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On a system of hyperbolic partial fractional differential inclusions

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Abstract. We first investigate a coupled system of nonlinear hyperbolic partial fractional differential equations under some boundary value conditions. After that, we investigate a related coupled system of nonlinear hyperbolic partial fractional differential inclusions under some assumptions.

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1. Introduction

As it well known, it has been published many works on fractional delayed equations or time-fractional partial differential equations (see for example, [1, 2, 7, 9, 13, 17, 18]). It is interesting to work in two space variables or hyperbolic fractional partial differential equations (see for example, [3, 4, 5, 8]).

Let $\alpha = \alpha_1 + \alpha_2 \notin \mathbb{N}$ with $m-1 < \alpha_1 \leq m$ and $n-1 < \alpha_2 \leq n$. The Riemann-Liouville partial fractional order integral of a function $u \in L^1(J_a \times J_b := [0,a] \times [0,b])$ is defined by

$$(I_\theta^\alpha u)(x,y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1} u(s,t)\ dt\ ds$$

whenever the integral exists ([21, 22, 23]). Also, the Caputo partial derivative of fractional of order α for a function $u \in L^1(J_a \times J_b)$ is defined by

$$({}^cD_\theta^\alpha u)(x,y) = \frac{1}{\Gamma(m-\alpha_1)\Gamma(n-\alpha_2)} \int_0^x \int_0^y \frac{D_{st}^{m+n} u(s,t)}{(x-s)^{\alpha_1-m+1}(y-t)^{\alpha_2-n+1}}\ dt\ ds,$$

where $\theta = (0,0)$ denotes the lower limits of the integral and $D_{xy}^{m+n} := \dfrac{\partial^{m+n}}{\partial x^m \partial y^n}$ is the mixed partial derivative of order $m+n$ ([21, 22, 23]). It is known that $({}^cD_\theta^\alpha u)(x,y) = I_\theta^{m+n-\alpha}(D_{xy}^{m+n} u)(x,y)$ for all $(x,y) \in J_a \times J_b$ ([7]). In

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this paper, we first study the existence of solutions for the coupled system of nonlinear hyperbolic partial fractional differential equations

(1.1)
$$\begin{cases} ({}^cD_\theta^\alpha u)(x,y) \\ = f_1(x,y,u(x,y),v(x,y),D_x^2u(x,y),D_x^2v(x,y),D_y^2u(x,y),D_y^2v(x,y)) \\ ({}^cD_\theta^\beta v)(x,y) \\ = f_2(x,y,u(x,y),v(x,y),D_x^2u(x,y),D_x^2v(x,y),D_y^2u(x,y),D_y^2v(x,y)) \end{cases}$$

with the boundary value conditions

$$(1.2)\quad u(x,0)=\phi_1(x),\ u(0,y)=\psi_1(y),\ v(x,0)=\phi_2(x),\ v(0,y)=\psi_2(y),$$

where $\alpha=\alpha_1+\alpha_2$, $\beta=\beta_1+\beta_2$ with $2<\alpha_1,\beta_1\leq 3$, $2<\alpha_2,\beta_2\leq 3$, $\alpha\notin\mathbb{N}$, $\beta\notin\mathbb{N}$, $\theta=(0,0)$, $(x,y)\in J_a\times J_b$ with $a,b>0$, the mappings $f_1,f_2: J_a\times J_b\times\mathbb{R}^6\to\mathbb{R}$ are continuous, the real-valued functions $\phi_1,\phi_2:J_a\to\mathbb{R}$ and $\psi_1,\psi_2:J_b\to\mathbb{R}$ are absolutely continuous with $\phi_i(0)=\psi_i(0)$. Here, ${}^cD_\theta^\alpha$ denotes the Caputo fractional partial derivative of order α, $D_x^2=\dfrac{\partial^2}{\partial x^2}$ and $D_y^2=\dfrac{\partial^2}{\partial y^2}$. Also, we investigate its related coupled system of nonlinear hyperbolic partial fractional differential inclusions

(1.3)
$$\begin{cases} ({}^cD_\theta^{\alpha_1}u_1)(x,y)\in \\ F_1(x,y,u_1(x,y),u_2(x,y),D_x^2u_1(x,y),D_x^2u_2(x,y),D_y^2u_1(x,y),D_y^2u_2(x,y)) \\ ({}^cD_\theta^{\alpha_2}u_2)(x,y)\in \\ F_2(x,y,u_1(x,y),u_2(x,y),D_x^2u_1(x,y),D_x^2u_2(x,y),D_y^2u_1(x,y),D_y^2u_2(x,y)) \end{cases}$$

with the boundary value conditions

$$(1.4)\quad u_1(x,0)=\phi_1(x),\ u_1(0,y)=\psi_1(y),\ u_2(x,0)=\phi_2(x),\ u_2(0,y)=\psi_2(y)$$

under some conditions on the multifunctions F_1 and F_2.

Let (X,d) be a metric space. Denote by $\mathcal{P}(X)$, 2^X, $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{bd}(X)$, $\mathcal{P}_{cv}(X)$, $\mathcal{P}_{cp}(X)$, $\mathcal{P}_{cp,cv}(X)$, the class of all subsets, the set consisting of all nonempty subsets of X, the set consisting of all closed subsets of X, the set consisting of all bounded subsets of X, the set consisting of all convex subsets of X, the set consisting of all compact subsets of X and the set consisting of all compact convex subsets of X respectively. Let $F:X\to 2^X$ be a multifunction. We say that $u\in X$ is a fixed point of F whenever $u\in Fu$ ([14]). A multifunction $F: J_a\times J_b\to\mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable whenever $(x,y)\mapsto d(w,F(x,y))=\inf\{\|w-v\|:\ v\in F(x,y)\}$ is a measurable function for all $w\in\mathbb{R}$ ([12]). The Pompeiu-Hausdorff metric $H_d:2^X\times 2^X\to[0,\infty)$ is defined by $H_d(A,B)=\max\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(A,b)\}$, where $d(A,b)=\inf_{a\in A}d(a,b)$ ([6] and [10]). Then $(CB(X),H_d)$ is a metric space and $(CB(X),H_d)$ is a generalized metric space, where $CB(X)$is the set of closed and bounded subsets of X and $C(X)$ is the set of closed subsets of X ([6] and [10]). We say that a multifunction $F:X\to 2^X$ is convex valued (compact valued) whenever Fu is convex (compact) set for all $u \in X$ ([12]). A multifunction $F : X \to \mathcal{P}_{cl}(X)$ is called a contraction whenever there exists a constant $\gamma \in (0,1)$ such that $H_d(Fu, Fv) \leq \gamma d(u,v)$ for all $u, v \in X$ ([11]). In 1970, Covitz and Nadler proved that each closed-valued contractive multifunction on a complete metric space has a fixed point ([11]). We say that $F : J_a \times J_b \times \mathbb{R}^6 \to 2^{\mathbb{R}}$ is a Caratheodory multifunction whenever the map

