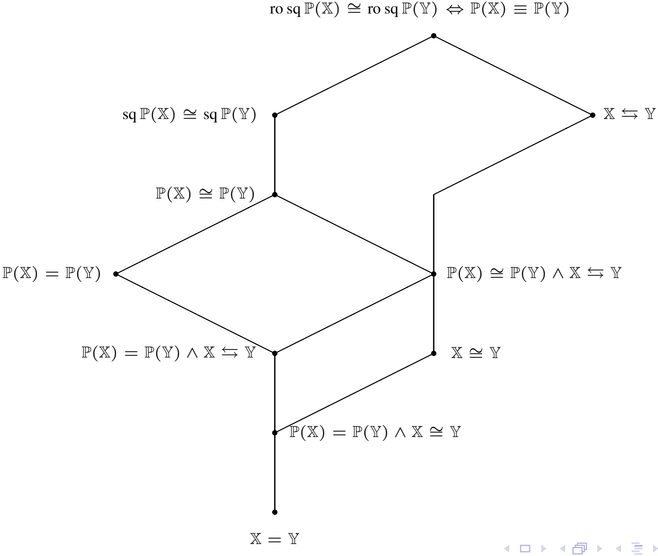
The hierarchy of similarities between relational structures



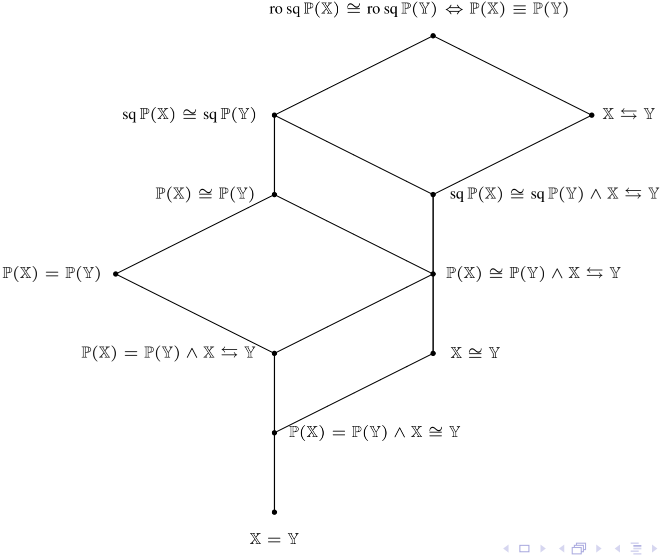
The hierarchy of similarities between relational structures



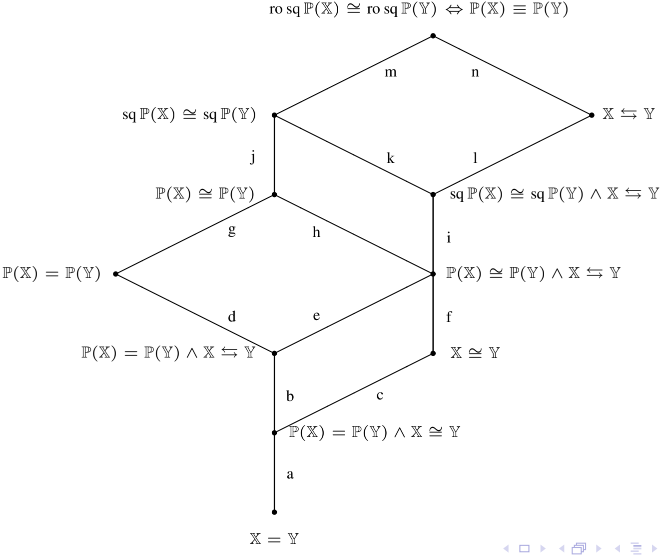
The hierarchy of similarities between relational structures



The hierarchy of similarities between relational structures



The hierarchy of similarities between relational structures



Example: Countable scattered l. o.’ s are in Column D

Example: Countable scattered l. o.' s are in Column D

Theorem ([5])

For each countable scattered linear order $\mathbb{X}$

- The poset $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is atomless and σ-closed

Example: Countable scattered l. o.' s are in Column D

Theorem ([5])

For each countable scattered linear order $\mathbb{X}$

- The poset $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is atomless and σ-closed
- Under CH we have $\langle \mathbb{P}(\mathbb{X}), \subset \rangle \equiv (P(\omega)/\,\mathrm{Fin})^+$.

Sub-example: Countable ordinals

Sub-example: Countable ordinals

Theorem ([6])

If

$$\alpha = \omega^{\gamma_n + r_n} s_n + \cdots + \omega^{\gamma_0 + r_0} s_0 + k$$

is a countably infinite ordinal presented in the Cantor normal form, then

Sub-example: Countable ordinals

Theorem ([6])

If

$$\alpha = \omega^{\gamma_n + r_n} s_n + \cdots + \omega^{\gamma_0 + r_0} s_0 + k$$

is a countably infinite ordinal presented in the Cantor normal form, then

$$\mathrm{sq}\langle \mathbb{P}(\alpha), \subset \rangle \cong \prod_{i=0}^{n} \left(\left(\mathrm{rp}^{r_i} \left(P(\omega^{\gamma_i}) / \mathcal{I}_{\omega^{\gamma_i}} \right) \right)^{+} \right)^{s_i}$$

Sub-example: Countable ordinals

Theorem ([6])

If

$$\alpha = \omega^{\gamma_n + r_n} s_n + \cdots + \omega^{\gamma_0 + r_0} s_0 + k$$

is a countably infinite ordinal presented in the Cantor normal form, then

$$\mathrm{sq}\langle \mathbb{P}(\alpha), \subset \rangle \cong \prod_{i=0}^{n} \left(\left(\mathrm{rp}^{r_i} (P(\omega^{\gamma_i}) / \mathcal{I}_{\omega^{\gamma_i}}) \right)^{+} \right)^{s_i}$$

$$\langle \mathbb{P}(\alpha), \subset \rangle \equiv \begin{cases} (P(\omega) / \operatorname{Fin})^{+} & \text{if } \alpha < \omega + \omega \\ & \end{cases}$$

Sub-example: Countable ordinals

Theorem ([6])

If

$$\alpha = \omega^{\gamma_n + r_n} s_n + \cdots + \omega^{\gamma_0 + r_0} s_0 + k$$

is a countably infinite ordinal presented in the Cantor normal form, then

$$\mathrm{sq}\langle \mathbb{P}(\alpha), \subset \rangle \cong \prod_{i=0}^{n} \left(\left(\mathrm{rp}^{r_i} (P(\omega^{\gamma_i}) / \mathcal{I}_{\omega^{\gamma_i}}) \right)^{+} \right)^{s_i}$$

$$\langle \mathbb{P}(\alpha), \subset \rangle \equiv \begin{cases} (P(\omega)/\,\mathrm{Fin})^{+} & \text{if } \alpha < \omega + \omega \\ (P(\omega)/\,\mathrm{Fin})^{+} * \pi & \text{if } \alpha \geq \omega + \omega \end{cases}$$

where $[\omega] \Vdash$ "π is an ω_1-closed, separative atomless forcing".

Example: Countable non-scattered l. o.'s are in Column C

Example: Countable non-scattered l. o.'s are in Column C

Theorem (with S. Todorčević, [10])

For each countable non-scattered linear order $\mathbb{X}$ we have

$$\langle \mathbb{P}(\mathbb{X}), \subset \rangle \equiv \mathbb{S} * \pi$$

where

Example: Countable non-scattered l. o.'s are in Column C

Theorem (with S. Todorčević, [10])

For each countable non-scattered linear order $\mathbb{X}$ we have

$$\langle \mathbb{P}(\mathbb{X}), \subset \rangle \equiv \mathbb{S} * \pi$$

where

- $\mathbb{S}$ is the Sacks forcing

Example: Countable non-scattered l. o.'s are in Column C

Theorem (with S. Todorčević, [10])

For each countable non-scattered linear order $\mathbb{X}$ we have

$$\langle \mathbb{P}(\mathbb{X}), \subset \rangle \equiv \mathbb{S} * \pi$$

where

- $\mathbb{S}$ is the Sacks forcing
- π codes a σ-closed forcing

Example: Countable non-scattered l. o.'s are in Column C

Theorem (with S. Todorčević, [10])

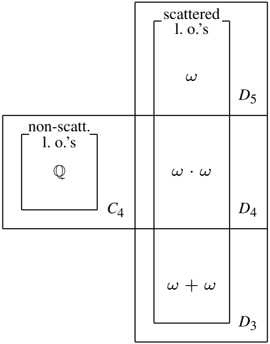
For each countable non-scattered linear order $\mathbb{X}$ we have

$$\langle \mathbb{P}(\mathbb{X}), \subset \rangle \equiv \mathbb{S} * \pi$$

where

- $\mathbb{S}$ is the Sacks forcing
- π codes a σ-closed forcing
- $1_{\mathbb{S}} \Vdash \pi \equiv (P(\omega)/\,\mathrm{Fin})^{+}$, under CH or PFA.

Countable linear orders in the $A_1 - D_5$ diagram



The poset $\mathbb{P}(\mathbb{X})$ and the monoid $\mathbb{E}\mathrm{mb}(\mathbb{X})$

The poset $\mathbb{P}(\mathbb{X})$ and the monoid $\mathbb{E}\mathrm{mb}(\mathbb{X})$

If $\mathbb{M} = \langle M, \cdot, e\rangle$ is a monoid (a semigroup with unity), the **right Green's preorder** on M is defined by

The poset $\mathbb{P}(\mathbb{X})$ and the monoid $\mathbb{E}\mathrm{mb}(\mathbb{X})$

If $\mathbb{M} = \langle M, \cdot, e \rangle$ is a monoid (a semigroup with unity), the **right Green's preorder** on M is defined by

$$x \preceq^R y \quad \Leftrightarrow \quad \exists z \in M \;\; xz = y$$

The poset $\mathbb{P}(\mathbb{X})$ and the monoid $\mathbb{E}\mathrm{mb}(\mathbb{X})$

If $\mathbb{M} = \langle M, \cdot, e\rangle$ is a monoid (a semigroup with unity), the **right Green's preorder** on M is defined by

$$x \preceq^R y \quad \Leftrightarrow \quad \exists z \in M \;\; xz = y$$

Fact

If $\mathbb{X}$ is a relational structure and $\mathbb{E}\mathrm{mb}(\mathbb{X}) = \langle \mathrm{Emb}(\mathbb{X}), \circ, \mathrm{id}_X\rangle$ the corresponding monoid of self-embeddings of $\mathbb{X}$, then

The poset $\mathbb{P}(\mathbb{X})$ and the monoid $\mathbb{E}\mathrm{mb}(\mathbb{X})$

If $\mathbb{M} = \langle M, \cdot, e \rangle$ is a monoid (a semigroup with unity), the **right Green's preorder** on M is defined by

$$x \preceq^R y \quad \Leftrightarrow \quad \exists z \in M \;\; xz = y$$

Fact

If $\mathbb{X}$ is a relational structure and $\mathbb{E}\mathrm{mb}(\mathbb{X}) = \langle \mathrm{Emb}(\mathbb{X}), \circ, \mathrm{id}_X \rangle$ the corresponding monoid of self-embeddings of $\mathbb{X}$, then

- $\mathbb{P}(\mathbb{X}) = \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\}$

The poset $\mathbb{P}(\mathbb{X})$ and the monoid $\mathbb{E}\text{mb}(\mathbb{X})$

If $\mathbb{M} = \langle M, \cdot, e\rangle$ is a monoid (a semigroup with unity), the **right Green's preorder** on M is defined by

$$x \preceq^R y \quad \Leftrightarrow \quad \exists z \in M \;\; xz = y$$

Fact

If $\mathbb{X}$ is a relational structure and $\mathbb{E}\text{mb}(\mathbb{X}) = \langle \text{Emb}(\mathbb{X}), \circ, \text{id}_X\rangle$ the corresponding monoid of self-embeddings of $\mathbb{X}$, then

- $\mathbb{P}(\mathbb{X}) = \{f[X] : f \in \text{Emb}(\mathbb{X})\}$
- $\langle \mathbb{P}(\mathbb{X}), \subset\rangle \cong \text{asq}\langle \text{Emb}(\mathbb{X}), (\preceq^R)^{-1}\rangle$

The poset $\mathbb{P}(\mathbb{X})$ and the monoid $\mathbb{E}\mathrm{mb}(\mathbb{X})$