$$(x,y) \mapsto F(x,y,u_1,u_2,u_3,u_4,u_5,u_6)$$

is measurable for all $u_1,u_2,u_3,u_4,u_5,u_6 \in \mathbb{R}$ and $(u_1,u_2,u_3,u_4,u_5,u_6) \mapsto F(x,y,u_1,u_2,u_3,u_4,u_5,u_6)$ is upper semi-continuous for almost all $(x,y) \in J_a \times J_b$ and all $u_1,u_2,u_3,u_4,u_5,u_6 \in \mathbb{R}$ ([12]). A Caratheodory multifunction $F : J_a \times J_b \times \mathbb{R}^6 \to 2^{\mathbb{R}}$ is called L^1-Caratheodory whenever for each $\rho > 0$ there exists a map $\phi_\rho \in L^1(J_a \times J_b, \mathbb{R}^+)$ such that

$$\|F(x,y,u_1,u_2,u_3,u_4,u_5,u_6)\| = \sup_{(x,y)\in J_a\times J_b} \{|s| : s \in F(x,y,u_1,u_2,u_3,u_4,u_5,u_6)\} \leq \phi_\rho(x,y)$$

for all $|u_i| \leq \rho$ $(i = 1,\dots,6)$ and for almost all $(x,y) \in J_a \times J_b$ ([12]). The set of selections of F is defined by

$$S_{F,u} := \{v \in L^1(J_a \times J_b, \mathbb{R}) : \ v(x,y) \in F(x,y,u(x,y), D_x^2 u(x,y), D_y^2 u(x,y) \text{ for almost all } (x,y) \in J_a \times J_b\}$$

for all $u \in C(J_a \times J_b, \mathbb{R})$. It has been proved that $S_{F,u} \neq \varnothing$ for all $u \in C(J_a \times J_b, X)$ whenever $\dim X < \infty$ ([19]). The graph of a multifunction F is defined by $Gr(F) = \{(x,y) \in X \times Y : \ y \in F(x)\}$ ([12]). We say that the graph of $F : X \to \mathcal{P}_{cl}(Y)$ is closed whenever for each sequence $\{u_n\}_{n\geq 1}$ in X and $\{y_n\}_{n\geq 1}$ in Y with $u_n \to u_0$, $y_n \to y_0$ and $y_n \in F(u_n)$ for all n, we have $y_0 \in F(u_0)$ ([12]). We shall use next theorems in our main results.

Theorem 1.1. *(Schaefer's fixed point theorem [23]) Suppose that X is a Banach space, $T : X \to X$ a completely continuous operator and the set $K = \{u \in X : \ u = \lambda Tu$ for some $\lambda \in [0,1]\}$ is bounded. Then T has a fixed point.*

Lemma 1.2. *([15]) If $F : X \to \mathcal{P}_{cl}(Y)$ is an upper semi-continuous multifunction, then $Gr(F)$ is a closed subset of $X \times Y$. If F is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 1.3. *[20] Let X be a separable Banach space, $F : [0,a] \times [0,b] \times X \times X \times X \to \mathcal{P}_{cp,cv}(X)$ an L^1-Caratheodory multifunction. Then the operator $\Theta \circ S_F : C(J_a \times J_b, X) \to \mathcal{P}_{cp,cv}(C(J_a \times J_b, X))$ defined by $u \mapsto (\Theta \circ S_F)(u) = \Theta(S_{F,u})$ is a closed graph operator, where Θ is a linear continuous mapping from $L^1(J_a \times J_b, X)$ into $C(J_a \times J_b, X)$.*

Theorem 1.4. *[16] Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,cv}(C)$ is an upper semi-continuous compact map, where $\mathcal{P}_{cp,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either F has a fixed point in $\overline{U}$ or there exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in \lambda F(u)$.*

2. Main results

Let $X = \{u : \ u, D_x^2 u, D_y^2 u \in C(J_a \times J_b, \mathbb{R})\}$. Define the norm

$$\|u\| = \|u\|_X := \sup_{(x,y)\in J_a\times J_b} |u(x,y)|$$
$$+ \sup_{(x,y)\in J_a\times J_b} |D_x^2 u(x,y)| + \sup_{(x,y)\in J_a\times J_b} |D_y^2 u(x,y)|.$$

It is clear that $(X, \|\cdot\| = \|\cdot\|_X)$ and $(X \times X, \|(u,v)\|_{X\times X})$ are Banach spaces, where $\|(u,v)\|_{X\times X} := \|u\|_X + \|v\|_X$.

Lemma 2.1. *Let $m-1 < \alpha_1 \le m$ and $n-1 < \alpha_2 \le n$ with $\alpha = \alpha_1 + \alpha_2 \notin \mathbb{N}$, $a > 0$, $b > 0$ and $g \in L^1(J_a \times J_b, X)$. Then $u_0 \in C(J_a \times J_b, X)$ is a solution for the fractional integral equation*

$$u(x,y) = \mu(x,y) + \int_0^x \int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} g(s,t)\ dt\ ds \tag{2.1}$$

if and only if u_0 is a solution for the hyperbolic partial fractional problem

$$({}^c D_\theta^\alpha u)(x,y) = g(x,y) \tag{2.2}$$

with the boundary conditions $u(x,0) = \phi(x)$ and $u(0,y) = \psi(y)$, where $\mu(x,y) = \phi(x) + \psi(y) - \phi(0)$ and $(x,y) \in J_a \times J_b$.

Proof. By using some calculations, it is easy to check that

$$u_0(x,y) = \phi(x) + \psi(y) - \phi(0) + \int_0^x \int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} g(s,t)\ dt\ ds$$

is a solution for the fractional integral equation (2.2). Let u_0 be a solution for hyperbolic partial fractional differential equation (2.2). Then,

$$I_\theta^{m+n-\alpha}(D_{xy}^{m+n} u_0)(x,y) = g(x,y)$$

for all $(x,y) \in J_a \times J_b$. Hence, $I_\theta^\alpha[I_\theta^{m+n-\alpha}(D_{xy}^{m+n} u_0)](x,y) = I_\theta^\alpha[g(x,y)]$ and so $I_\theta^{m+n}(D_{xy}^{m+n} u_0)(x,y) = I_\theta^\alpha[g(x,y)]$. Since

$$I_\theta^{m+n}(D_{xy}^{m+n} u_0)(x,y) = u_0(x,y) - u_0(x,0) - u_0(0,y) - u_0(0,0),$$

we get $u_0(x,y) - u_0(x,0) - u_0(0,y) - u_0(0,0) = I_\theta^\alpha[g(x,y)]$. Now by using the boundary value conditions, we obtain $u_0(x,y) - \phi(x) - \psi(y) - \phi(0) = I_\theta^\alpha[g(x,y)]$ and so

$$u_0(x,y) = \mu(x,y) + I_\theta^\alpha[g(x,y)]$$
$$= \mu(x,y) + \int_0^x \int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} g(s,t)\ dt\ ds.$$