If $\mathbb{M} = \langle M, \cdot, e \rangle$ is a monoid (a semigroup with unity), the **right Green's preorder** on M is defined by

$$x \preceq^R y \quad \Leftrightarrow \quad \exists z \in M \;\; xz = y$$

Fact

If $\mathbb{X}$ is a relational structure and $\mathbb{E}\mathrm{mb}(\mathbb{X}) = \langle \mathrm{Emb}(\mathbb{X}), \circ, \mathrm{id}_X \rangle$ the corresponding monoid of self-embeddings of $\mathbb{X}$, then

- $\mathbb{P}(\mathbb{X}) = \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\}$
- $\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \mathrm{asq}\langle \mathrm{Emb}(\mathbb{X}), (\preceq^R)^{-1} \rangle$
- $\{f \in \mathrm{Emb}(\mathbb{X}) : f \text{ is invertible}\} = \{f \in \mathrm{Emb}(\mathbb{X}) : f \text{ is regular}\} = \mathrm{Aut}(\mathbb{X})$

The poset $\mathbb{P}(\mathbb{X})$ and the monoid $\mathbb{E}\mathrm{mb}(\mathbb{X})$

If $\mathbb{M} = \langle M, \cdot, e\rangle$ is a monoid (a semigroup with unity), the **right Green's preorder** on M is defined by

$$x \preceq^R y \quad \Leftrightarrow \quad \exists z \in M \;\; xz = y$$

Fact

If $\mathbb{X}$ is a relational structure and $\mathbb{E}\mathrm{mb}(\mathbb{X}) = \langle \mathrm{Emb}(\mathbb{X}), \circ, \mathrm{id}_X\rangle$ the corresponding monoid of self-embeddings of $\mathbb{X}$, then

- $\mathbb{P}(\mathbb{X}) = \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\}$
- $\langle \mathbb{P}(\mathbb{X}), \subset\rangle \cong \mathrm{asq}\langle \mathrm{Emb}(\mathbb{X}), (\preceq^R)^{-1}\rangle$
- $\{f \in \mathrm{Emb}(\mathbb{X}) : f \text{ is invertible}\} = \{f \in \mathrm{Emb}(\mathbb{X}) : f \text{ is regular}\} = \mathrm{Aut}(\mathbb{X})$
- $\{\mathrm{id}_X\} = \{f \in \mathrm{Emb}(\mathbb{X}) : f \text{ is idempotent}\}$.

The poset $\mathbb{P}(\mathbb{X})$ and the monoid $\mathbb{E}\mathrm{mb}(\mathbb{X})$

If $\mathbb{M} = \langle M, \cdot, e\rangle$ is a monoid (a semigroup with unity), the **right Green's preorder** on M is defined by

$$x \preceq^R y \quad \Leftrightarrow \quad \exists z \in M \;\; xz = y$$

Fact

If $\mathbb{X}$ is a relational structure and $\mathbb{E}\mathrm{mb}(\mathbb{X}) = \langle \mathrm{Emb}(\mathbb{X}), \circ, \mathrm{id}_X\rangle$ the corresponding monoid of self-embeddings of $\mathbb{X}$, then

- $\mathbb{P}(\mathbb{X}) = \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\}$
- $\langle \mathbb{P}(\mathbb{X}), \subset\rangle \cong \mathrm{asq}\langle \mathrm{Emb}(\mathbb{X}), (\preceq^R)^{-1}\rangle$
- $\{f \in \mathrm{Emb}(\mathbb{X}) : f \text{ is invertible}\} = \{f \in \mathrm{Emb}(\mathbb{X}) : f \text{ is regular}\} = \mathrm{Aut}(\mathbb{X})$
- $\{\mathrm{id}_X\} = \{f \in \mathrm{Emb}(\mathbb{X}) : f \text{ is idempotent}\}$.

Theorem

If $\mathbb{X}$ and $\mathbb{Y}$ are relational structures, then

$$\mathbb{X} \cong \mathbb{Y} \Rightarrow \mathbb{E}\mathrm{mb}(\mathbb{X}) \cong \mathbb{E}\mathrm{mb}(\mathbb{Y}) \Rightarrow \langle \mathbb{P}(\mathbb{X}), \subset\rangle \cong \langle \mathbb{P}(\mathbb{Y}), \subset\rangle$$

An application of $\mathbb{E}\mathrm{mb}(\mathbb{X}) \cong \mathbb{E}\mathrm{mb}(\mathbb{Y}) \Rightarrow \langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \langle \mathbb{P}(\mathbb{Y}), \subset \rangle$

An application of $\mathbb{E}\mathrm{mb}(\mathbb{X}) \cong \mathbb{E}\mathrm{mb}(\mathbb{Y}) \Rightarrow \langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \langle \mathbb{P}(\mathbb{Y}), \subset \rangle$

Since $\mathbb{P}((0,1)_{\mathbb{Q}}, <) \ncong \mathbb{P}([0,1]_{\mathbb{Q}}, <)$ we have

$$\mathbb{E}\mathrm{mb}((0,1)_{\mathbb{Q}}, <) \ncong \mathbb{E}\mathrm{mb}([0,1]_{\mathbb{Q}}, <).$$

An application of $\mathbb{E}\mathrm{mb}(\mathbb{X}) \cong \mathbb{E}\mathrm{mb}(\mathbb{Y}) \Rightarrow \langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \langle \mathbb{P}(\mathbb{Y}), \subset \rangle$

Since $\mathbb{P}((0,1)_{\mathbb{Q}}, <) \not\cong \mathbb{P}([0,1]_{\mathbb{Q}}, <)$ we have

$$\mathbb{E}\mathrm{mb}((0,1)_{\mathbb{Q}}, <) \not\cong \mathbb{E}\mathrm{mb}([0,1]_{\mathbb{Q}}, <).$$

(Comment: but $\mathrm{sq}\, \mathbb{P}((0,1)_{\mathbb{Q}}, <) \cong \mathrm{sq}\, \mathbb{P}([0,1]_{\mathbb{Q}}, <)$)

Cancellativity commutativity and reversibility of $\mathrm{Emb}(\mathbb{X})$

Cancellativity commutativity and reversibility of $\mathrm{Emb}(\mathbb{X})$

A monoid $\mathbb{M} = \langle M, \cdot, e \rangle$ is

Cancellativity commutativity and reversibility of $\mathrm{Emb}(\mathbb{X})$

A monoid $\mathbb{M} = \langle M, \cdot, e\rangle$ is

$$\textbf{left reversible} \quad \Leftrightarrow \quad \forall x, y \;\; \exists u, v \;\; xu = yv$$

Cancellativity commutativity and reversibility of $\mathrm{Emb}(\mathbb{X})$

A monoid $\mathbb{M} = \langle M, \cdot, e\rangle$ is

$$\begin{aligned} \textbf{left reversible} \quad &\Leftrightarrow \quad \forall x, y \ \ \exists u, v \ \ xu = yv \\ \textbf{right reversible} \quad &\Leftrightarrow \quad \forall x, y \ \ \exists u, v \ \ ux = vy \end{aligned}$$

Cancellativity commutativity and reversibility of $\mathrm{Emb}(\mathbb{X})$

A monoid $\mathbb{M} = \langle M, \cdot, e \rangle$ is

$$\begin{array}{rcl} \textbf{left reversible} & \Leftrightarrow & \forall x, y \;\; \exists u, v \;\; xu = yv \\ \textbf{right reversible} & \Leftrightarrow & \forall x, y \;\; \exists u, v \;\; ux = vy \end{array}$$

If $\mathbb{X}$ is a relational structure, a set $A \subset X$ will be called **embedding-dense**, we will write $A \in \mathrm{EDense}(\mathbb{X})$ iff

Cancellativity commutativity and reversibility of $\mathrm{Emb}(\mathbb{X})$

A monoid $\mathbb{M} = \langle M, \cdot, e \rangle$ is

$$\begin{aligned} \textbf{left reversible} \quad &\Leftrightarrow \quad \forall x, y \;\; \exists u, v \;\; xu = yv \\ \textbf{right reversible} \quad &\Leftrightarrow \quad \forall x, y \;\; \exists u, v \;\; ux = vy \end{aligned}$$

If $\mathbb{X}$ is a relational structure, a set $A \subset X$ will be called **embedding-dense**, we will write $A \in \mathrm{EDense}(\mathbb{X})$ iff

$$\forall g, h \in \mathrm{Emb}(\mathbb{X}) \;\; (g \restriction A = h \restriction A \Rightarrow g = h).$$

Cancellativity commutativity and reversibility of $\mathrm{Emb}(\mathbb{X})$

A monoid $\mathbb{M} = \langle M, \cdot, e \rangle$ is

$$\begin{array}{rcl} \textbf{left reversible} & \Leftrightarrow & \forall x, y \ \ \exists u, v \ \ xu = yv \\ \textbf{right reversible} & \Leftrightarrow & \forall x, y \ \ \exists u, v \ \ ux = vy \end{array}$$

If $\mathbb{X}$ is a relational structure, a set $A \subset X$ will be called **embedding-dense**, we will write $A \in \mathrm{EDense}(\mathbb{X})$ iff

$$\forall g, h \in \mathrm{Emb}(\mathbb{X}) \ \ (g \restriction A = h \restriction A \Rightarrow g = h).$$

Theorem

If $\mathbb{X}$ is a relational structure, then

Cancellativity commutativity and reversibility of $\mathrm{Emb}(\mathbb{X})$

A monoid $\mathbb{M} = \langle M, \cdot, e\rangle$ is

$$\begin{array}{rcl} \textbf{left reversible} & \Leftrightarrow & \forall x, y \ \ \exists u, v \ \ xu = yv \\ \textbf{right reversible} & \Leftrightarrow & \forall x, y \ \ \exists u, v \ \ ux = vy \end{array}$$

If $\mathbb{X}$ is a relational structure, a set $A \subset X$ will be called **embedding-dense**, we will write $A \in \mathrm{EDense}(\mathbb{X})$ iff

$$\forall g, h \in \mathrm{Emb}(\mathbb{X}) \ \ (g \restriction A = h \restriction A \Rightarrow g = h).$$

Theorem

If $\mathbb{X}$ is a relational structure, then

- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is cancellative $\Leftrightarrow \mathbb{P}(\mathbb{X}) \subset \mathrm{EDense}(\mathbb{X})$

Cancellativity commutativity and reversibility of $\mathrm{Emb}(\mathbb{X})$

A monoid $\mathbb{M} = \langle M, \cdot, e\rangle$ is

$$\begin{array}{rcl} \textbf{left reversible} & \Leftrightarrow & \forall x, y\ \ \exists u, v\ \ xu = yv \\ \textbf{right reversible} & \Leftrightarrow & \forall x, y\ \ \exists u, v\ \ ux = vy \end{array}$$

If $\mathbb{X}$ is a relational structure, a set $A \subset X$ will be called **embedding-dense**, we will write $A \in \mathrm{EDense}(\mathbb{X})$ iff

$$\forall g, h \in \mathrm{Emb}(\mathbb{X}) \ \ (g \restriction A = h \restriction A \Rightarrow g = h).$$

Theorem

If $\mathbb{X}$ is a relational structure, then

- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is cancellative $\Leftrightarrow \mathbb{P}(\mathbb{X}) \subset \mathrm{EDense}(\mathbb{X})$
- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is left reversible $\Leftrightarrow$ the poset $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is atomic (Column A)

Cancellativity commutativity and reversibility of Emb($\mathbb{X}$)

A monoid $\mathbb{M} = \langle M, \cdot, e\rangle$ is

$$\begin{aligned} \textbf{left reversible} \quad &\Leftrightarrow \quad \forall x, y \ \ \exists u, v \ \ xu = yv \\ \textbf{right reversible} \quad &\Leftrightarrow \quad \forall x, y \ \ \exists u, v \ \ ux = vy \end{aligned}$$

If $\mathbb{X}$ is a relational structure, a set $A \subset X$ will be called **embedding-dense**, we will write $A \in \mathrm{EDense}(\mathbb{X})$ iff

$$\forall g, h \in \mathrm{Emb}(\mathbb{X}) \ \ (g \restriction A = h \restriction A \Rightarrow g = h).$$

Theorem

If $\mathbb{X}$ is a relational structure, then

- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is cancellative $\Leftrightarrow \mathbb{P}(\mathbb{X}) \subset \mathrm{EDense}(\mathbb{X})$
- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is left reversible $\Leftrightarrow$ the poset $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is atomic (Column A)
- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is right reversible $\Leftrightarrow \mathbb{X}$ has the amalgamation property for embeddings