This completes the proof. □

Now for the problem (1.1) with boundary conditions (1.2), define the operators $T_1, T_2 : X \to X$ by

$$(T_1 v)(x,y) = \mu_1(x,y) + \int_0^x \int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$$
$$\times f_1(s,t,u(s,t),v(s,t),D_x^2 u(s,t),D_x^2 v(s,t),D_y^2 u(s,t),D_y^2 v(s,t))\,dt\,ds$$

and

$$(T_2 u)(x,y) = \mu_2(x,y) + \int_0^x \int_0^y \frac{(x-s)^{\beta_1-1}(y-t)^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}$$
$$\times f_2(s,t,u(s,t),v(s,t),D_x^2 u(s,t),D_x^2 v(s,t),D_y^2 u(s,t),D_y^2 v(s,t))\,dt\,ds,$$

where $\mu_1(x,y) = \phi_1(x) + \psi_1(y) - \phi_1(0)$, $\mu_2(x,y) = \phi_2(x) + \psi_2(y) - \phi_2(0)$ and $(x,y) \in J_a \times J_b$. Now, put $N_1 = \frac{a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}$, $N_2 = \frac{a^{\alpha_1-2} b^{\alpha_2}}{\Gamma(\alpha_1-1)\Gamma(\alpha_2+1)}$, $N_3 = \frac{a^{\alpha_1} b^{\alpha_2-2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-1)}$, $N_4 = \frac{a^{\beta_1} b^{\beta_2}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}$, $N_5 = \frac{a^{\beta_1-2} b^{\beta_2}}{\Gamma(\beta_1-1)\Gamma(\beta_2+1)}$, $N_6 = \frac{a^{\beta_1} b^{\beta_2-2}}{\Gamma(\beta_1+1)\Gamma(\beta_2-1)}$, $M_1 = \mu_1(a,b) + \phi_1''(a) + \psi_1''(b)$ and $M_2 = \mu_2(a,b) + \phi_2''(a) + \psi_2''(b)$.

Theorem 2.2. *Consider the system of nonlinear hyperbolic partial fractional differential equations (1.1) with boundary conditions (1.2). Suppose that* $f_1, f_2 : J_a \times J_b \times X^6 \to X$ *be continuous mappings and there exist positive constants* L_1 *and* L_2 *such that* $|f_1(x,y,u_1,u_2,\cdots,u_6)| \le L_1$ *and* $|f_2(x,y,u_1,u_2,\cdots,u_6)| \le L_2$ *for all* $(x,y) \in J_a \times J_b$ *and* $u_1,\cdots,u_6 \in X$*. Then the problem (1.1) with boundary conditions (1.2) has a solution.*

Proof. Consider the continuous operator $T : X \times X \to X \times X$ defined by $T(u,v)(x,y) := ((T_1 v)(x,y), (T_2 u)(x,y))$ for all $(x,y) \in J_a \times J_b$. First, we show that T maps bounded sets to bounded subsets of $X \times X$. Let Ω be a bounded subset of $X \times X$, $(u,v) \in \Omega$ and $(x,y) \in J_a \times J_b$. Then, we have

$$|(T_1 v)(x,y)| = \Big|\mu_1(x,y) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}$$
$$f_1(s,t,u(s,t),v(s,t),D_x^2 u(s,t),D_x^2 v(s,t),D_y^2 u(s,t),D_y^2 v(s,t))\,dt\,ds\Big|$$
$$\le |\mu_1(x,y)| + \Big|\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}$$
$$f_1(s,t,u(s,t),v(s,t),D_x^2 u(s,t),D_x^2 v(s,t),D_y^2 u(s,t),D_y^2 v(s,t))\,dt\,ds\Big|$$
$$\le |\mu_1(x,y)| + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}$$
$$|f_1(s,t,u(s,t),v(s,t),D_x^2 u(s,t),D_x^2 v(s,t),D_y^2 u(s,t),D_y^2 v(s,t))|\,dt\,ds$$
$$\le |\mu_1(x,y)| + L_1\Big\{\frac{x^{\alpha_1} y^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Big\}$$
$$\le \mu_1(a,b) + L_1\Big\{\frac{a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Big\} = \mu_1(a,b) + L_1 N_1,$$

$$|D_x^2(T_1v)(x,y)| = \Big|D_x^2\mu_1(x,y) + \frac{1}{\Gamma(\alpha_1-2)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-3}(y-t)^{\alpha_2-1}$$
$$f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))\ dt\ ds\Big|$$
$$\leq |D_x^2\mu_1(x,y)| + \Big|\frac{1}{\Gamma(\alpha_1-2)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-3}(y-t)^{\alpha_2-1}$$
$$f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))\ dt\ ds\Big|$$
$$\leq |\phi_1''(x)| + \frac{1}{\Gamma(\alpha_1-2)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-3}(y-t)^{\alpha_2-1}$$
$$|f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))|\ dt\ ds$$
$$\leq |\phi_1''(x)| + L_1\Big\{\frac{x^{\alpha_1-2}y^{\alpha_2}}{\Gamma(\alpha_1-1)\Gamma(\alpha_2+1)}\Big\}$$
$$\leq \phi_1''(a) + L_1\Big\{\frac{a^{\alpha_1-2}b^{\alpha_2}}{\Gamma(\alpha_1-1)\Gamma(\alpha_2+1)}\Big\} = \phi_1''(a) + L_1N_2$$

and

$$|D_y^2(T_1v)(x,y)| = \Big|D_y^2\mu_1(x,y) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2-2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-3}$$
$$f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))\ dt\ ds\Big|$$
$$\leq |D_y^2\mu_1(x,y)| + \Big|\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2-2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-3}$$
$$f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))\ dt\ ds\Big|$$
$$\leq |\psi_1''(y)| + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2-2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-3}$$
$$|f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))|\ dt\ ds$$
$$\leq |\psi_1''(y)| + L_1\Big\{\frac{x^{\alpha_1}y^{\alpha_2-2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-1)}\Big\}$$
$$\leq \psi_1''(b) + L_1\Big\{\frac{a^{\alpha_1}b^{\alpha_2-2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-1)}\Big\} = \psi_1''(b) + L_1N_3.$$

Hence, $\|(T_1v)(x,y)\|_X \leq L_1(N_1+L_2+L_3)+M_1$. On the other hand, we have

$$|(T_2u)(x,y)| \leq \mu_2(a,b) + L_2N_4,\ |D_x^2(T_2u)(x,y)|$$
$$\leq \phi_2''(a) + L_2N_5,\ |D_y^2(T_2u)(x,y)| \leq \psi_2'' + L_2N_6$$

and so $\|(T_2u)(x,y)\|_X \leq L_2(N_4+L_5+L_6)+M_2$. This implies that

$$\|T(u,v)(x,y)\|_{X\times X} \leq L_1(N_1+L_2+L_3)+L_2(N_4+L_5+L_6)+M_1+M_2.$$

This shows that T maps bounded sets to bounded subsets of $X \times X$. Now, we prove that T is equicontinuous. Let $(x_1, y_1), (x_2, y_2) \in J_a \times J_b$ with $x_1 < x_2$ and $y_1 < y_2$. Then, we get