Cancellativity commutativity and reversibility of $\mathrm{Emb}(\mathbb{X})$

A monoid $\mathbb{M} = \langle M, \cdot, e\rangle$ is

$$\begin{array}{rcl} \textbf{left reversible} & \Leftrightarrow & \forall x, y \ \exists u, v \ \ xu = yv \\ \textbf{right reversible} & \Leftrightarrow & \forall x, y \ \exists u, v \ \ ux = vy \end{array}$$

If $\mathbb{X}$ is a relational structure, a set $A \subset X$ will be called **embedding-dense**, we will write $A \in \mathrm{EDense}(\mathbb{X})$ iff

$$\forall g, h \in \mathrm{Emb}(\mathbb{X}) \ \ (g \restriction A = h \restriction A \Rightarrow g = h).$$

Theorem

If $\mathbb{X}$ is a relational structure, then

- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is cancellative $\Leftrightarrow \mathbb{P}(\mathbb{X}) \subset \mathrm{EDense}(\mathbb{X})$
- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is left reversible $\Leftrightarrow$ the poset $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is atomic (Column A)
- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is right reversible $\Leftrightarrow \mathbb{X}$ has the amalgamation property for embeddings
- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is commutative $\Rightarrow \mathbb{E}\mathrm{mb}(\mathbb{X})$ is cancellative, left reversible, and right reversible.

Embeddability of Emb($\mathbb{X}$) into a group

Embeddability of $\mathrm{Emb}(\mathbb{X})$ into a group

Theorem

$\mathbb{E}\mathrm{mb}(\mathbb{X})$ is a (retract of a) group $\Leftrightarrow \mathbb{P}(\mathbb{X}) = \{X\}$

Embeddability of Emb($\mathbb{X}$) into a group

Theorem

$\mathbb{E}\mathrm{mb}(\mathbb{X})$ is a (retract of a) group $\Leftrightarrow \mathbb{P}(\mathbb{X}) = \{X\}$

Theorem

Each of the following conditions implies that $\mathbb{E}\mathrm{mb}(\mathbb{X})$ embeds into a group

Embeddability of $\mathrm{Emb}(\mathbb{X})$ into a group

Theorem

$\mathbb{E}\mathrm{mb}(\mathbb{X})$ is a (retract of a) group $\Leftrightarrow \mathbb{P}(\mathbb{X}) = \{X\}$

Theorem

Each of the following conditions implies that $\mathbb{E}\mathrm{mb}(\mathbb{X})$ embeds into a group

- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is commutative

Embeddability of $\mathbb{E}\mathrm{mb}(\mathbb{X})$ into a group

Theorem

$\mathbb{E}\mathrm{mb}(\mathbb{X})$ is a (retract of a) group $\Leftrightarrow \mathbb{P}(\mathbb{X}) = \{X\}$

Theorem

Each of the following conditions implies that $\mathbb{E}\mathrm{mb}(\mathbb{X})$ embeds into a group

- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is commutative
- $\mathbb{P}(\mathbb{X}) \subset \mathrm{EDense}(\mathbb{X})$ and $\mathbb{P}(\mathbb{X})$ is atomic

Embeddability of $\mathrm{Emb}(\mathbb{X})$ into a group

Theorem

$\mathbb{E}\mathrm{mb}(\mathbb{X})$ is a (retract of a) group $\Leftrightarrow \mathbb{P}(\mathbb{X}) = \{X\}$

Theorem

Each of the following conditions implies that $\mathbb{E}\mathrm{mb}(\mathbb{X})$ embeds into a group

- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is commutative
- $\mathbb{P}(\mathbb{X}) \subset \mathrm{EDense}(\mathbb{X})$ and $\mathbb{P}(\mathbb{X})$ is atomic
- $\mathbb{P}(\mathbb{X}) \subset \mathrm{EDense}(\mathbb{X})$ and $\mathbb{X}$ has amalgamation for embeddings

Embeddability of $\mathrm{Emb}(\mathbb{X})$ into a group

Theorem

$\mathbb{E}\mathrm{mb}(\mathbb{X})$ is a (retract of a) group $\Leftrightarrow \mathbb{P}(\mathbb{X}) = \{X\}$

Theorem

Each of the following conditions implies that $\mathbb{E}\mathrm{mb}(\mathbb{X})$ embeds into a group

- $\mathbb{E}\mathrm{mb}(\mathbb{X})$ is commutative
- $\mathbb{P}(\mathbb{X}) \subset \mathrm{EDense}(\mathbb{X})$ and $\mathbb{P}(\mathbb{X})$ is atomic
- $\mathbb{P}(\mathbb{X}) \subset \mathrm{EDense}(\mathbb{X})$ and $\mathbb{X}$ has amalgamation for embeddings

Proof. Using theorems of Grothendieck, Ore, and Dubreil.

Everything is possible

Everything is possible

$\mathbb{X}$	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ a group	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ commutative	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ embeddable into a group

Everything is possible

$\mathbb{X}$	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ a group	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ commutative	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ embeddable into a group
$G_{\mathbb{Z}}$	+	+	of course

Everything is possible

$\mathbb{X}$	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ a group	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ commutative	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ embeddable into a group
$G_{\mathbb{Z}}$	+	+	of course
$\bigcup_{n\geq 3} C_n$	+	-	of course

Everything is possible

$\mathbb{X}$	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ a group	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ commutative	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ embeddable into a group
$G_{\mathbb{Z}}$	+	+	of course
$\bigcup_{n\geq 3} C_n$	+	-	of course
G_ω	-	+	+

Everything is possible

$\mathbb{X}$	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ a group	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ commutative	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ embeddable into a group
$G_{\mathbb{Z}}$	+	+	of course
$\bigcup_{n\geq 3} C_n$	+	-	of course
G_ω	-	+	+
$G_\omega \cup \bigcup_{n\geq 3} C_n$	-	-	+

Everything is possible

$\mathbb{X}$	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ a group	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ commutative	$\mathbb{E}\mathrm{mb}(\mathbb{X})$ embeddable into a group
$G_{\mathbb{Z}}$	+	+	of course
$\bigcup_{n\geq 3} C_n$	+	-	of course
G_ω	-	+	+
$G_\omega \cup \bigcup_{n\geq 3} C_n$	-	-	+
$\langle \omega, < \rangle$	-	-	-

Bi-interpretable structures

Bi-interpretable structures

Let $\mathbb{X}$ and $\mathbb{Y}$ be relational structures of languages $L_{\mathbb{X}}$ and $L_{\mathbb{Y}}$.