$$|(T_1 v)(x_2, y_2) - (T_1 v)(x_1, y_1)|$$

$$= \Big|\mu_1(x_2, y_2) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_2} \int_0^{y_2} (x_2 - s)^{\alpha_1 - 1} (y_2 - t)^{\alpha_2 - 1}$$

$$f_1(s, t, u(s,t), v(s,t), D_x^2 u(s,t), D_x^2 v(s,t), D_y^2 u(s,t), D_y^2 v(s,t))\, dt\, ds$$

$$-\mu_1(x_1, y_1) - \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} \int_0^{y_1} (x_1 - s)^{\alpha_1 - 1} (y_1 - t)^{\alpha_2 - 1}$$

$$f_1(s, t, u(s,t), v(s,t), D_x^2 u(s,t), D_x^2 v(s,t), D_y^2 u(s,t), D_y^2 v(s,t))\, dt\, ds\Big|$$

$$\leq |\mu_1(x_2, y_2) - \mu_1(x_1, y_1)| + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2 - s)^{\alpha_1 - 1} (y_2 - t)^{\alpha_2 - 1}$$

$$|f_1(s, t, u(s,t), v(s,t), D_x^2 u(s,t), D_x^2 v(s,t), D_y^2 u(s,t), D_y^2 v(s,t))|\, dt\, ds$$

$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} \int_0^{y_1} [(x_1 - s)^{\alpha_1 - 1} (y_1 - t)^{\alpha_2 - 1} - (x_2 - s)^{\alpha_1 - 1} (y_2 - t)^{\alpha_2 - 1}]$$

$$|f_1(s, t, u(s,t), v(s,t), D_x^2 u(s,t), D_x^2 v(s,t), D_y^2 u(s,t), D_y^2 v(s,t))|\, dt\, ds$$

$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} \int_{y_1}^{y_2} (x_2 - s)^{\alpha_1 - 1} (y_2 - t)^{\alpha_2 - 1}$$

$$|f_1(s, t, u(s,t), v(s,t), D_x^2 u(s,t), D_x^2 v(s,t), D_y^2 u(s,t), D_y^2 v(s,t))|\, dt\, ds$$

$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2 - s)^{\alpha_1 - 1} (y_2 - t)^{\alpha_2 - 1}$$

$$|f_1(s, t, u(s,t), v(s,t), D_x^2 u(s,t), D_x^2 v(s,t), D_y^2 u(s,t), D_y^2 v(s,t))|\, dt\, ds$$

$$\leq \|\mu_1(x_2, y_2) - \mu_1(x_1, y_1)\| + \frac{L_1}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)}$$

$$\Big\{2y_2^{\alpha_2}(x_2 - x_1)^{\alpha_1} + 2x_2^{\alpha_1}(y_2 - y_1)^{\alpha_2} + x_2^{\alpha_1} y_2^{\alpha_2} - x_1^{\alpha_1} y_1^{\alpha_2} - 2(x_2 - x_1)^{\alpha_1}(y_2 - y_1)^{\alpha_2}\Big\}$$

and so $|(T_1 v)(x_2, y_2) - (T_1 v)(x_1, y_1)| \to 0$ as $(x_2, y_2) \to (x_1, y_1)$. Also, we have

$$|D_x^2(T_1 v)(x_2, y_2) - D_x^2(T_1 v)(x_1, y_1)|$$

$$= \Big|D_x^2 \mu_1(x_2, y_2) + \frac{1}{\Gamma(\alpha_1 - 2)\Gamma(\alpha_2)} \int_0^{x_2} \int_0^{y_2} (x_2 - s)^{\alpha_1 - 3} (y_2 - t)^{\alpha_2 - 1}$$

$$f_1(s, t, u(s,t), v(s,t), D_x^2 u(s,t), D_x^2 v(s,t), D_y^2 u(s,t), D_y^2 v(s,t))\, dt\, ds$$

$$-D_x^2 \mu_1(x_1, y_1) - \frac{1}{\Gamma(\alpha_1 - 2)\Gamma(\alpha_2)} \int_0^{x_1} \int_0^{y_1} (x_1 - s)^{\alpha_1 - 3} (y_1 - t)^{\alpha_2 - 1}$$

$$f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))\ dt\ ds\Big|$$

$$\leq |\phi_1''(x_2)-\phi_1''(x_1)| + \frac{1}{\Gamma(\alpha_1-2)\Gamma(\alpha_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{\alpha_1-3}(y_2-t)^{\alpha_2-1}$$

$$|f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))|\ dt\ ds$$

$$+\frac{1}{\Gamma(\alpha_1-2)\Gamma(\alpha_2)}\int_0^{x_1}\int_0^{y_1}[(x_1-s)^{\alpha_1-3}(y_1-t)^{\alpha_2-1}-(x_2-s)^{\alpha_1-3}(y_2-t)^{\alpha_2-1}]$$

$$|f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))|\ dt\ ds$$

$$+\frac{1}{\Gamma(\alpha_1-2)\Gamma(\alpha_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{\alpha_1-3}(y_2-t)^{\alpha_2-1}$$

$$|f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))|\ dt\ ds$$

$$+\frac{1}{\Gamma(\alpha_1-2)\Gamma(\alpha_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{\alpha_1-3}(y_2-t)^{\alpha_2-1}$$

$$|f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))|\ dt\ ds$$

$$\leq \|\phi_1''(x_2)-\phi_1''(x_1)\| + \frac{L_1}{\Gamma(\alpha_1-1)\Gamma(\alpha_2+1)}$$

$$\Big\{2y_2^{\alpha_2}(x_2-x_1)^{\alpha_1-2}+2x_2^{\alpha_1-2}(y_2-y_1)^{\alpha_2}+x_2^{\alpha_1-2}y_2^{\alpha_2}-x_1^{\alpha_1-2}y_1^{\alpha_2}$$

$$-2(x_2-x_1)^{\alpha_1-2}(y_2-y_1)^{\alpha_2}\Big\}$$

and so $|D_x^2(T_1v)(x_2,y_2)-D_x^2(T_1v)(x_1,y_1)|\to 0$ as $(x_2,y_2)\to(x_1,y_1)$. Finally,

$$|D_y^2(T_1v)(x_2,y_2)-D_y^2(T_1v)(x_1,y_1)|$$

$$=\Big|D_y^2\mu_1(x_2,y_2)+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2-2)}\int_0^{x_2}\int_0^{y_2}(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-3}$$

$$f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))\ dt\ ds$$

$$-D_y^2\mu_1(x_1,y_1)-\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2-2)}\int_0^{x_1}\int_0^{y_1}(x_1-s)^{\alpha_1-1}(y_1-t)^{\alpha_2-3}$$

$$f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))\ dt\ ds\Big|$$

$$\leq |\psi_1''(y_2)-\psi_1''(y_1)| + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2-2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-3}$$

$$|f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))|\ dt\ ds$$

$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2-2)}\int_0^{x_1}\int_0^{y_1}[(x_1-s)^{\alpha_1-1}(y_1-t)^{\alpha_2-3}-(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-3}]$$

$$|f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))|\ dt\ ds$$

$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2-2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-3}$$

$$|f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))|\,dt\,ds$$

$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2-2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-3}$$