Bi-interpretable structures

Let $\mathbb{X}$ and $\mathbb{Y}$ be relational structures of languages $L_{\mathbb{X}}$ and $L_{\mathbb{Y}}$.

- An **interpretation of $\mathbb{X}$ in $\mathbb{Y}$ (without parameters)** is a triple $\langle n, S, f \rangle$, where

Bi-interpretable structures

Let $\mathbb{X}$ and $\mathbb{Y}$ be relational structures of languages $L_{\mathbb{X}}$ and $L_{\mathbb{Y}}$.

- An **interpretation of** $\mathbb{X}$ **in** $\mathbb{Y}$ **(without parameters)** is a triple $\langle n, S, f\rangle$, where
  - $n \in \mathbb{N}$,

Bi-interpretable structures

Let $\mathbb{X}$ and $\mathbb{Y}$ be relational structures of languages $L_{\mathbb{X}}$ and $L_{\mathbb{Y}}$.

- An **interpretation of** $\mathbb{X}$ **in** $\mathbb{Y}$ **(without parameters)** is a triple $\langle n, S, f \rangle$, where
  - $n \in \mathbb{N}$,
  - $S \subset Y^n$,

Bi-interpretable structures

Let $\mathbb{X}$ and $\mathbb{Y}$ be relational structures of languages $L_{\mathbb{X}}$ and $L_{\mathbb{Y}}$.

- An **interpretation of** $\mathbb{X}$ **in** $\mathbb{Y}$ **(without parameters)** is a triple $\langle n, S, f \rangle$, where
  - $n \in \mathbb{N}$,
  - $S \subset Y^n$,
  - $f : S \longrightarrow X$ is a surjection,

Bi-interpretable structures

Let $\mathbb{X}$ and $\mathbb{Y}$ be relational structures of languages $L_{\mathbb{X}}$ and $L_{\mathbb{Y}}$.

- An **interpretation of** $\mathbb{X}$ **in** $\mathbb{Y}$ **(without parameters)** is a triple $\langle n, S, f \rangle$, where
  - $n \in \mathbb{N}$,
  - $S \subset Y^n$,
  - $f : S \to X$ is a surjection,
  - the pre-image of each set definable in $\mathbb{X}$ is definable in $\mathbb{Y}$.

Bi-interpretable structures

Let $\mathbb{X}$ and $\mathbb{Y}$ be relational structures of languages $L_{\mathbb{X}}$ and $L_{\mathbb{Y}}$.

- An **interpretation of** $\mathbb{X}$ **in** $\mathbb{Y}$ **(without parameters)** is a triple $\langle n, S, f\rangle$, where
  - $n \in \mathbb{N}$,
  - $S \subset Y^n$,
  - $f : S \to X$ is a surjection,
  - the pre-image of each set definable in $\mathbb{X}$ is definable in $\mathbb{Y}$.

  Then we write $f : \mathbb{Y} \rightsquigarrow \mathbb{X}$.

Bi-interpretable structures

Let $\mathbb{X}$ and $\mathbb{Y}$ be relational structures of languages $L_{\mathbb{X}}$ and $L_{\mathbb{Y}}$.

- An **interpretation of $\mathbb{X}$ in $\mathbb{Y}$ (without parameters)** is a triple $\langle n, S, f\rangle$, where
  - $n \in \mathbb{N}$,
  - $S \subset Y^n$,
  - $f : S \to X$ is a surjection,
  - the pre-image of each set definable in $\mathbb{X}$ is definable in $\mathbb{Y}$.

  Then we write $f : \mathbb{Y} \rightsquigarrow \mathbb{X}$.
- Structures $\mathbb{X}$ and $\mathbb{Y}$ are **bi-interpretable** iff there are interpretations

$$f : \mathbb{Y} \rightsquigarrow \mathbb{X} \text{ and } g : \mathbb{X} \rightsquigarrow \mathbb{Y}$$

  such that the compositions $f * g$ and $g * f$ are definable (in $\mathbb{X}$ and $\mathbb{Y}$ respectively).

Bi-interpretable structures

Let $\mathbb{X}$ and $\mathbb{Y}$ be relational structures of languages $L_{\mathbb{X}}$ and $L_{\mathbb{Y}}$.

- An **interpretation of $\mathbb{X}$ in $\mathbb{Y}$ (without parameters)** is a triple $\langle n, S, f\rangle$, where
  - $n \in \mathbb{N}$,
  - $S \subset Y^n$,
  - $f : S \longrightarrow X$ is a surjection,
  - the pre-image of each set definable in $\mathbb{X}$ is definable in $\mathbb{Y}$.

  Then we write $f : \mathbb{Y} \rightsquigarrow \mathbb{X}$.
- Structures $\mathbb{X}$ and $\mathbb{Y}$ are **bi-interpretable** iff there are interpretations

$$f : \mathbb{Y} \rightsquigarrow \mathbb{X} \text{ and } g : \mathbb{X} \rightsquigarrow \mathbb{Y}$$

  such that the compositions $f * g$ and $g * f$ are definable (in $\mathbb{X}$ and $\mathbb{Y}$ respectively).

Theorem

$\mathbb{X}$ and $\mathbb{Y}$ are quantifier-free bi-interpretable $\Rightarrow$ $\mathrm{Emb}(\mathbb{X}) \cong \mathrm{Emb}(\mathbb{Y})$.

An application: a dichotomy for ultrahomogeneous str.

An application: a dichotomy for ultrahomogeneous str.

The **enlargement** of a binary structure $\mathbb{X} = \langle X, \rho \rangle$ is the structure $\mathbb{X}_e := \langle X, \rho_e \rangle$, where

An application: a dichotomy for ultrahomogeneous str.

The **enlargement** of a binary structure $\mathbb{X} = \langle X, \rho \rangle$ is the structure $\mathbb{X}_e := \langle X, \rho_e \rangle$, where

$$x \rho_e y \Leftrightarrow x \rho y \vee (x \neq y \wedge \neg x \rho y \wedge \neg y \rho x)$$

An application: a dichotomy for ultrahomogeneous str.

The **enlargement** of a binary structure $\mathbb{X} = \langle X, \rho \rangle$ is the structure $\mathbb{X}_e := \langle X, \rho_e \rangle$, where

$$x \rho_e y \Leftrightarrow x \rho y \vee (x \neq y \wedge \neg x \rho y \wedge \neg y \rho x)$$

Theorem

(I) For each countable ultrahomogeneous reflexive or irreflexive binary structure $\mathbb{X}$ we have

An application: a dichotomy for ultrahomogeneous str.

The **enlargement** of a binary structure $\mathbb{X} = \langle X, \rho \rangle$ is the structure $\mathbb{X}_e := \langle X, \rho_e \rangle$, where

$$x \rho_e y \;\Leftrightarrow\; x \rho y \vee (x \neq y \wedge \neg x \rho y \wedge \neg y \rho x)$$

Theorem

(I) For each countable ultrahomogeneous reflexive or irreflexive binary structure $\mathbb{X}$ we have

- Either $\mathbb{X}$ is biconnected

An application: a dichotomy for ultrahomogeneous str.

The **enlargement** of a binary structure $\mathbb{X} = \langle X, \rho \rangle$ is the structure $\mathbb{X}_e := \langle X, \rho_e \rangle$, where

$$x \rho_e y \;\Leftrightarrow\; x \rho y \vee (x \neq y \wedge \neg x \rho y \wedge \neg y \rho x)$$

Theorem

(I) For each countable ultrahomogeneous reflexive or irreflexive binary structure $\mathbb{X}$ we have

- Either $\mathbb{X}$ is biconnected
- Or there is an ultrahomogeneous digraph $\mathbb{Y}$ (i.e. $\mathbb{Y} \in$ Cherlin's list) and a cardinal $2 \leq \kappa \leq \omega$ such that $\mathbb{X}$ is isomorphic to one of the following structures:

An application: a dichotomy for ultrahomogeneous str.

The **enlargement** of a binary structure $\mathbb{X} = \langle X, \rho \rangle$ is the structure $\mathbb{X}_e := \langle X, \rho_e \rangle$, where

$$x \rho_e y \Leftrightarrow x \rho y \vee (x \neq y \wedge \neg x \rho y \wedge \neg y \rho x)$$

Theorem

(I) For each countable ultrahomogeneous reflexive or irreflexive binary structure $\mathbb{X}$ we have

- Either $\mathbb{X}$ is biconnected
- Or there is an ultrahomogeneous digraph $\mathbb{Y}$ (i.e. $\mathbb{Y} \in$ Cherlin's list) and a cardinal $2 \leq \kappa \leq \omega$ such that $\mathbb{X}$ is isomorphic to one of the following structures:
  - $\bigcup_\kappa \mathbb{Y}_e$

An application: a dichotomy for ultrahomogeneous str.

The **enlargement** of a binary structure $\mathbb{X} = \langle X, \rho \rangle$ is the structure $\mathbb{X}_e := \langle X, \rho_e \rangle$, where

$$x \rho_e y \;\Leftrightarrow\; x \rho y \vee (x \neq y \wedge \neg x \rho y \wedge \neg y \rho x)$$

Theorem

(I) For each countable ultrahomogeneous reflexive or irreflexive binary structure $\mathbb{X}$ we have

- Either $\mathbb{X}$ is biconnected
- Or there is an ultrahomogeneous digraph $\mathbb{Y}$ (i.e. $\mathbb{Y} \in$ Cherlin's list) and a cardinal $2 \leq \kappa \leq \omega$ such that $\mathbb{X}$ is isomorphic to one of the following structures:
  - $\bigcup_\kappa \mathbb{Y}_e$
  - $(\bigcup_\kappa \mathbb{Y}_e)^c$

An application: a dichotomy for ultrahomogeneous str.

The **enlargement** of a binary structure $\mathbb{X} = \langle X, \rho\rangle$ is the structure $\mathbb{X}_e := \langle X, \rho_e\rangle$, where

$$x\rho_e y \iff x\rho y \vee (x \neq y \wedge \neg x\rho y \wedge \neg y\rho x)$$

Theorem

(I) For each countable ultrahomogeneous reflexive or irreflexive binary structure $\mathbb{X}$ we have

- Either $\mathbb{X}$ is biconnected
- Or there is an ultrahomogeneous digraph $\mathbb{Y}$ (i.e. $\mathbb{Y} \in$ Cherlin's list) and a cardinal $2 \leq \kappa \leq \omega$ such that $\mathbb{X}$ is isomorphic to one of the following structures:
  - $\bigcup_\kappa \mathbb{Y}_e$
  - $(\bigcup_\kappa \mathbb{Y}_e)^c$
  - $(\bigcup_\kappa \mathbb{Y}_e)_{re}$

An application: a dichotomy for ultrahomogeneous str.

The **enlargement** of a binary structure $\mathbb{X} = \langle X, \rho \rangle$ is the structure $\mathbb{X}_e := \langle X, \rho_e \rangle$, where

$$x \rho_e y \;\Leftrightarrow\; x\rho y \vee (x \neq y \wedge \neg x \rho y \wedge \neg y \rho x)$$

Theorem

(I) For each countable ultrahomogeneous reflexive or irreflexive binary structure $\mathbb{X}$ we have

- Either $\mathbb{X}$ is biconnected
- Or there is an ultrahomogeneous digraph $\mathbb{Y}$ (i.e. $\mathbb{Y} \in$ Cherlin's list) and a cardinal $2 \leq \kappa \leq \omega$ such that $\mathbb{X}$ is isomorphic to one of the following structures:
  - $\bigcup_\kappa \mathbb{Y}_e$
  - $(\bigcup_\kappa \mathbb{Y}_e)^c$
  - $(\bigcup_\kappa \mathbb{Y}_e)_{re}$
  - $((\bigcup_\kappa \mathbb{Y}_e)_{re})^c$.

An application: a dichotomy for ultrahomogeneous str.

The **enlargement** of a binary structure $\mathbb{X} = \langle X, \rho \rangle$ is the structure $\mathbb{X}_e := \langle X, \rho_e \rangle$, where

$$x \rho_e y \iff x\rho y \vee (x \neq y \wedge \neg x \rho y \wedge \neg y \rho x)$$

Theorem

(I) For each countable ultrahomogeneous reflexive or irreflexive binary structure $\mathbb{X}$ we have

- Either $\mathbb{X}$ is biconnected
- Or there is an ultrahomogeneous digraph $\mathbb{Y}$ (i.e. $\mathbb{Y} \in$ Cherlin's list) and a cardinal $2 \leq \kappa \leq \omega$ such that $\mathbb{X}$ is isomorphic to one of the following structures:
  - $\bigcup_\kappa \mathbb{Y}_e$
  - $(\bigcup_\kappa \mathbb{Y}_e)^c$
  - $(\bigcup_\kappa \mathbb{Y}_e)_{re}$
  - $((\bigcup_\kappa \mathbb{Y}_e)_{re})^c$.