$$|f_1(s,t,u(s,t),v(s,t),D_x^2u(s,t),D_x^2v(s,t),D_y^2u(s,t),D_y^2v(s,t))|\,dt\,ds$$

$$\leq\|\psi_1''(y_2)-\psi_1''(y_1)\|+\frac{L_1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-1)}$$

$$\Big\{2y_2^{\alpha_2-2}(x_2-x_1)^{\alpha_1}+2x_2^{\alpha_1}(y_2-y_1)^{\alpha_2-2}+x_2^{\alpha_1}y_2^{\alpha_2-2}-x_1^{\alpha_1}y_1^{\alpha_2-2}$$

$$-2(x_2-x_1)^{\alpha_1}(y_2-y_1)^{\alpha_2-2}\Big\}$$

and so $|D_y^2(T_1v)(x_2,y_2)-D_y^2(T_1v)(x_1,y_1)|\to 0$ as $(x_2,y_2)\to(x_1,y_1)$. Thus,

$$\|(T_1v)(x_2,y_2)-(T_1v)(x_1,y_1)\|$$

$$\leq\|\mu_1(x_2,y_2)-\mu_1(x_1,y_1)\|+\|\phi_1''(x_2)-\phi_1''(x_1)\|+\|\psi_1''(y_2)-\psi_1''(y_1)\|$$

$$+\frac{L_1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Big\{2y_2^{\alpha_2}(x_2-x_1)^{\alpha_1}+2x_2^{\alpha_1}(y_2-y_1)^{\alpha_2}+x_2^{\alpha_1}y_2^{\alpha_2}-x_1^{\alpha_1}y_1^{\alpha_2}$$

$$-2(x_2-x_1)^{\alpha_1}(y_2-y_1)^{\alpha_2}\Big\}$$

$$+\frac{L_1}{\Gamma(\alpha_1-1)\Gamma(\alpha_2+1)}\Big\{2y_2^{\alpha_2}(x_2-x_1)^{\alpha_1-2}+2x_2^{\alpha_1-2}(y_2-y_1)^{\alpha_2}+x_2^{\alpha_1-2}y_2^{\alpha_2}$$

$$-x_1^{\alpha_1-2}y_1^{\alpha_2}-2(x_2-x_1)^{\alpha_1-2}(y_2-y_1)^{\alpha_2}\Big\}$$

$$+\frac{L_1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-1)}\Big\{2y_2^{\alpha_2-2}(x_2-x_1)^{\alpha_1}+2x_2^{\alpha_1}(y_2-y_1)^{\alpha_2-2}+x_2^{\alpha_1}y_2^{\alpha_2-2}$$

$$-x_1^{\alpha_1}y_1^{\alpha_2-2}-2(x_2-x_1)^{\alpha_1}(y_2-y_1)^{\alpha_2-2}\Big\}.$$

By using the Arzela-Ascoli theorem, the operator T_1 is completely continuous. Similarly, we obtain

$$\|(T_2u)(x_2,y_2)-(T_2u)(x_1,y_1)\|$$

$$\leq\|\mu_2(x_2,y_2)-\mu_2(x_1,y_1)\|+\|\phi_2''(x_2)-\phi_2''(x_1)\|+\|\psi_2''(y_2)-\psi_2''(y_1)\|$$

$$+\frac{L_2}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}\Big\{2y_2^{\beta_2}(x_2-x_1)^{\beta_1}+2x_2^{\beta_1}(y_2-y_1)^{\beta_2}+x_2^{\beta_1}y_2^{\beta_2}-x_1^{\beta_1}y_1^{\beta_2}$$

$$-2(x_2-x_1)^{\beta_1}(y_2-y_1)^{\beta_2}\Big\}$$

$$+\frac{L_2}{\Gamma(\beta_1-1)\Gamma(\beta_2+1)}\Big\{2y_2^{\beta_2}(x_2-x_1)^{\beta_1-2}+2x_2^{\beta_1-2}(y_2-y_1)^{\beta_2}+x_2^{\beta_1-2}y_2^{\beta_2}$$

$$-x_1^{\beta_1-2}y_1^{\beta_2}-2(x_2-x_1)^{\beta_1-2}(y_2-y_1)^{\beta_2}\Big\}$$

$$+\frac{L_2}{\Gamma(\beta_1+1)\Gamma(\beta_2-1)}\Big\{2y_2^{\beta_2-2}(x_2-x_1)^{\beta_1}+2x_2^{\beta_1}(y_2-y_1)^{\beta_2-2}+x_2^{\beta_1}y_2^{\beta_2-2}$$

$$-x_1^{\beta_1}y_1^{\beta_2-2}-2(x_2-x_1)^{\beta_1}(y_2-y_1)^{\beta_2-2}\Big\}$$

and so by the Arzela-Ascoli theorem, we get the operator T_2 is completely continuous. Hence, $\|T(u,v)(x_2,y_2)-T(u,v)(x_2,y_2)\|_{X\times X}\to 0$ as (x_2,y_2) tends to (x_1,y_1). This shows that the operator T is completely continuous. Now, we show that $\Omega=\{(u,v)\in X\times X:\ (u,v)=\lambda T(u,v)$ for some $\lambda\in[0,1]\}$ is a bounded set. Let $(u,v)\in\Omega$. Choose $\lambda\in[0,1]$ such that $(u,v)=\lambda T(u,v)$. Then, $v(x,y)=\lambda(T_1v)(x,y)$ and $u(x,y)=\lambda(T_2u)(x,y)$ for all $(x,y)\in J_a\times J_b$. Since $\frac{1}{\lambda}|v(x,y)|=|(T_1v)(x,y)|\le\mu_1(a,b)+L_1N_1$,

$$\frac{1}{\lambda}|D_x^2v(x,y)|=|D_x^2(T_1v)(x,y)|\le\phi_1''(a)+L_1N_2$$

and $\frac{1}{\lambda}|D_y^2v(x,y)|=|D_y^2(T_1v)(x,y)|\le\psi_1''(b)+L_1N_3$, we get

$$|v(x,y)|\le\lambda\mu_1(a,b)+\lambda L_1N_1,$$

$|D_x^2v(x,y)|\le\lambda\phi_1''(a)+\lambda L_1N_2$ and $|D_y^2v(x,y)|\le\lambda\psi_1''(b)+\lambda L_1N_3$. This implies that $\|v(x,y)\|_X\le\lambda[M_1+L_1(N_1+N_2+N_3)]$. Similarly, we obtain

$$\|u(x,y)\|_X\le\lambda[M_2+L_2(N_4+N_5+N_6)].$$

Thus, $\|(u,v)\|_{X\times X}\le\lambda[M_1+L_1(N_1+N_2+N_3)]+\lambda[M_2+L_2(N_4+N_5+N_6)]$ and so Ω is a bounded set. Now by using the Schaefer's fixed point theorem, the operator T has a fixed point which is a solution for the system of hyperbolic partial fractional differential equations (1.1) with boundary conditions (1.2). □