(II) In the second case we have

An application: a dichotomy for ultrahomogeneous str.

The **enlargement** of a binary structure $\mathbb{X} = \langle X, \rho \rangle$ is the structure $\mathbb{X}_e := \langle X, \rho_e \rangle$, where

$$x \rho_e y \;\Leftrightarrow\; x \rho y \vee (x \neq y \wedge \neg x \rho y \wedge \neg y \rho x)$$

Theorem

(I) For each countable ultrahomogeneous reflexive or irreflexive binary structure $\mathbb{X}$ we have

- Either $\mathbb{X}$ is biconnected
- Or there is an ultrahomogeneous digraph $\mathbb{Y}$ (i.e. $\mathbb{Y} \in$ Cherlin's list) and a cardinal $2 \leq \kappa \leq \omega$ such that $\mathbb{X}$ is isomorphic to one of the following structures:
  - $\bigcup_\kappa \mathbb{Y}_e$
  - $(\bigcup_\kappa \mathbb{Y}_e)^c$
  - $(\bigcup_\kappa \mathbb{Y}_e)_{re}$
  - $((\bigcup_\kappa \mathbb{Y}_e)_{re})^c$.

(II) In the second case we have

- Either $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{Z})^n$, for some biconnected $\mathbb{Z} \in$ Cherlin's list and $n \geq 2$,

An application: a dichotomy for ultrahomogeneous str.

The **enlargement** of a binary structure $\mathbb{X} = \langle X, \rho \rangle$ is the structure $\mathbb{X}_e := \langle X, \rho_e \rangle$, where

$$x \rho_e y \Leftrightarrow x \rho y \vee (x \neq y \wedge \neg x \rho y \wedge \neg y \rho x)$$

Theorem

(I) For each countable ultrahomogeneous reflexive or irreflexive binary structure $\mathbb{X}$ we have

- Either $\mathbb{X}$ is biconnected
- Or there is an ultrahomogeneous digraph $\mathbb{Y}$ (i.e. $\mathbb{Y} \in$ Cherlin's list) and a cardinal $2 \leq \kappa \leq \omega$ such that $\mathbb{X}$ is isomorphic to one of the following structures:
  - $\bigcup_\kappa \mathbb{Y}_e$
  - $(\bigcup_\kappa \mathbb{Y}_e)^c$
  - $(\bigcup_\kappa \mathbb{Y}_e)_{re}$
  - $((\bigcup_\kappa \mathbb{Y}_e)_{re})^c$.

(II) In the second case we have

- Either $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{Z})^n$, for some biconnected $\mathbb{Z} \in$ Cherlin's list and $n \geq 2$,
- Or sq $\mathbb{P}(\mathbb{X})$ is an atomless σ-closed poset (Column D) and, hence, $\mathbb{P}(\mathbb{X}) \equiv (P(\omega)/\operatorname{Fin})^+$, under CH.

Proof

(I) Using Ramsey's theorem.

Proof

(I) Using Ramsey's theorem.
(II) If $\mathbb{Y} = \bigcup_k \mathbb{Z}$ then

Proof

(I) Using Ramsey's theorem.
(II) If $\mathbb{Y} = \bigcup_k \mathbb{Z}$ then

$((\bigcup_m(\bigcup_k \mathbb{Z})_e)_{re})^c$ and $\bigcup_{mk} \mathbb{Z}$ are quantifier free bi-interpretable and, hence,

Proof

(I) Using Ramsey's theorem.
(II) If $\mathbb{Y} = \bigcup_k \mathbb{Z}$ then

$((\bigcup_m (\bigcup_k \mathbb{Z})_e)_{re})^c$ and $\bigcup_{mk} \mathbb{Z}$ are quantifier free bi-interpretable and, hence,

$$\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\bigcup_{mk} \mathbb{Z}) \cong \mathbb{P}(\mathbb{Z})^{mk}.$$

B. Balcar, P. Vopěnka, On systems of almost disjoint sets, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 20 (1972) 421–424.

B. Balcar, P. Simon, Disjoint refinement, in: J. D. Monk and R. Bonnet (Eds.), Handbook of Boolean algebras, Vol. 2, 333–388, Elsevier Science Publishers B.V., Amsterdam, 1989.

M. S. Kurilić, From A_1 to D_5: Towards a forcing-related classification of relational structures, J. Symbolic Logic (to appear).

M. S. Kurilić, Maximally embeddable components, Arch. Math. Logic 52,7 (2013) 793-808.

M. S. Kurilić, Posets of copies of countable scattered linear orders, Ann. Pure Appl. Logic, 165 (2014) 895–912.

M. S. Kurilić, Forcing with copies of countable ordinals, Proc. Amer. Math. Soc. (to appear).

M. S. Kurilić, Isomorphic and strongly connected components, Arch. Math. Logic (to appear).

M. S. Kurilić, Embedding-minimal and embedding-maximal structures, in preparation.

M. S. Kurilić, Different Similarities, submitted.

M. S. Kurilić, S. Todorčević, Forcing by non-scattered sets, Ann. Pure Appl. Logic 163 (2012) 1299–1308.

M. S. Kurilić, S. Todorčević, Forcing by isomorphic substructures of the Rado graph, submitted.

R. Laver, On Fraïssé's order type conjecture, Ann. of Math. 93,2 (1971) 89-111.

S. Shelah, O. Spinas, The distributivity numbers of $P(\omega)/\mathrm{fin}$ and its square, Trans. Amer. Math. Soc. 352,5 (2000) 2023–2047.

A. Szymański, Zhou Hao Xua, The behaviour of ω^{2^*} under some consequences of Martin's axiom, General topology and its relations to modern analysis and algebra, V (Prague, 1981), 577-584, Sigma Ser. Pure Math., 3, Heldermann, Berlin, 1983.

P. Vopěnka, A. Pultr, Z. Hedrlín, A rigid relation exists on any set, Comment. Math. Univ. Carolinae 6 (1965) 149–155.