Note that, one can extend the problem (1.1) with boundary conditions (1.2) to a n-dimensional system of nonlinear hyperbolic partial fractional differential equations as following. Suppose that $\alpha_i=\alpha_{i1}+\alpha_{i2}\notin\mathbb{N}$ with $2<\alpha_{i1},\alpha_{i2}\le 3$ for $i=1,2,\dots,n$, $(x,y)\in J_a\times J_b:=[0,a]\times[0,b]$ with $a,b>0$ and the functions $f_i:J_a\times J_b\times\mathbb{R}^{3n}\to\mathbb{R}$ are continuous for $i=1,2,\dots,n$. Consider the $n-$dimensional system of nonlinear hyperbolic partial fractional differential equations

$$\begin{cases}({}^cD_\theta^{\alpha_1}u_1)(x,y)=f_1(x,y,u_1(x,y),\cdots,u_n(x,y),D_x^2u_1(x,y),\cdots,D_x^2u_n(x,y),\\ D_y^2u_1(x,y),\cdots,D_y^2u_n(x,y))\\ ({}^cD_\theta^{\alpha_2}u_2)(x,y)=f_2(x,y,u_1(x,y),\cdots,u_n(x,y),D_x^2u_1(x,y),\cdots,D_x^2u_n(x,y),\\ D_y^2u_1(x,y),\cdots,D_y^2u_n(x,y))\\ .\quad(2.3)\\ .\\ .\\ ({}^cD_\theta^{\alpha_n}u_n)(x,y)=f_n(x,y,u_1(x,y),\cdots,u_n(x,y),D_x^2u_1(x,y),\cdots,D_x^2u_n(x,y),\\ D_y^2u_1(x,y),\cdots,D_y^2u_n(x,y))\end{cases}$$

with the boundary value conditions

$$u_i(x,0) = \phi_i(x),\ u_i(0,y) = \psi_i(y),\ (i = 1,2,\dots,n) \tag{2.4}$$

where $\phi_i : J_a \to \mathbb{R}$ and $\psi_i : J_b \to \mathbb{R}$ are absolutely continuous functions with $\phi_i(0) = \psi_i(0)$. For record, we state the related result.

Theorem 2.3. *Suppose that the functions* $f_1, \dots, f_{3n} : J_a \times J_b \times X^{3n} \to X$ *are continuous mappings and there exist positive constants* $L_1, L_2, \dots, L_n$ *such that* $|f_i(x,y,u_1,u_2,\cdots,u_{3n})| \le L_i$ *for* $i = 1,\dots,3n$*. Then, the* n*-dimensional system of nonlinear hyperbolic partial fractional differential equations (2.3) with boundary conditions (2.4) has a solution.*

Now, we investigate the coupled system of inclusions (1.3) with boundary conditions (1.4).

Definition 2.4. We say that $(u_1,u_2) \in C(J_a \times J_b, X) \times C(J_a \times J_b, X)$ is a solution for the system of hyperbolic partial fractional differential inclusions (1.3) with boundary conditions (1.4) whenever it satisfies (1.4) and there exists a function $(w_1,w_2) \in L^1(J_a \times J_b) \times L^1(J_a \times J_b)$ such that

$$w_1(x,y) \in$$
$$F_1(x,y,u_1(x,y),u_2(x,y),D_x^2u_1(x,y),D_x^2u_2(x,y),D_y^2u_1(x,y),D_y^2u_2(x,y))$$

and

$$w_2(x,y) \in$$
$$F_2(x,y,u_1(x,y),u_2(x,y),D_x^2u_1(x,y),D_x^2u_2(x,y),D_y^2u_1(x,y),D_y^2u_2(x,y))$$

for almost all $(x,y) \in J_a \times J_b$ and

$$u_i(x,y) = \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})}\int_0^x\int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1}w_1(s,t)\,dt\,ds + \mu_i(x,y)$$

for all $(x,y) \in J_a \times J_b$ and $i = 1,2$, where $\mu_i(x,y) = \phi_i(x) + \psi_i(y) - \phi_i(0)$.

Theorem 2.5. *Suppose that multifunctions* $F_1, F_2 : J_a \times J_b \times \mathbb{R}^6 \to \mathcal{P}_{cp,cv}(\mathbb{R})$ *are* L^1*-Caratheodory multifunctions and there exist a bounded continuous non-decreasing map* $\psi : [0,\infty) \to (0,\infty)$ *and a continuous function* $p : J_a \times J_b \to (0,\infty)$ *such that* $\|F_i(x,y,u_i(x,y),D_x^2u_i(x,y),D_y^2u_i(x,y))\| \le p(x,y)\psi(\|u_i\|)$ *for all* $(x,y) \in J_a \times J_b$ *and* $u_i \in X$ $(i = 1,2)$ *. Then, the coupled system of non-linear hyperbolic partial fractional differential inclusions (1.3) with boundary conditions (1.4) has at least one solution.*

Proof. Define the operator $N : X\times X \to 2^{X\times X}$ by $N(u_1,u_2) = \begin{pmatrix} N_1(u_1,u_2) \\ N_2(u_1,u_2) \end{pmatrix}$, where $N_1(u_1,u_2) = \{h_1 \in X \times X :$ there exists $v_1 \in S_{F_1,u_1}$ such that $h_1(x,y) = v_1(x,y)$ for all $(x,y) \in J_a \times J_b\}$, $N_2(u_1,u_2) = \{h_2 \in X \times X :$ there exists $v_2 \in S_{F_2,u_2}$ such that $h_2(x,y) = v_2(x,y)$ for all $(x,y) \in J_a \times J_b\}$,

$$h_1(x,y) = \frac{1}{\Gamma(\alpha_{11})\Gamma(\alpha_{22})}\int_0^x\int_0^y (x-s)^{\alpha_{11}-1}(y-t)^{\alpha_{22}-1}v_1(s,t)\,dt\,ds + \mu_1(x,y),$$

and $h_2(x,y) = \frac{1}{\Gamma(\beta_{11})\Gamma(\beta_{22})}\int_0^x\int_0^y (x-s)^{\beta_{11}-1}(y-t)^{\beta_{22}-1}v_2(s,t)\,dt\,ds + \mu_2(x,y)$. By using Lemma 2.1, it is easy to check that each fixed point of the operator N is a solution for the system of hyperbolic partial fractional differential inclusions (1.3). First, we show that the multifunction N has convex values. Let $(u_1,u_2) \in X \times X$ and $(h_1,h_2),(h_1',h_2') \in N(u_1,u_2)$. Choose $v_i, v_i' \in S_{F_i,(u_1,u_2)}$ such that

$$h_i(x,y) = \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})}\int_0^x\int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1}v_i(s,t)\,dt\,ds + \mu_i(x,y)$$

and $h_i'(x,y) = \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})}\int_0^x\int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1}v_i'(s,t)\,dt\,ds + \mu_i'(x,y)$ for almost all $(x,y) \in J_a \times J_b$ and $i = 1,2$. If $0 \le \lambda \le 1$, then we have

$$[\lambda h_i + (1-\lambda)h_i'](x,y) = \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})}$$

$$\times \int_0^x\int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1}[\lambda v_i(s,t)+(1-\lambda)v_i'(s,t)]\,dt\,ds+\mu_i(x,y)+\mu_i'(x,y).$$

Since F_i has convex values, S_{F_i,u_i} is convex and so $[\lambda h_i+(1-\lambda)h_i'] \in N(u_1,u_2)$. Thus, the operator N has convex values. Now, we prove that N maps bounded sets of X into bounded subsets. Let $r > 0$ and $B_r = \{(u_1,u_2) \in X \times X : \|(u_1,u_2)\| \le r\}$ be a bounded subset of $X \times X$. Suppose that $(h_1,h_2) \in N(u_1,u_2)$ and $(u_1,u_2) \in B_r$. Choose $(v_1,v_2) \in S_{F_1,u_1} \times S_{F_2,u_2}$ such that

$$h_i(x,y) = \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})}\int_0^x\int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1}v_i(s,t)\,dt\,ds + \mu_i(x,y)$$

for almost all $(x,y) \in J_a \times J_b$ and $i = 1,2$. If $\|p\|_\infty = \sup_{(x,y)\in J_a\times J_b}|p(x,y)|$, then

$$|h_i(x,y)| = \Big|\frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})}\int_0^x\int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1}v_i(s,t)\,dt\,ds+\mu_i(x,y)\Big|$$

$$\le \Big|\frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})}\int_0^x\int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1}v_i(s,t)\,dt\,ds\Big| + \Big|\mu_i(x,y)\Big|$$

$$\le \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})}\int_0^x\int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1}|v_i(s,t)|\,dt\,ds + |\mu_i(x,y)|$$

$$\le \frac{x^{\alpha_{i1}}y^{\alpha_{i2}}}{\Gamma(\alpha_{i1}+1)\Gamma(\alpha_{i2}+1)}p(x,y)\psi(\|u_i\|) + |\mu_i(x,y)|$$

$$\le \frac{a^{\alpha_{i1}}b^{\alpha_{i2}}}{\Gamma(\alpha_{i1}+1)\Gamma(\alpha_{i2}+1)}\|p\|_\infty\psi(\|u_i\|) + \mu_i(a,b)$$

$$= \Lambda_{i1}\|p\|_\infty\psi(\|u_i\|) + \mu_i(a,b),$$

$$|D_x^2 h_i(x,y)|$$

$$= \left| \frac{1}{\Gamma(\alpha_{i1}-2)\Gamma(\alpha_{i2})} \int_0^x \int_0^y (x-s)^{\alpha_{i1}-3}(y-t)^{\alpha_{i2}-1} v_i(s,t)\, dt\, ds + D_x^2 \mu_i(x,y) \right|$$
$$\leq \frac{a^{\alpha_{i1}-2} b^{\alpha_{i2}}}{\Gamma(\alpha_{i1}-1)\Gamma(\alpha_{i2}+1)} \|p\|_\infty \psi(\|u_i\|) + \phi_i''(a)$$
$$= \Lambda_{i2} \|p\|_\infty \psi(\|u_i\|) + \phi_i''(a)$$

and

$$|D_y^2 h_i(x,y)|$$
$$= \left| \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2}-2)} \int_0^x \int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-3} v_i(s,t)\, dt\, ds + D_y^2 \mu_i(x,y) \right|$$
$$\leq \frac{a^{\alpha_{i1}} b^{\alpha_{i2}-2}}{\Gamma(\alpha_{i1}+1)\Gamma(\alpha_{i2}-1)} \|p\|_\infty \psi(\|u_i\|) + \psi_i''(b)$$
$$= \Lambda_{i3} \|p\|_\infty \psi(\|u_i\|) + \psi_i''(b),$$

where the constants Λ_{i1}, Λ_{i2} and Λ_{i3} are defined by $\Lambda_{i1} = \frac{a^{\alpha_{i1}} b^{\alpha_{i2}}}{\Gamma(\alpha_{i1}+1)\Gamma(\alpha_{i2}+1)}$, $\Lambda_{i2} = \frac{a^{\alpha_{i1}-2} b^{\alpha_{i2}}}{\Gamma(\alpha_{i1}-1)\Gamma(\alpha_{i2}+1)}$ and $\Lambda_{i3} = \frac{a^{\alpha_{i1}} b^{\alpha_{i2}-2}}{\Gamma(\alpha_{i1}+1)\Gamma(\alpha_{i2}-1)}$. Thus,

$$\|h_i\| \leq (\Lambda_{i1} + \Lambda_{i2} + \Lambda_{i3}) \|p\|_\infty \psi(\|u_i\|) + M_i$$

for $i = 1, 2$, where $M_i = \mu_i(a,b) + \phi_i''(a) + \psi_i''(b)$. This implies that

$$\|(h_1, h_2)\| \leq \|p\|_\infty \psi(\|(u_1, u_2)\| \sum_{i=1}^{2} (\Lambda_{i1} + \Lambda_{i2} + \Lambda_{i3}) + \sum_{i=1}^{2} M_i.$$

Now, we show that N is completely continuous. Suppose that $(u_1, u_2) \in B_r$ and $(x_1, y_1), (x_2, y_2) \in J_a \times J_b$ with $x_1 < x_2$ and $y_1 < y_2$. Then, we obtain

$$|h_i(x_2, y_2) - h_i(x_1, y_1)|$$
$$\leq \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2-s)^{\alpha_{i1}-1}(y_2-t)^{\alpha_{i2}-1} |v_i(s,t)|\, dt\, ds$$
$$+ \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})}$$
$$\times \int_0^{x_1} \int_0^{y_1} [(x_1-s)^{\alpha_{i1}-1}(y_1-t)^{\alpha_{i2}-1} - (x_2-s)^{\alpha_{i1}-1}(y_2-t)^{\alpha_{i2}-1}] |v_i(s,t)|\, dt\, ds$$
$$+ \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})} \int_0^{x_1} \int_{y_1}^{y_2} (x_2-s)^{\alpha_{i1}-1}(y_2-t)^{\alpha_{i2}-1} |v_i(s,t)|\, dt\, ds$$
$$+ \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})}$$
$$\times \int_{x_1}^{x_2} \int_0^{y_1} (x_2-s)^{\alpha_{i1}-1}(y_2-t)^{\alpha_{i2}-1} |v_i(s,t)|\, dt\, ds + |\mu_i(x_2, y_2) - \mu_i(x_1, y_1)|$$

$$\leq \frac{\|p\|_\infty \psi(\|(u_1,u_2)\|)}{\Gamma(\alpha_{i1}+1)\Gamma(\alpha_{i2}+1)}$$

$$\times\Big\{2y_2^{\alpha_{i2}}(x_2-x_1)^{\alpha_{i1}}+2x_2^{\alpha_{i1}}(y_2-y_1)^{\alpha_{i2}}-x_2^{\alpha_{i1}}y_2^{\alpha_{i2}}+x_1^{\alpha_{i1}}y_1^{\alpha_{i2}}$$

$$-2(x_2-x_1)^{\alpha_{i1}}(y_2-y_1)^{\alpha_{i2}}\Big\}+\|\mu_i(x_2,y_2)-\mu_i(x_1-y_1)\|$$

and so $|h_i(x_2,y_2)-h_i(x_1,y_1)|\to 0$ as $(x_2,y_2)\to(x_1,y_1)$. Since ϕ_1'' and ϕ_2'' are absolutely continuous, we get

$$|D_x^2h_i(x_2,y_2)-D_x^2h_i(x_1,y_1)|$$

$$\leq \frac{\|p\|_\infty \psi(\|(u_1,u_2)\|)}{\Gamma(\alpha_{i1}-2)\Gamma(\alpha_{i2}+1)}\Big\{2y_2^{\alpha_{i2}}(x_2-x_1)^{\alpha_{i1}-2}+2x_2^{\alpha_{i1}-2}(y_2-y_1)^{\alpha_{i2}}$$

$$-x_2^{\alpha_{i1}-2}y_2^{\alpha_{i2}}+x_1^{\alpha_{i1}-2}y_1^{\alpha_{i2}}-2(x_2-x_1)^{\alpha_{i1}-2}(y_2-y_1)^{\alpha_{i2}}\Big\}+\|\phi_i''(x_2)-\phi_i''(x_1)\|$$

and so $|D_x^2h_i(x_2,y_2)-D_x^2h_i(x_1,y_1)|\to 0$ as $(x_2,y_2)\to(x_1,y_1)$. Similarly, we obtain

$$|D_y^2h_i(x_2,y_2)-D_y^2h_i(x_1,y_1)|$$

$$\leq \frac{\|p\|_\infty \psi(\|(u_1,u_2)\|)}{\Gamma(\alpha_{i1}+1)\Gamma(\alpha_{i2}-1)}\Big\{2y_2^{\alpha_{i2}-2}(x_2-x_1)^{\alpha_{i1}}+2x_2^{\alpha_{i1}}(y_2-y_1)^{\alpha_{i2}-2}$$

$$-x_2^{\alpha_{i1}}y_2^{\alpha_{i2}-2}+x_1^{\alpha_{i1}}y_1^{\alpha_{i2}-2}-2(x_2-x_1)^{\alpha_{i1}}(y_2-y_1)^{\alpha_{i2}-2}\Big\}+\|\psi_i''(y_2)-\psi_i''(y_1)\|$$

and so $|D_y^2h_i(x_2,y_2)-D_y^2h_i(x_1,y_1)|\to 0$ as $(x_2,y_2)\to(x_1,y_1)$. Now by using the Arzela-Ascoli theorem, the operator N is completely continuous. Now, we show that N is upper semi-continuous. By using Lemma 1.2, it is sufficient to show that N has a closed graph. Let $(u_1^n,u_2^n)\in X\times X$, $(u_1^n,u_2^n)\to(u_1^0,u_2^0)$, $(h_1^n,h_2^n)\in N(u_1^n,u_2^n)$ and $(h_1^n,h_2^n)\to(h_1^0,h_2^0)$. We have to show that $(h_1^0,h_2^0)\in N(u_1^0,u_2^0)$. Since (h_1^n,h_2^n) is an element of $N(u_1^n,u_2^n)$, there is $(v_1^n,v_2^n)\in S_{F_1,u_1^n}\times S_{F_2,u_2^n}$ such that

$$h_i^n(x,y)=\frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})}\int_0^x\int_0^y(x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1}v_i^n(s,t)\,dt\,ds+\mu_i(x,y)$$

for all $(x,y)\in J_a\times J_b$ and $i=1,2$. We show that $(v_1^0,v_2^0)\in S_{F_1,u_1^0}\times S_{F_2,u_2^0}$ and

$$h_i^0(x,y)=\frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})}\int_0^x\int_0^y(x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1}v_i^0(s,t)\,dt\,ds+\mu_i(x,y)$$

for all $(x,y)\in J_a\times J_b$ and $i=1,2$. Consider the linear operators

$$\Theta_i: L^1(J_a\times J_b,X)\to C(J_a\times J_b,X)$$

defined by

$$\Theta_i(v)(x,y) = \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})} \int_0^x \int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1} v(s,t)\, dt\, ds + \mu_i(x,y)$$

for $i = 1, 2$. Since

$$\|h_i^n(x,y) - h_i^0(x,y)\|$$

$$= \left\| \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})} \int_0^x \int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1}[v_i^n(s,t) - v_i^0(s,t)]\, dt\, ds \right\| \to 0,$$

by using Lemma 1.3 we get $\Theta_i \circ S_{F_i}$ is a closed graph operator and so $h_i^n(x,y) \in \Theta_i(S_{F_i,u_i^n})$. Since $u_i^n \to u_i^0$,

$$h_i^0(x,y) = \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})} \int_0^x \int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1} v_i^0(s,t)\, dt\, ds + \mu_i(x,y)$$

for some $v_i^0 \in S_{F_i,u_i^0}$. This implies that N has a closed graph. Finally, suppose that $\lambda \in (0,1)$ and $(u_1, u_2) \in \lambda N(u_1, u_2)$. Choose $v_i \in S_{F_i,u_i}$ such that

$$u_i(x,y) = \frac{1}{\Gamma(\alpha_{i1})\Gamma(\alpha_{i2})} \int_0^x \int_0^y (x-s)^{\alpha_{i1}-1}(y-t)^{\alpha_{i2}-1} v_i(s,t)\, dt\, ds + \mu_i(x,y)$$

for all $(x,y) \in J_a \times J_b$ and $i = 1, 2$. As we have proved in the second step, we get $\|u_i\| \leq (\Lambda_{i1} + \Lambda_{i2} + \Lambda_{i3})\|p\|_\infty \psi(\|u_i\|) + M_i$ for $i = 1, 2$ and so $\frac{\|u_i\|}{(\Lambda_{i1} + \Lambda_{i2} + \Lambda_{i3})\|p\|_\infty \psi(\|u_i\|) + M_i} \leq 1$ for $i = 1, 2$. Choose constants $L_i > 0$ such that

$$\frac{L_i}{(\Lambda_{i1} + \Lambda_{i2} + \Lambda_{i3})\|p\|_\infty \psi(L_i) + M_i} > 1$$

and $\|u_i\| \neq L_i$ for $i = 1, 2$. Put $U = \{(u_1, u_2) \in X \times X : \|(u_1, u_2)\| < \min\{L_1, L_2\}\}$. Note that, the operator $N : \overline{U} \to \mathcal{P}(X)$ is upper semi-continuous, completely continuous and there is no $(u_1, u_2) \in \partial U$ such that $(u_1, u_2) \in \lambda N(u_1, u_2)$ for some $\lambda \in (0,1)$. By using Theorem 1.4, N has a fixed point $(u_1, u_2) \in \overline{U}$ which is a solution for the coupled system of hyperbolic partial fractional differential inclusions (1.3) with boundary conditions (1.4). □

One can extend the coupled system of hyperbolic partial fractional differential inclusions (1.3) with boundary conditions (1.4) to $n-$dimensional case.

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